$\mathcal{I}_{\mathbf{g}}^*$-CLOSED SETS VIA IDEAL TOPOLOGICAL SPACES

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Abstract. In this paper, aspects of generalized continuity and generalized closedness are explored. The standard material on the notions of $*g$-open, $\mathbf{g}$-open sets and some definitions and results that are needed are presented first. Then the class of $\mathcal{I}_{\mathbf{g}}^*$-closed sets is introduced and its fundamental properties are studied. Also, $\mathcal{I}_{\mathbf{g}}^*$-regular, *-additive, *-multiplicative, $\mathcal{I}_{\mathbf{g}}^*$-additive, and $\mathcal{I}_{\mathbf{g}}^*$-multiplicative spaces are introduced and their properties are investigated.

1. Introduction

Ideal topological spaces have been considered since 1930. This topic has earned its importance via a paper by Vaidyanathaswamy [45]. Jankovic and Hamlett [19] began the generalization of some important properties in general topology via topological ideals such as decomposition of continuity, separation axioms, connectedness, compactness, and resolvability. Several types of generalized closed sets are investigated in the literature of topological spaces [39, 28, 27, 26, 42, 40, 44, 18, 2, 4]. Using the concept of generalized closed sets, several separation axioms [13] are introduced which are found to be useful in the study of digital topology (digital line) [23]. Dontchev et al. [12, 24] obtained several characterizations of extremally disconnectedness in terms of generalized closed sets in ideal topological space. The purpose of this paper is to show that these diagrams can be obtained in the setting of generalized topological spaces (GTSs) via ideal topology introduced by Ravi, Helen, et al. [44, 18].

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An ideal $\mathcal{I}$ on a nonempty set $\tilde{X}$ is a nonempty collection of subsets of $\tilde{X}$ which satisfies the following conditions: $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$, $A \in \mathcal{I}$ and $B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$ [29]. Applications to various fields were further investigated by Jankovic and Hamlett [19], Mukherjee et al. [35], Arenas et al. [7], Nasef, and Mahmoud [37], etc. Given a topological space $(X, \mathcal{T})$ with an ideal $\mathcal{I}$ on X, $\wp(\tilde{X})$ is the set of all subsets of X and a set operator $(.)^* : \wp(X) \to \wp(X)$, called a local function [29] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subseteq X$,

$$A^*(\mathcal{I}, \tau) = \left\{ x \in \tilde{X} \mid U \cap A \notin \mathcal{I} \, \text{for every} \, U \in \mathcal{T}(x) \right\},$$

where $\tau(x) = \{U \in \tau \mid x \in U\}$. Furthermore, $Cl^*(A) = A \cup A^*(\mathcal{I}, \mathcal{T})$ defines a
Kuratowski closure operator for the topology $\mathcal{T}^*$, finer than $\mathcal{T}$. When there is no chance for confusion, we will simply write A^* for $A^*(\mathcal{I}, \mathcal{T})$. $\tilde{X}^*$ is often a proper subset of $\tilde{X}$. By a space, we always mean a topological space $\left(\tilde{X}, \mathcal{T}\right)$ with no separation properties assumed. If $A \subset \tilde{X}$, let $Cl(A)$ and $Int(A)$ denote the closure and interior of A in $\left(\tilde{X}, \mathcal{T}\right)$, respectively.

A subset A of an ideal space $\left(\tilde{X}, \mathcal{T}\right)$ is said to be $R_{\mathcal{I}}$-open (resp. $R_{\mathcal{I}}$-closed) [46] if $A = Int(Cl^*(A))$ (resp. $A = Cl^*(Int(A))$. A point $\tilde{x} \in \tilde{X}$ is called a $\delta\mathcal{I}$-cluster point of A if $Int(Cl^*(U)) \cap A \neq \emptyset$ for each open set V containing $\tilde{x}$. The family of all $\delta\mathcal{I}$-cluster points of A is called the $\delta\mathcal{I}$-closure of A and is denoted by $\delta Cl_I(A)$. The set $\delta\mathcal{I}$-interior of A is the union of all $R_{\mathcal{I}}$-open sets of $\tilde{X}$ contained in A and it is denoted by $\delta Int_I(A)$. A is said to be $\delta\mathcal{I}$-closed if $\delta Cl_I(A) = A$ [46].

A set operator $()^{*S} : P(\tilde{X}) \to P(\tilde{X})$ is called a semi local function and Cl^{*S} [1] of A with respect to $\mathcal{T}$ and $\mathcal{I}$ are defined as follows:
For $A \subset \tilde{X}$,

$$A^{*S}(\mathcal{I}, \mathcal{T}) = \left\{ \tilde{x} \in \tilde{X} \mid \tilde{U} \cap A \notin \mathcal{I} \text{ for each semi-open} \, \tilde{U} \, \text{containing} \, \tilde{x} \right\}$$

and $Cl^{*S} = A \cup A^{*S}$.

This paper is arranged as follows. Section 2 contains some necessary concepts of general topology and recollections of generalized closed sets in ideal topological space. In Section 3, we give definitions to present $\mathcal{I}_g^*$-closed sets and $G{*}_I$-space. Additionally, we give their characterizations and explain a number of properties about them. In Section 4, we propose the relationship with regular $\mathcal{I}_g^*$-closed sets, $\mathcal{I}_g^*$-open neighborhood, *-finitely additive, *-countably additive, and *-additive in ideal topological space. Theoretical applications of $\mathcal{I}_g^*$-closed sets are also presented.

2. Preliminaries

Yüksel introduced the class of **g**-closed sets [32], a super class of closed sets in 1970. S. P. Arya and T. Nour [8] defined gs-closed sets in 1990, which were used for characterizing s-normal spaces. Dontchev [11], Gnanambal [14], and Palaniappan and Rao [41] introduced gsp-closed sets, gpr-closed sets, and rg-closed sets, respectively. Kumar [27] introduced a new class of sets (using a new technique) called g^*-closed sets, which is properly placed in between the class of closed sets and the class of **g**-closed sets.

Definition 2.1. A subset A of a topological space $(\tilde{X}, \mathcal{T})$ is said to be

(1) α-open [5] if $A \subset Int(Cl(Int(A)))$.
(2) semi-open [30] if $A \subset Cl(Int(A))$.
(3) pre-open [4] if $A \subset Int(Cl(A))$.
(4) generalized closed [32] if $Cl(A) \subseteq V$, $A \subseteq V$, and V is open in $(\tilde{X}, \mathcal{T})$. The complement of a g-closed set is called a g-open set.
(5) semi-generalized closed [9] if $sCl(A) \subseteq V$, $A \subseteq V$, and V is semi-open in $(\tilde{X}, \mathcal{T})$.
(6) generalized semi closed [8] if $sCl(A) \subseteq V$, $A \subseteq V$, and V is open in $(\tilde{X}, \mathcal{T})$.
(7) g^*-closed [27] if $Cl(A) \subseteq V$, $A \subseteq V$, and V is **g**-open in $(\tilde{X}, \mathcal{T})$.
(8) $\hat{g}$-closed [27] if $Cl(A) \subseteq V$, $A \subseteq V$, and V is semi-open in $(\tilde{X}, \mathcal{T})$. The complement of a $\hat{g}$-closed set is called a $\hat{g}$-open set.
(9) ${}^*g$-closed [28] if $sCl(A) \subseteq V$, $A \subseteq V$, and V is semi-open in $(\tilde{X}, \mathcal{T})$.
(10) α-generalized closed (briefly αg-closed) [33] if $\alpha Cl(A) \subseteq V$, $A \subseteq V$, and V is α-open in $(\tilde{X}, \mathcal{T})$.
(11) pg-closed [39] if $pCl(A) \subseteq V$, $A \subseteq V$, and V is pre-open in $(\tilde{X}, \mathcal{T})$.
(12) sg^*-closed [36] if $Cl(A) \subseteq V$, $A \subseteq V$, and V is semi-open in $(\tilde{X}, \mathcal{T})$.
(13) pg^*-closed [36] if $Cl(A) \subseteq V$, $A \subseteq V$, and V is pre-open in $(\tilde{X}, \mathcal{T})$.
(14) αg^*-closed [36] if $Cl(A) \subseteq V$, $A \subseteq V$, and V is α-open in $(\tilde{X}, \mathcal{T})$.
(15) βg^*-closed [36] if $Cl(A) \subseteq V$, $A \subseteq V$, and V is semi-pre-open in $(\tilde{X}, \mathcal{T})$.

Definition 2.2. A function $f : (\tilde{X}, \mathcal{T}) \longrightarrow (Y, \sigma)$ is said to be

(1) weakly continuous [31] if for all $\tilde{x} \in \tilde{X}$ and for all open sets U in Y containing $f(x)$, there exists an open set V containing $\tilde{x}$ such that $f(V) \subset Cl(U)$.
(2) weakly pre-continuous [34] if for all $\tilde{x} \in \tilde{X}$ and for all open sets U in Y containing $f(x)$, there exists a pre-open set V containing $\tilde{x}$ such that $f(V) \subset Cl(U)$.

(3) weakly semi-continuous [22] if for all $\tilde{x} \in \tilde{X}$ and for all open set U in Y containing $f(x)$, there exists an semi-open set V containing $\tilde{x}$ such that $f(V) \subset Cl(U)$.
(4) weakly pre-$\mathcal{I}$-continuous [16] if for all $\tilde{x} \in \tilde{X}$ and for all open set U in Y containing $f(x)$, there exists a pre-$\mathcal{I}$-open set V containing $\tilde{x}$ such that $f(V) \subset Cl(U)$.
(5) weakly semi-$\mathcal{I}$-continuous [17] if for all $\tilde{x} \in \tilde{X}$ and for all open set U in Y containing $f(x)$, there exists a semi-$\mathcal{I}$-open set V containing $\tilde{x}$ such that $f(V) \subset Cl(U)$.

Definition 2.3. A subset A of a space $(\tilde{X}, \mathcal{T}, \mathcal{I})$ is said to be

(1) β-open [15] if $A \subset Cl(Int(Cl(A)))$.
(2) semi-$\mathcal{I}$-open [3] if $A \subset Cl^*(Int(A))$.
(3) pre-$\mathcal{I}$-open [6] if $A \subset Int(Cl^*(A))$.
(4) $g\mathcal{I}$-closed [12] if $A^{*S} \subseteq V$, $A \subseteq V$, and V is open in $(\tilde{X}, \mathcal{T})$.
(5) $\mathcal{I}_g$-closed [24] if $A^* \subseteq V$, $A \subseteq V$, and V is open in $(\tilde{X}, \mathcal{T})$.
(6) $sg\mathcal{I}$-closed [25] if $A^{*S} \subseteq V$, $A \subseteq V$, and V is semi open in $(\tilde{X}, \mathcal{T})$.
(7) $\mathcal{I}_{s*g}$-closed [26] if $A^* \subseteq V$, $A \subseteq V$, and V is semi open in $(\tilde{X}, \mathcal{T})$.
(8) $rg\mathcal{I}$-closed [42] if $A_* \subseteq V$, $A \subseteq V$, and V is open in $(\tilde{X}, \mathcal{T})$.
(9) $\mathcal{I} - sg$-closed [40] if $A^* \subseteq V$, $A \subseteq V$, and V is semi open in $(\tilde{X}, \mathcal{T})$.
(10) $\mathcal{I} - pg$-closed [40] if $A^* \subseteq V$, $A \subseteq V$, and V is pre open in $(\tilde{X}, \mathcal{T})$.
(11) $\mathcal{I} - \alpha g$-closed [40] if $A^* \subseteq V$, $A \subseteq V$, and V is α-open in $(\tilde{X}, \mathcal{T})$.
(12) $\mathcal{I} - \beta g$-closed [40] if $A^* \subseteq V$, $A \subseteq V$, and V is β-open in $(\tilde{X}, \mathcal{T})$.
(13) $*g\mathcal{I}$-closed [44] if $A_* \subseteq V$, $A \subseteq V$, and V is $\hat{g}$-open in $(\tilde{X}, \mathcal{T})$.
(14) $\mathcal{I}*g$-closed [44] if $A^* \subseteq V$, $A \subseteq V$, and V is $\hat{g}$-open in $(\tilde{X}, \mathcal{T})$.
(15) $g^{*S}\mathcal{I}$-closed [18] if $A^{s*} \subseteq V$, $A \subseteq V$, and V is $\mathbf{g}$-open in $(\tilde{X}, \mathcal{T})$.

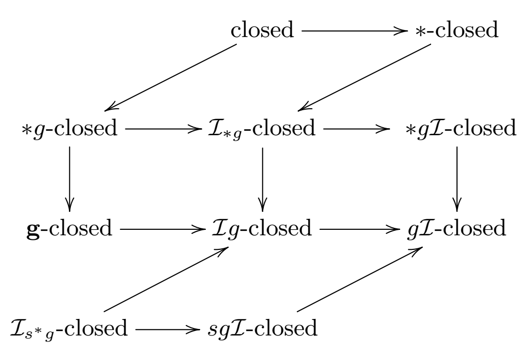

Diagram 2.2.1 (*Generalized closed sets*)

3. $\mathcal{I}_g^*$-Closed Sets

Definition 3.1. A subset F of an ideal space $(\tilde{X}, \mathcal{T}, \mathcal{I})$ is said to be $\mathcal{I}_{\mathbf{g}}^*$-closed if $Cl^*(F) \subset V$, $F \subset V$, and V is **g**-open in $\tilde{X}$.

The complement of $\mathcal{I}_{\mathbf{g}}^*$-closed set is said to be $\mathcal{I}_{\mathbf{g}}^*$-open. The collection of all $\mathcal{I}_{\mathbf{g}}^*$-closed sets (resp. $\mathcal{I}_{\mathbf{g}}^*$-open sets) is denoted by $G^*C(\tilde{X})$ (resp. $G^*O(\tilde{X})$).

Remark 3.2. *Let $(\tilde{X}, \mathcal{T}, \mathcal{I})$ be an ideal topological space and F a subset of $\tilde{X}$. Then the following are true.*

1. *If $\mathcal{I} = P(\tilde{X})$, then $Cl^*(F) = F$ for all $F \subset \tilde{X}$ and hence, every subset of $\tilde{X}$ is $\mathcal{I}_{\mathbf{g}}^*$-closed.*
2. *Since $F^* = \emptyset$ for every $F \subset \tilde{X}$, every member of $\mathcal{I}$ is $\mathcal{I}_{\mathbf{g}}^*$-closed.*
3. *Since every open set is **g**-open, every $\mathcal{I}_{\mathbf{g}}^*$-closed set is $g\mathcal{I}$-closed.*
4. *$\mathcal{I}_{\mathbf{g}}^*$-closed and $\mathcal{I}_{s^*g}$-closed are independent concepts.*
5. *$\mathcal{I}_{\mathbf{g}}^*$-closed and $\mathcal{I}_g$-closed are independent concepts.*
6. *$\mathcal{I}_{\mathbf{g}}^*$-closed and $sg\mathcal{I}$-closed are independent concepts.*
7. *Every $\mathcal{I}_{\mathbf{g}}^*$-closed set is $g^{*S}\mathcal{I}$-closed set.*
8. *Every $\mathcal{I}_{\mathbf{g}}^*$-closed set is $\mathcal{I}{*}g$-closed set.*

The following examples show the converse is not true in general for Remark 3.2.

Example 3.3. *Let $\tilde{X} = \{a, b, c, d\}$ with topology $\mathcal{T} = \{\emptyset, \tilde{X}, \{a, b\}\}$ and ideal $\mathcal{I} = \{\emptyset, \{a\}\}$. Let $F = \{a, c\}$. Then $F^* = F_* = \{c, d\}$. Thus, F is $\mathcal{I}_{\mathbf{g}}^*$-closed and $g^{*S}\mathcal{I}$-closed.*

Example 3.4. *Let $\tilde{X} = \{a, b, c\}$ with topology $\mathcal{T} = \{\emptyset, \tilde{X}, \{c\}, \{a, b\}\}$ and ideal $\mathcal{I} = \{\emptyset, \{a\}\}$. Then the $\mathcal{I}{*}g$-closed sets are $\emptyset, \tilde{X}, \{a\}, \{a, b\}, \{a, c\}$. If $F = \{a\}$, then $F^* = \{\emptyset\}$. It is clear that $\{a\}$ is $\mathcal{I}{*}g$-closed but not $\mathcal{I}_{\mathbf{g}}^*$-closed.*

Example 3.5. *Let $\tilde{X} = \{a, b, c, d\}$ with topology $\mathcal{T} = \{\emptyset, \tilde{X}, \{a\}, \{b\}, \{a, b\}\}$ and ideal $\mathcal{I} = \{\emptyset, \{b\}\{c\}, \{b, c\}\}$. Then $F = \{a\}$ is $g^{*S}\mathcal{I}$-closed but not $\mathcal{I}_{\mathbf{g}}^*$-closed. Then $F^* = \{a, c, d\}$ and $F_* = \{a\}$, where F_* is contained in the **g**-open set $U = \{a\}$ but $F^* \nsubseteq U$.*

Example 3.6. *Let $\tilde{X} = \{a, b, c, d\}$ with topology $\mathcal{T} = \{\emptyset, \tilde{X}, \{a\}, \{b\}, \{a, b\}\}$ and ideal $\mathcal{I} = \{\emptyset\}$. Then $F = \{a\}$ is $g^{*S}\mathcal{I}$-closed and $\mathcal{I}_{\mathbf{g}}^*$-closed, since $F^* = F_*$ but not $\mathcal{I}g$-closed.*

Example 3.7. *Let $(\tilde{X}, \mathcal{T})$ be an indiscrete space, $x_0 \in \tilde{X}$, and $\mathcal{I} = \{\emptyset, \{x_0\}\}$. Then $F^* = \tilde{X}$ if $F \neq \{x_0\}$, and $F^* = \emptyset$ if $F = \{x_0\}$. For any subset $F \neq \{x_0\}$ is $\mathcal{I}_g$-closed, $g\mathcal{I}$-closed, $sg\mathcal{I}$-closed, and $\mathcal{I}_{s^*g}$-closed, but not $\mathcal{I}_{\mathbf{g}}^*$-closed.*

Example 3.8. *Let $(\tilde{X}, \mathcal{T})$ be an indiscrete space, $p \in \tilde{X}$, and $\mathcal{I} = \{F \subset \tilde{X}/p \notin F\}$. Then $F^* = \tilde{X}$ if $p \in F$, and $F^* = \emptyset$ if $p \notin F$. $F = \{p\}$ is $\mathcal{I}_g$-closed, $g\mathcal{I}$-closed, $sg\mathcal{I}$-closed, $\mathcal{I}_{s^*g}$-closed, but not $\mathcal{I}_{\mathbf{g}}^*$-closed.*

Theorem 3.9. *If $(\tilde{X}, \mathcal{T}, \mathcal{I})$ be an ideal topological space and $F \subset \tilde{X}$. Then the following are true.*

(1) F is $\mathcal{I}_{\mathbf{g}}^*$-closed,
(2) $Cl^*(F) \subseteq V$ if $F \subseteq V$ and V is $\mathbf{g}$-open in $\tilde{X}$,
(3) For all $\tilde{x} \in Cl^*(F)$, $gCl(\{\tilde{x}\}) \cap F \neq \emptyset$,
(4) $Cl^*(F) - F$ does not contain a $\mathbf{g}$-closed set and
(5) $F^* - F$ does not contain a $\mathbf{g}$-closed set.

Proof. (1) $\Rightarrow$ (2): If F is $\mathcal{I}_{\mathbf{g}}^*$-closed, then $F^* \subset V$ if $F \subseteq V$ and V is $\mathbf{g}$-open in $\tilde{X}$ and so $Cl^*(F) = F \cup F^* \subseteq V$ if $F \subseteq V$ and V is $\mathbf{g}$-open in $\tilde{X}$. This proves (2).

(2) $\Rightarrow$ (3): Suppose $\tilde{x} \in Cl^*(F)$. If $gCl(\{\tilde{x}\}) \cap F = \emptyset$, then $F \subseteq \tilde{X} - gCl(\{\tilde{x}\})$. By (2), $Cl^*(F) \subseteq \tilde{X} - gCl(\{\tilde{x}\})$, a contradiction since $\tilde{x} \in Cl^*(F)$.

(3) $\Rightarrow$ (4): Suppose $F \subseteq Cl^*(F) - F$, F is $\mathbf{g}$-closed, and $\tilde{x} \in F$. Since $F \subseteq \tilde{X} - F$ and F is $\mathbf{g}$-closed, $F \subseteq \tilde{X} - F$, F is $\mathbf{g}$-closed, and $gCl(\{\tilde{x}\}) \cap F = \emptyset$. Since $\tilde{x} \in Cl^*(F)$, by (3), $gCl(\{\tilde{x}\}) \cap F \neq \emptyset$. Therefore, $Cl^*(F) - F$ does not contain an $\mathbf{g}$-closed set.

(4) $\Rightarrow$ (5): We know

$$Cl^*(F) - F = (F \cup F^*) - F = (F \cup F^*) \cap F^c = (F \cap F^c) \cup (F^* \cap F^c) = F^* \cap F^c = F^* - F.$$

Therefore, $F^* - F$ does not contains a $\mathbf{g}$-closed set.

(5) $\Rightarrow$ (1): Let $F \subseteq V$, where V is $\mathbf{g}$-open set. Therefore, $\tilde{X} - V \subseteq \tilde{X} - F$ and is a $F^* \cap (\tilde{X} - V) \subseteq F^* \cap (\tilde{X} - F)$=$F^* - F$. Therefore, $F^* \cap (\tilde{X} - V) \subseteq F^* - F$. Since F^* is always a closed set, F^* is a $\mathbf{g}$-closed set and so $F^* \cap (\tilde{X} - V)$ is a $\mathbf{g}$-closed set contained in $F^* - F$. Therefore, $F^* \cap (\tilde{X} - V) = \emptyset$ and hence, $F^* \subseteq V$. Therefore, F is $\mathcal{I}_{\mathbf{g}}^*$-closed. □

Theorem 3.10. *Every $*$-closed set is $\mathcal{I}_{\mathbf{g}}^*$-closed.*

Proof. Let F be $*$-closed. Then $F^* \subseteq F$. Let $F \subseteq V$, where V is $\mathbf{g}$-open. Hence, $F^* \subseteq V$ whenever $F \subseteq V$ and V is $\mathbf{g}$-open. Therefore, F is $\mathcal{I}_{\mathbf{g}}^*$-closed. □

Example 3.11. *Let $\tilde{X} = \{a, b, c\}$ with topology $\mathcal{T} = \{\emptyset, \tilde{X}, \{a\}, \{b, c\}\}$ and ideal $\mathcal{I} = \{\emptyset, \{c\}\}$. Then the $\mathcal{I}_{\mathbf{g}}^*$-closed sets are the powerset of $\tilde{X}$ and $*$-closed sets are $\{\emptyset, \tilde{X}, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$. It clear that $\{b\}$ is a $\mathcal{I}_{\mathbf{g}}^*$-closed set, but it is not $*$-closed.*

Definition 3.12. A space is $(\tilde{X}, \mathcal{T}, \mathcal{I})$ called $G*_I$-space if every $\mathcal{I}_{\mathbf{g}}^*$-closed set in it is closed.

Theorem 3.13. *Let $(\tilde{X}, \mathcal{T}, \mathcal{I})$ be an ideal space. For every $F \in \mathcal{I}$, F is $\mathcal{I}_{\mathbf{g}}^*$-closed.*

Proof. Let $F \subseteq V$, where V is a $\mathbf{g}$-open set. Since $F^* = \emptyset$ for every $F \in \mathcal{I}$, then $Cl^*(F) = F \cup F^* = F \subseteq V$. Therefore, by Theorem 3.9, F is $\mathcal{I}_{\mathbf{g}}^*$-closed. □

Theorem 3.14. *If $(\tilde{X}, \mathcal{T}, \mathcal{I})$ is any ideal space, then F^* is always $\mathcal{I}_{\mathbf{g}}^*$-closed for every subset F of $\tilde{X}$.*

Proof. Let $F^* \subseteq V$, where V is $\mathbf{g}$-open. Since $(F^*)^* \subseteq F^*$, we have $(F^*)^* \subseteq V$ if $F^* \subset V$ and V is $\mathbf{g}$-open. Hence, F^* is $\mathcal{I}^*_{\mathbf{g}}$-closed. □

Theorem 3.15. *Let $(\tilde{X}, \mathcal{T}, \mathcal{I})$ be an ideal space. Then every $\mathcal{I}^*_{\mathbf{g}}$-closed, $\mathbf{g}$-open set is a $*$-closed set.*

Proof. We have F is $\mathcal{I}^*_{\mathbf{g}}$-closed and $\mathbf{g}$-open. Then $F^* \subseteq F$ whenever $F \subseteq F$ and F is $\mathbf{g}$-open. Hence, F is $*$-closed. □

Corollary 3.16. *If $(\tilde{X}, \mathcal{T}, \mathcal{I})$ is a $T_{\mathcal{I}}$ ideal space and F is an $\mathcal{I}^*_{\mathbf{g}}$-closed set, then F is $*$-closed.*

Corollary 3.17. *If $(\tilde{X}, \mathcal{T}, \mathcal{I})$ is an ideal space and F is an $\mathcal{I}^*_{\mathbf{g}}$-closed set, then the following are equivalent.*
1. *F is a $*$-closed set.*
2. *$Cl^*(F) - F$ is a $\mathbf{g}$-closed set.*
3. *$F^* - F$ is a $\mathbf{g}$-closed set.*

Proof. (1) ⇒ (2): If F is $*$-closed, then $F^* \subseteq F$ and so $Cl^*(F) - F = (F \cup F^*) - F = \emptyset$. Hence, $Cl^*(F) - F$ is $\mathbf{g}$-closed set.

(2) ⇒ (3): Since $Cl^*(F) - F = F^* - F$, we have $F^* - F$ is a $\mathbf{g}$-closed set.

(3) ⇒ (1): If $F^* - F$ is a $\mathbf{g}$-closed set, since F is $\mathcal{I}^*_{\mathbf{g}}$-closed set, by Theorem 3.9, $F^* - F = \emptyset$ and so F is $*$-closed. □

Theorem 3.18. *If $(\tilde{X}, \mathcal{T}, \mathcal{I})$ is an ideal space, then every $\mathbf{g}$-closed set is a $\mathcal{I}^*_{\mathbf{g}}$-closed set, but not conversely.*

Proof. Let F be a $\mathbf{g}$-closed set. Then $Cl(F) \subseteq V$ if $F \subseteq V$ and V is $\mathbf{g}$-open. We have $Cl^*(F) \subseteq Cl(F) \subseteq V$ if $F \subseteq V$ and V is $\mathbf{g}$-open. Hence, F is $\mathcal{I}^*_{\mathbf{g}}$-closed. □

Example 3.19. *Let $\tilde{X} = \{a, b, c\}$ with topology $\mathcal{T} = \{\emptyset, \tilde{X}, \{a\}, \{a, c\}\}$ and ideal $\mathcal{I} = \{\emptyset, \{a\}\}$. Then the $\mathcal{I}^*_{\mathbf{g}}$-closed sets are $\emptyset, \tilde{X}, \{a\}, \{b\}, \{a, b\}, \{b, c\}$ and the $\mathbf{g}$-closed sets are $\emptyset, \tilde{X}, \{b\}, \{b, c\}$. It is clear that $\{a\}$ is a $\mathcal{I}^*_{\mathbf{g}}$-closed set, but it is not $\mathbf{g}$-closed in $(\tilde{X}, \mathcal{T})$.*

Example 3.20. *Let $\tilde{X} = \{a, b, c\}$ with topology $\mathcal{T} = \{\emptyset, \tilde{X}, \{a\}, \{a, c\}\}$ and ideal $\mathcal{I} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$. Clearly, the set $\{c\}$ is $\mathcal{I}^*_{\mathbf{g}}$-closed, but it is not $\mathbf{g}$-closed in $(\tilde{X}, \mathcal{T})$.*

Lemma 3.21. [10] *Let $(\tilde{X}, \mathcal{T}, \mathcal{I})$ be an ideal space and $F \subseteq \tilde{X}$. If $F \subseteq F^*$, then $F^* = Cl(F^*) = Cl(F) = Cl^*(F)$.*

Theorem 3.22. *If $(\tilde{X}, \mathcal{T}, \mathcal{I})$ is any topological ideal space, F is $*$-dense in itself and an $\mathcal{I}^*_{\mathbf{g}}$-closed subset of $\tilde{X}$, then F is g^*-closed.*

Proof. Suppose F is $*$-dense in itself and an $\mathcal{I}^*_{\mathbf{g}}$-closed subset of $\tilde{X}$. Let $F \subseteq V$, where V is $\mathbf{g}$-open. Then by Theorem 3.9 (2), $Cl^*(F) \subseteq V$ if $F \subseteq V$ and V is $\mathbf{g}$-open. Since F is $*$-dense in itself, by Lemma 3.21, $Cl(F) = Cl^*(F)$. Therefore, $Cl(F) \subseteq V$ if $F \subseteq V$ and V is $\mathbf{g}$-open. Hence, F is g^*-closed. □

Corollary 3.23. *If $(\tilde{X}, \mathcal{T}, \mathcal{I})$ is an ideal topological space where $\mathcal{I} = \{\emptyset\}$, then F is $\mathcal{I}_{\mathbf{g}}^*$-closed if and only if F is $\mathbf{g}$-closed.*

Proof. Since $\mathcal{I} = \{\emptyset\}$, $F^* = Cl(F) \supseteq F$. Therefore, F is $*$-dense in itself. Since F is $\mathcal{I}_{\mathbf{g}}^*$-closed, by Theorem 3.22, F is $\mathbf{g}$-closed. Conversely, by Theorem 3.18, every $\mathbf{g}$-closed set is $\mathcal{I}_{\mathbf{g}}^*$-closed. □

Corollary 3.24. *If $(\tilde{X}, \mathcal{T}, \mathcal{I})$ is an ideal topological space, where $\mathcal{I}$ is codense and F is a semi-open, $\mathcal{I}_{\mathbf{g}}^*$-closed subset of $\tilde{X}$, then F is $\mathbf{g}$-closed.*

Proof. Since $\mathcal{I} = \{\emptyset\}$, $F^* = Cl(F) \supseteq F$. Therefore, F is $*$-dense in itself. Since F is $\mathcal{I}_{\mathbf{g}}^*$-closed, by Theorem 3.22, F is $\mathbf{g}$-closed. Conversely, by Theorem 3.18, every $\mathbf{g}$-closed set is $\mathcal{I}_{\mathbf{g}}^*$-closed set. □

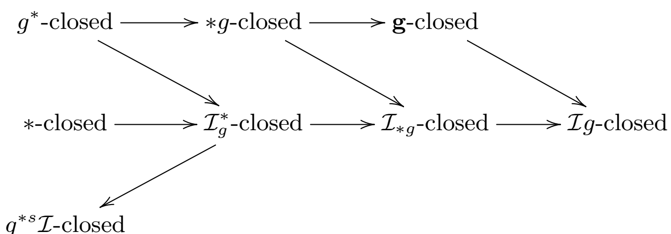

Diagram 3.1.1 (Relation between generalized closed sets)

Theorem 3.25. *Let $(\tilde{X}, \mathcal{T}, \mathcal{I})$ be an ideal space and $F \subseteq \tilde{X}$. Then F is $\mathcal{I}_{\mathbf{g}}^*$-closed if and only if $F = S - N$, where S is $*$-closed and N contains no nonempty $\mathbf{g}$-closed set.*

Proof. If F is $\mathcal{I}_{\mathbf{g}}^*$-closed, then by Theorem 3.9 (5), $N = F^* - F$ contains no nonempty $\mathbf{g}$-closed set. If $S = Cl^*(F)$, then S is $*$-closed and

$$\begin{aligned} S - N &= (F \cup F^*) - (F^* - F) = (F \cup F^*) \cap (F^* \cap F^c)^c = (F \cup F^*) \cap ((F^*)^c \cup F) \\ &= (F \cup F^*) \cap (F \cup (F^*)^c) = F \cup (F^* \cap (F^*)^c) = F. \end{aligned}$$

Conversely, suppose $F = S - N$, where S is $*$-closed and N contains no nonempty $\mathbf{g}$-closed set. Let V be an $\mathbf{g}$-open set such that $F \subseteq V$. Then $S - N \subseteq V$ which implies that $S \cap (\tilde{X} - V) \subseteq N$. Now $F \subseteq S$ and since $S^* \subseteq S$, $F^* \subseteq S^*$ and so $F^* \cap (\tilde{X} - V) \subseteq S^* \cap (\tilde{X} - V) \subseteq S \cap (\tilde{X} - V) \subseteq N$. By hypothesis, since $F^* \cap (\tilde{X} - V)$ is $\mathbf{g}$-closed, $F^* \cap (\tilde{X} - V) = \emptyset$ and so $F^* \subseteq V$. Hence, F is $\mathcal{I}_{\mathbf{g}}^*$-closed. □

Theorem 3.26. *Let $(\tilde{X}, \mathcal{T}, \mathcal{I})$ be an ideal space and $F \subseteq \tilde{X}$. If $F \subseteq D \subseteq F^*$, then $F^* = D^*$ and D is $*$-dense in itself.*

Proof. Since $F \subseteq D$, then $F^* \subseteq D^*$ and since $D \subseteq F^*$, then $D^* \subseteq (F^*)^* \subseteq F^*$. Therefore, $F^* = D^*$ and $D \subseteq F^* \subseteq D^*$. This completes the proof. □

Theorem 3.27. *Let $(\tilde{X}, \mathcal{T}, \mathcal{I})$ be an ideal space and $F \subseteq \tilde{X}$. If F and D are subsets of $\tilde{X}$ such that $F \subseteq D \subseteq Cl^*(F)$ and F is $\mathcal{I}_{\mathbf{g}}^*$-closed, then D is $\mathcal{I}_{\mathbf{g}}^*$-closed.*

Proof. Since F is $\mathcal{I}^*_{\mathbf{g}}$-closed, then by Theorem 3.9 (5), $Cl^*(F) - F$ contains no nonempty $\mathbf{g}$-closed set. We have $Cl^*(D) - D \subseteq Cl^*(F) - F$ and so $Cl^*(D) - D$ contains no nonempty $\mathbf{g}$-closed set. Hence, D is $\mathcal{I}^*_{\mathbf{g}}$-closed. □

Corollary 3.28. *Let $(\tilde{X}, \mathcal{T}, \mathcal{I})$ be an ideal space. If F and B are subsets of $\tilde{X}$ such that $F \subseteq B \subseteq F^*$ and F is $\mathcal{I}^*_{\mathbf{g}}$-closed, then F and B are g^*-closed sets.*

Proof. Let F and B be subsets of $\tilde{X}$ such that $F \subseteq B \subseteq F^*$. This implies that $F \subseteq B \subseteq F^* \subseteq Cl^*(F)$ and F is $\mathcal{I}^*_{\mathbf{g}}$-closed. By Theorem 3.27, B is $\mathcal{I}^*_{\mathbf{g}}$-closed. Since $F \subseteq B \subseteq F^*$, $F^* = B^*$ and so F and B are $*$-dense in itself. By Theorem 3.22, F and B are g^*-closed. □

Theorem 3.29. *Let $(\tilde{X}, \mathcal{T}, \mathcal{I})$ be an ideal space and $F \subseteq \tilde{X}$. Then F is $\mathcal{I}^*_{\mathbf{g}}$-open if and only if $S \subseteq Int^*(F)$, whenever S is $\mathbf{g}$-closed and $S \subseteq F$.*

Proof. Suppose F is $\mathcal{I}^*_{\mathbf{g}}$-open. If S is $\mathbf{g}$-closed and $S \subseteq F$, then $\tilde{X} - F \subseteq \tilde{X} - S$ and so, $Cl^*(\tilde{X} - F) \subseteq \tilde{X} - S$ by Theorem 3.9 (2). Therefore, $S \subseteq \tilde{X} - Cl^*(\tilde{X} - F) = Int^*(F)$. Hence, $S \subseteq Int^*(F)$. Conversely, suppose the condition holds. Let V be a $\mathbf{g}$-open set such that $\tilde{X} - F \subseteq V$. Then $\tilde{X} - V \subseteq F$ and so $\tilde{X} - V \subseteq Int^*(F)$. Therefore, $Cl^*(\tilde{X} - F) \subseteq V$. By Theorem 3.9 (2), $\tilde{X} - F$ is $\mathcal{I}^*_{\mathbf{g}}$-closed. Hence, F is $\mathcal{I}^*_{\mathbf{g}}$-open. □

Corollary 3.30. *Let $(\tilde{X}, \mathcal{T}, \mathcal{I})$ be an ideal space and $F \subseteq \tilde{X}$. If F is $\mathcal{I}^*_{\mathbf{g}}$-open, then $F \subseteq Int^*(F)$ whenever S is closed and $S \subseteq F$.*

Theorem 3.31. *Let $(\tilde{X}, \mathcal{T}, \mathcal{I})$ be an ideal space and $F \subseteq \tilde{X}$. If F is $\mathcal{I}^*_{\mathbf{g}}$-open and $Int^*(F) \subseteq D \subseteq F$, then D is $\mathcal{I}^*_{\mathbf{g}}$-open.*

Proof. Since F is $\mathcal{I}^*_{\mathbf{g}}$-open, $\tilde{X} - F$ is $\mathcal{I}^*_{\mathbf{g}}$-closed. By Theorem 3.9 (4), $Cl^*(\tilde{X} - F) - (\tilde{X} - F)$ contains no nonempty $\mathbf{g}$-closed set. Since $Int^*(F) \subseteq Int^*(D)$, we have $Cl^*(\tilde{X} - D) \subseteq Cl^*(\tilde{X} - F)$ and so $Cl^*(\tilde{X} - D) - (\tilde{X} - D) \subseteq Cl^*(\tilde{X} - F) - (\tilde{X} - F)$. Hence, D is $\mathcal{I}^*_{\mathbf{g}}$-open. □

Theorem 3.32. *Let $(\tilde{X}, \mathcal{T}, \mathcal{I})$ be any ideal topological space and $F \subseteq \tilde{X}$. Then the following are true.*

(1) F is $\mathcal{I}^*_{\mathbf{g}}$-closed.
(2) $F \cup (\tilde{X} - F^*)$ is $\mathcal{I}^*_{\mathbf{g}}$-closed.
(3) $F - F^*$ is $\mathcal{I}^*_{\mathbf{g}}$-closed.

Proof. (1) $\Rightarrow$ (2): Suppose F is $\mathcal{I}^*_{\mathbf{g}}$-closed. If V is any $\mathbf{g}$-open set such that $F \cup (\tilde{X} - F^*) \subseteq V$, then $\tilde{X} - V \subseteq \tilde{X} - (F \cup (\tilde{X} - F^*)) = \tilde{X} \cap (F \cup (F^*)^c)^c = F^* \cap F^c = F^* - F$. Since F is $\mathcal{I}^*_{\mathbf{g}}$-closed, by Theorem 3.9 (5), it follows that $\tilde{X} - V = \emptyset$ and so $\tilde{X} = V$. Therefore, $F \cup (\tilde{X} - F^*) \subseteq V$ which implies that $F \cup (\tilde{X} - F^*) \subseteq \tilde{X}$ and so $(F \cup (\tilde{X} - F^*))^* \subseteq \tilde{X}^* \subseteq \tilde{X} = V$. Hence, $F \cup (\tilde{X} - F^*)$ is $\mathcal{I}^*_{\mathbf{g}}$-closed.

(2) $\Rightarrow$ (1): Suppose $F \cup (\tilde{X} - F^*)$ is $\mathcal{I}^*_{\mathbf{g}}$-closed. If S is any $\mathbf{g}$-closed set such that $S \subseteq F^* - F$, then $S \subseteq F^*$ and $S \subseteq \tilde{X} \backslash F$ which implies that $\tilde{X} - F^* \subseteq \tilde{X} - S$

and $F \subseteq \tilde{X} - S$. Therefore, $F \cup (\tilde{X} - F^*) \subseteq F \cup (\tilde{X} - S) = \tilde{X} - S$ and $\tilde{X} - S$ is **g**-open. Since $(F \cup (\tilde{X} - F^*))^* \subseteq \tilde{X} - S$, we have $F^* \cup (\tilde{X} - F^*)^* \subseteq \tilde{X} - S$ and so $F^* \subseteq \tilde{X} - S$ which implies that $S \subseteq \tilde{X} - F^*$. Since $S \subseteq F^*$, it follows that $S = \emptyset$. Hence, F is $\mathcal{I}_{\mathbf{g}}^*$-closed.

(2) ⇒ (3): Since

$$\tilde{X}-(F^*-F) = \tilde{X}\cap(F^*\cap F^c)^c = \tilde{X}\cap((F^*)^c\cup F) = (\tilde{X}\cap(F^*)c)\cup(\tilde{X}\cap F) = F\cup(\tilde{X}-F^*),$$

the equivalence is clear. □

Theorem 3.33. *Let $(\tilde{X}, \mathcal{T}, \mathcal{I})$ be an ideal topological space. Then every subset of $\tilde{X}$ is $\mathcal{I}_{\mathbf{g}}^*$-closed if and only if every **g**-open set is $*$-closed.*

Proof. Suppose every subset of $\tilde{X}$ is $\mathcal{I}_{\mathbf{g}}^*$-closed. If $V \subseteq \tilde{X}$ is **g**-open, then V is $\mathcal{I}_{\mathbf{g}}^*$-closed and so $V^* \subseteq V$. Hence, V is $*$-closed. Conversely, suppose that every **g**-open set is $*$-closed. If V is **g**-open set such that $F \subseteq V \subseteq \tilde{X}$, then $F^* \subseteq V^* \subseteq V$ and so F is $\mathcal{I}_{\mathbf{g}}^*$-closed. □

Lemma 3.34 ([10], Theorem 6)**.** *Let $(\tilde{X}, \mathcal{T}, \mathcal{I})$ be an ideal space. If $\mathcal{I}$ is completely codense, then $\mathcal{T}^* \subseteq \mathcal{T}^\alpha$.*

Theorem 3.35. *Let $(\tilde{X}, \mathcal{T}, \mathcal{I})$ be an ideal topological space where $\mathcal{I}$ is completely dense. Then the following are equivalent.*

- *(1) $\tilde{X}$ is normal.*
- *(2) For any disjoint closed sets F and D, there exist disjoint $\mathcal{I}_{\mathbf{g}}^*$-open sets V and U such that $F \subseteq V$ and $D \subseteq U$.*
- *(3) For any closed set F and open set U containing F, there exists an $\mathcal{I}_{\mathbf{g}}^*$-open set V such that $F \subseteq V \subseteq Cl^*(V) \subseteq U$.*

Proof. (1) ⇒ (2): The proof follows from the fact that every open set is $\mathcal{I}_{\mathbf{g}}^*$-open.

(2) ⇒ (3): Suppose F is closed and U is an open set containing F. Since F and $\tilde{X} - U$ are disjoint closed sets, there exist disjoint $\mathcal{I}_{\mathbf{g}}^*$-open sets V and ω such that $F \subseteq V$ and $\tilde{X} - U \subseteq \omega$. Since $\tilde{X} - U$ is **g**-closed and ω is $\mathcal{I}_{\mathbf{g}}^*$-open, $\tilde{X} - U \subseteq Int^*(\omega)$ and so $\tilde{X} - Int^*(\omega) \subseteq U$. Again, $V \cap \omega = \emptyset$ which implies that $V \cap Int^*(\omega) = \emptyset$ and so, $V \subseteq \tilde{X} - Int^*(\omega)$ which implies that $Cl^*(V) \subseteq \tilde{X} - Int^*(\omega) \subseteq U$. V is the required $\mathcal{I}_{\mathbf{g}}^*$-open sets with $F \subseteq V \subseteq Cl^*(V) \subseteq U$.

(2) ⇒ (3): Let F and D be two disjoint closed subsets of $\tilde{X}$. By hypothesis, there exists an $\mathcal{I}_{\mathbf{g}}^*$-open set V such that $F \subseteq V \subseteq Cl^*(V) \subseteq \tilde{X} - D$. Since V is $\mathcal{I}_{\mathbf{g}}^*$-open, $F \subseteq Int^*(V)$. Since $\mathcal{I}$ is completely codense, by Lemma 3.34, $\mathcal{T}^* \subseteq \mathcal{T}^\alpha$ and so, $Int^*(V)$ and $\tilde{X} - Cl^*(V) \in \mathcal{T}^\alpha$. Hence, $F \subseteq Int^*(V) \subseteq Int(Cl(Int(Int^*(V)))) = A$ and $D \subseteq \tilde{X} - Cl^*(V) \subseteq Int(Cl(Int(\tilde{X} - Cl^*(V)))) = B$, which proves (1). □

If $\mathcal{I} = N$, it is not difficult to see $\mathcal{I}_{\mathbf{g}}^*$-closed sets coincide with g^*p-closed sets and so we have the following corollary.

Corollary 3.36. *Let $(\tilde{X}, \mathcal{T}, \mathcal{I})$ be an ideal space where $\mathcal{I} = N$. Then the following are equivalent.*

(1) *$\tilde{X}$ is normal.*
(2) *For any disjoint closed sets F and D, there exist disjoint g^*p-open sets V and U such that $F \subseteq V$ and $D \subseteq U$.*
(3) *For any closed set F and open set U containing F, there exists a g^*p-open set V such that $F \subseteq V \subseteq Cl^*(V) \subseteq U$.*

A subset F of an ideal topological space $(\tilde{X}, \mathcal{T}, \mathcal{I})$ is said to be $\mathcal{I}$-compact [38] if for every $\mathcal{T}$-open cover $\{\omega_\alpha : \alpha \in \Delta\}$ of F, there exists a finite subset Δ_o of Δ such that $(\tilde{X} - \cup\{\omega_\alpha : \alpha \in \Delta_o\}) \in \mathcal{I}$.

Corollary 3.37. *Let $(\tilde{X}, \mathcal{T}, \mathcal{I})$ be an ideal space. If F is an $\mathcal{I}^*_{\mathbf{g}}$-closed subset of $\tilde{X}$, then F is $\mathcal{I}$-compact.*

Proof. The proof follows from the fact that every $\mathcal{I}^*_{\mathbf{g}}$-closed is $\mathcal{I}g$-closed. □

4. Regular $\mathcal{I}^*_g$-Closed Sets

In this section, $\mathcal{I}^*_{\mathbf{g}}$-regular spaces are introduced and characterized.

Definition 4.1. An ideal space $(\tilde{X}, \mathcal{T}, \mathcal{I})$ is said to be an $\mathcal{I}^*_{\mathbf{g}}$-regular space if for each pair consisting of a point $\tilde{x}$ and closed set D not containing $\tilde{x}$, there exist disjoint $\mathcal{I}^*_{\mathbf{g}}$-open sets V and U such that $x \in V$ and $D \subset U$.

Definition 4.2. Let F be a subset of $(\tilde{X}, \mathcal{T}, \mathcal{I})$ and $\tilde{x} \in \tilde{X}$. The subset F of $\tilde{X}$ is called an $\mathcal{I}^*_{\mathbf{g}}$-open neighborhood of $\tilde{x}$ if there exists a $\mathcal{I}^*_{\mathbf{g}}$-open set V containing $\tilde{x}$ such that $V \subset F$.

Theorem 4.3. *Let $(\tilde{X}, \mathcal{T}, \mathcal{I})$ be an ideal topological space where $\mathcal{I}$ is completely dense. Then the following are equivalent.*

(1) *$\tilde{X}$ is $\mathcal{I}^*_{\mathbf{g}}$-regular.*
(2) *For every closed set D not containing $x \in \tilde{X}$, there exist disjoint $\mathcal{I}^*_{\mathbf{g}}$-open sets V and U such that $x \in V$ and $D \subset U$.*
(3) *For every open set U containing $x \in \tilde{X}$, there exists an $\mathcal{I}^*_{\mathbf{g}}$-open set V of $\tilde{X}$ such that $x \in V \subset Cl^*(V) \subset U$.*
(4) *For every closed set F, the intersection of all the $\mathcal{I}^*_{\mathbf{g}}$-closed neighborhoods of F is F.*
(5) *For every set F and an open set D such that $F \cap D \neq \emptyset$, there exists an $\mathcal{I}^*_{\mathbf{g}}$-open set F such that $F \cap F \neq \emptyset$ and $Cl^*(F) \subset D$.*
(6) *For every non empty set F and closed set D such that $F \cap D = \emptyset$, there exist disjoint $\mathcal{I}^*_{\mathbf{g}}$-open sets S and M such that $F \cap S \neq \emptyset$ and $D \subset M$.*

Proof. (1) and (2) are equivalent by definition.
(2) ⇒ (3): Let U be an open set such that $\tilde{x} \in U$. Then $\tilde{X} - U$ is a closed set not containing $\tilde{x}$. Therefore, there exist disjoint $\mathcal{I}^*_{\mathbf{g}}$-open sets V and ω such that $\tilde{x} \in V$ and $\tilde{X} - U \subset \omega$. Now $\tilde{X} - U \subset \omega$ implies $\tilde{X} - U \subset Int^*(\omega)$ and so $\tilde{X} - Int^*(\omega) \subset U$. Again, $V \cap \omega = \emptyset$ implies that $V \cap Int^*(\omega) = \emptyset$ and hence, $Cl^*(V) \subset \tilde{X} - Int^*(\omega)$. Therefore, $\tilde{x} \in V \subset Cl^*(V) \subset U$. This proves (3).

(3) $\Rightarrow$ (4): Let F be a closed set and $\tilde{x} \in \tilde{X} - F$. By (3), there exists an $\mathcal{I}_{\mathbf{g}}^*$-open set V such that $\tilde{x} \in V \subset Cl^*(V) \subset \tilde{X} - F$. Thus, $F \subset \tilde{X} - Cl^*(V) \subset \tilde{X} - V$. Consequently, $\tilde{X} - V$ is an $\mathcal{I}_{\mathbf{g}}^*$-closed neighborhood of F and $\tilde{x} \notin \tilde{X} - V$. This proves (4).
(4) $\Rightarrow$ (5): Let $F \cap D \neq \emptyset$, where D is open in $\tilde{X}$. Let $\tilde{x} \in F \cap D$. Then $\tilde{X} - D$ is closed and $\tilde{x} \notin \tilde{X} - D$. Then by (4), there exists an $\mathcal{I}_{\mathbf{g}}^*$-closed neighborhood U of $\tilde{X} - D$ such that $\tilde{x} \notin U$. Let $\tilde{X} - D \subset V \subset U$, where V is $\mathcal{I}_{\mathbf{g}}^*$-open. Then $F = \tilde{X} - U$ is $\mathcal{I}_{\mathbf{g}}^*$-open such that $\tilde{x} \in F$ and $F \cap F \neq \emptyset$. Also, $\tilde{X} - V$ is $\mathcal{I}_{\mathbf{g}}^*$-closed and $Cl^*(F) = Cl^*(\tilde{X} - U) \subset Cl^*(\tilde{X} - V) \subset D$. This proves (5).
(5) $\Rightarrow$ (6): Suppose $F \cap D \neq \emptyset$, where F is nonempty and D is closed. Then $\tilde{X} - D$ is open and $F \cap (\tilde{X} - D) \neq \emptyset$. By (5), there exists an $\mathcal{I}_{\mathbf{g}}^*$-open set S such that $F \cap S \neq \emptyset$ and $S \subset Cl^*(S) \subset \tilde{X} - D$. Let $M = \tilde{X} - Cl^*(S)$. Then $D \subset M$. S and M are $\mathcal{I}_{\mathbf{g}}^*$-open sets such that $M = \tilde{X} - Cl^*(S) \subset \tilde{X} - S$. This proves (6).
(6) $\Rightarrow$ (1): This is true and completes the proof. □

Definition 4.4. An ideal space $(\tilde{X}, \mathcal{T}, \mathcal{I})$ is said to be

(1) *-finitely additive if $\left[\bigcup_{i=1}^n F_i\right]^* = \bigcup_{i=1}^n (F_i)^*$ for every positive integer n.
(2) *-countably additive if $\left[\bigcup_{i=1}^n F_i\right]^* = \bigcup_{i=1}^\infty (F_i)^*$.
(3) *-additive if $\left[\bigcup_{\alpha\in\Omega} F_\alpha\right]^* = \bigcup_{\alpha\in\Omega} (F_\alpha)^*$ for all indexing sets Ω, where F_i s are subsets of $\tilde{X}$. Similarly, we define *-finitely multiplicative, countably multiplicative ideal spaces by taking intersections in the place of unions.
(4) $\mathcal{I}_{\mathbf{g}}^*$-finitely additive (resp. $\mathcal{I}_{\mathbf{g}}^*$-countable additive, $\mathcal{I}_{\mathbf{g}}^*$-additive) if finite union (resp.
countable union, arbitrary union) of $\mathcal{I}_{\mathbf{g}}^*$-closed sets is $\mathcal{I}_{\mathbf{g}}^*$-closed.

Similarly we define $\mathcal{I}_{\mathbf{g}}^*$-finitely multiplicative (resp. $\mathcal{I}_{\mathbf{g}}^*$-countably multiplicative, $\mathcal{I}_{\mathbf{g}}^*$-multiplicative) if finite intersection (resp. countable intersection, arbitrary intersection) of $\mathcal{I}_{\mathbf{g}}^*$-closed sets is $\mathcal{I}_{\mathbf{g}}^*$-closed.

Remark 4.5. *An ideal space $(\tilde{X}, \mathcal{T}, \mathcal{I})$ is said to be*

(1) a *-finitely additive space (respectively, $\mathcal{I}_{\mathbf{g}}^*$-countable additive space, additive space) if it is $\mathcal{I}_{\mathbf{g}}^*$-finitely additive (respectively, countably additive, additive), but not conversely.
(2) a *-additive space if it is *-countably additive and a *-countable additive space if it is *-finitely additive, but not conversely.
(3) a $\mathcal{I}_{\mathbf{g}}^*$-additive space if it is $\mathcal{I}_{\mathbf{g}}^*$-countably additive and a $\mathcal{I}_{\mathbf{g}}^*$-countably additive space if it is $\mathcal{I}_{\mathbf{g}}^*$-additive, but not conversely.

Example 4.6. *Let $\tilde{X} = R$, $\mathcal{I} = \{\emptyset\}$, and $\mathcal{T}$ a cofinite topology.*
Then $\mathcal{T} = \left\{\emptyset, \tilde{X}, F/F^c \text{ is finite}\right\}$. The closed sets are $\emptyset$, $\tilde{X}$, and all the finite subsets. $F^* = F$ if F is finite and $\tilde{X}$ if F is infinite.

For every positive integer n, if $F_n = \{-n, -n+1, \ldots, 0, \ldots, n-1, n\}$, then $F_n^* = F_n$ for all n. Also, $(UF_n)^* = (Z)^* = R$ and $U(F_n)^* = Z$. Let F and B be the set of all nonnegative and nonpositive integers, respectively.
Then $F^* = R = B^*$, $F^* \cap B^* = R$, and $(F \cap B)^* = \{0^*\} = \{0\}$. Therefore, this space is

(1) *not *-finitely multiplicative, and hence, not *-countably multiplicative, and not *-multiplicative.*
(2) **-finitely additive but not countably *-additive, and not *-additive.*
(3) *$\mathcal{I}_{\mathbf{g}}^*$-multiplicative, $\mathcal{I}_{\mathbf{g}}^*$-finitely multiplicative and $\mathcal{I}_{\mathbf{g}}^*$-countably multiplicative.*
(4) *$\mathcal{I}_{\mathbf{g}}^*$-finitely additive, but not $\mathcal{I}_{\mathbf{g}}^*$-countably additive and not $\mathcal{I}_{\mathbf{g}}^*$-additive.*

Remark 4.7. *In an ideal topological space $(\tilde{X}, \mathcal{T}, \mathcal{I})$ which is *-finitely additive, we have the following results:*

(1) $Cl^*(\phi) = \phi$,
(2) $Cl^*(\tilde{X}) = \tilde{X}$,
(3) $F \subset F^{a^*}(\mathcal{I})$,
(4) $F^{a^*} \subset Cl^*(F)$,
(5) $Cl^*(F \cup D) = Cl^*(F) \cup Cl^*(D)$,
(6) $Cl^*(Cl^*(F)) = Cl^*(F)$

for all subsets F, D, and $\tilde{X}$.
Therefore, $Cl^*()$ satisfies Kuratowski closure axioms (Jankovic & Hamlett [19]*) and hence, it defines a topology $\mathcal{T}^*$ whose closure operation is given by $Cl^*(F) = F \cup F^*$. Note that $\mathcal{T} \subset \mathcal{T}^*$. $Cl^*(F)$ and $int^*(F)$ denote the closure and interior of F in $(\tilde{X}, \mathcal{T}^*)$.*

Theorem 4.8. *A subset of a *-finitely additive ideal space $(\tilde{X}, \mathcal{T}, \mathcal{I})$ is $\mathcal{I}_{\mathbf{g}}^*$-open if and only if $F \subseteq Int^*(F)$, when $F \subseteq F$ and F is a $\mathbf{g}$-closed subset of $\tilde{X}$.*

Proof. Let F be $\mathcal{I}_{\mathbf{g}}^*$-open and F be a $\mathbf{g}$-closed subset of $\tilde{X}$ contained in F. Then $\tilde{X} - F$ is a $\mathbf{g}$-open set containing $\tilde{X} - F$ which implies $\tilde{X} - Int^*(F) = Cl^*(\tilde{X} - F) \subset \tilde{X} - F$. Conversely, let $F \subset Int^*(F)$ whenever $F \subseteq F$ and F is a $\mathbf{g}$-closed subset of $\tilde{X}$. Let V be $\mathbf{g}$-open and $\tilde{X} - F \subset V$. Then $\tilde{X} - V \subset Int^*(F) = \tilde{X} - Cl^*(\tilde{X} - F)$. Therefore, $Cl^*(\tilde{X} - F) \subset V$ which proves $\tilde{X} - F$ is $\mathcal{I}_{\mathbf{g}}^*$-closed. So F is $\mathcal{I}_{\mathbf{g}}^*$-open. □

Theorem 4.9. *For each $(\tilde{X}, \mathcal{T}, \mathcal{I})$, $\{x\}$ is $\mathbf{g}$-closed or $\{x\}^c$ is $\mathcal{I}_{\mathbf{g}}^*$-closed in $\tilde{X}$.*

Proof. If $\{x\}$ is not $\mathbf{g}$-closed, then $\{x\}^c$ is not $\mathbf{g}$-open. Therefore, the only $\mathbf{g}$-open sets containing $\{x\}^c$ are $\tilde{X}$ and $(\{x\}^c)^* \subseteq \tilde{X}$. This proves that $\{x\}^c$ is $\mathcal{I}_{\mathbf{g}}^*$closed. □

Theorem 4.10. *In an ideal space $(\tilde{X}, \mathcal{T}, \mathcal{I})$ which is *-finitely additive, if V is semi open and F is $\mathcal{I}_{\mathbf{g}}^*$-open, then $F \cap V$ is $\mathcal{I}_{\mathbf{g}}^*$-open.*

Proof. Let $\tilde{X} - (V \cap F) \subset A$ and A be $\mathbf{g}$-open.
Then $(\tilde{X} - F) \cup (\tilde{X} - V) \subset A$ and this implies $\tilde{X} - F \subset A$ and $\tilde{X} - V \subset A$. $(\tilde{X} - F)$ is $\mathcal{I}_{\mathbf{g}}^*$-closed and A is $\mathbf{g}$-open implies $Cl^*(\tilde{X} - F) \subset A$ and $Cl^*(\tilde{X} - V) \subset sCl(\tilde{X} -$

$V) = \tilde{X} - V \subset A$. Therefore, $Cl^*[\tilde{X} - (F \cap V)] = Cl^*\Big[(\tilde{X} - F) \cup (\tilde{X} - V)\Big] = Cl^*(\tilde{X} - F) \cup Cl^*(\tilde{X} - V) \subset A$.
Since the ideal space is *-finitely additive, this implies $F \cap V$ is $\mathcal{I}_{\mathbf{g}}^*$-open. □

Theorem 4.11. *If D is a subset of a *-finitely additive space $(\tilde{X}, \mathcal{T}, \mathcal{I})$ such that $F \subset D \subset Cl^*(F)$ and F is $\mathcal{I}_{\mathbf{g}}^*$-closed, then D is also $\mathcal{I}_{\mathbf{g}}^*$-closed in $\tilde{X}$.*

Proof. Let V be $\mathbf{g}$-open and $D \subset V$. Then $F \subset V$ implies $Cl^*(F) \subset V$. Therefore, $Cl^*(D) \subset Cl^*(Cl^*(F)) \subset Cl^*(F) \subset V$ which proves D is $\mathcal{I}_{\mathbf{g}}^*$-closed. □

Theorem 4.12. *If F and D are $\mathcal{I}_{\mathbf{g}}^*$-closed sets in an ideal space $(\tilde{X}, \mathcal{T}, \mathcal{I})$, then $F \cup D$ is also an $\mathcal{I}_{\mathbf{g}}^*$-closed set.*

Proof. Let V be a $\mathcal{I}_{\mathbf{g}}^*$-open subset of $(\tilde{X}, \mathcal{T}, \mathcal{I})$ containing $F \cup D$. Then $F \subset V$ and $D \subset V$. Since F and D are $\mathcal{I}_{\mathbf{g}}^*$-closed, $F^* \subset V$ and $D^* \subset V$. By Remark 4.7, $(F \cup D)^* = F^* \cup D^*$. Thus, $F \cup D \subset V$ which implies $F \cup D$ is $\mathcal{I}_{\mathbf{g}}^*$-closed. □

Definition 4.13. [18] A topological space $(\tilde{X}, \mathcal{T})$ is said to be a $\mathbf{g}$-multiplicative space if arbitrary intersections of $\mathbf{g}$-closed sets in $\tilde{X}$ are $\mathbf{g}$-closed.

Remark 4.14. [18]

(1) In $\mathbf{g}$-multiplicative spaces, $gCl(F)$ is the intersection of all $\mathbf{g}$-closed sets in $\tilde{X}$ containing F and is also $\mathbf{g}$-closed.
(2) Any indiscrete topological space $(\tilde{X}, \mathcal{T})$ is $\mathbf{g}$-multiplicative.
(3) If $\tilde{X} = \{a, b, c\}$ and $\mathcal{T} = \{\tilde{X}, \emptyset, \{a\}\}$, then $\{a, c\}$ and $\{a, b\}$ are $\mathbf{g}$-closed, but $\{a\}$ is not $\mathbf{g}$-closed and hence, $(\tilde{X}, \mathcal{T})$ is not $\mathbf{g}$-multiplicative.

Theorem 4.15. *Let $(\tilde{X}, \mathcal{T}, \mathcal{I})$ be a $\mathbf{g}$-multiplicative ideal space and let F be $\mathcal{I}_{\mathbf{g}}^*$-closed. Then F is $\mathcal{T}^*$-closed and $\Longleftrightarrow F^* - F$ is closed.*

Proof. Necessity: F is $\mathcal{T}^*$-closed and $\Longrightarrow F^* \subset F \Longrightarrow F^* - F = \emptyset$ is closed.
Sufficiency: Let $F^* - F$ be closed. Then it is $\mathbf{g}$-closed. By (5) of Theorem 3.9, $F^* - F = \emptyset$ which implies $F^* \subset F$. □

Theorem 4.16. *Let $(\tilde{X}, \mathcal{T}, \mathcal{I})$ be a $\mathbf{g}$-multiplicative ideal space and $F \subset \tilde{X}$. If F is $\mathcal{I}_{\mathbf{g}}^*$-closed, then $F \cup (\tilde{X} - F^*)$ is $\mathcal{I}_{\mathbf{g}}^*$-closed.*

Proof. Let V be $\mathbf{g}$-open and $F \cup (\tilde{X} - F^*) \subset V$.
Then $\tilde{X} - V \subset \tilde{X} - \Big[F \cup (\tilde{X} - F^*)\Big] = F^* - F$. Since F is $\mathcal{I}_{\mathbf{g}}^*$-closed, $F^* - F$ contains no nonempty $\mathbf{g}$-closed set. Therefore, $\tilde{X} - V = \emptyset$ which implies $\tilde{X} = V$. Thus, $\tilde{X}$ is the only $\mathbf{g}$-open set containing $F \cup (\tilde{X} - F^*)$ which proves $F \cup (\tilde{X} - F^*)$ is $\mathcal{I}_{\mathbf{g}}^*$-closed. □

Theorem 4.17. *Let F be a subset of a $\mathbf{g}$-multiplicative ideal space $(\tilde{X}, \mathcal{T}, \mathcal{I})$. If F is $\mathcal{I}_{\mathbf{g}}^*$-closed, then $F^* - F$ is $\mathcal{I}_{\mathbf{g}}^*$-open.*

Proof. Since $\tilde{X} - (F^* - F) = F \cup (\tilde{X} - F^*)$, the proof follows from Theorem 4.15. □

Theorem 4.18. *Let $(\tilde{\tilde{X}}, \mathcal{T}, \mathcal{I})$ be an ideal space. If every* **g***-open set is $\mathcal{T}^*$-closed, then every subset of $\tilde{X}$ is $\mathcal{I}^*_{\mathbf{g}}$-closed.*

Proof. Let $F \subset V$ and V be a **g**-open set in $\tilde{X}$. Then $Cl^*(F) \subset Cl^*(V) = V$ which proves F is $\mathcal{I}^*_{\mathbf{g}}$-closed. □

Theorem 4.19 ([12] Theorem 3.20)**.** *Let $(\tilde{X}, \mathcal{T}, \mathcal{I})$ be an ideal space and $F \subset \tilde{Y} \subset \tilde{X}$, where $\tilde{Y}$ is α-open in $\tilde{X}$. Then $F^*(\mathcal{I}_{\tilde{Y}}, \mathcal{T}\backslash\tilde{Y}) = F^*(\mathcal{I}, \mathcal{T}) \cap \tilde{Y}$.*

Theorem 4.20. *Let $(\tilde{X}, \mathcal{T}, \mathcal{I})$ be an ideal space and $F \subset Y \subset \tilde{X}$. If F is $\mathcal{I}^*_{\mathbf{g}}$-closed in $(Y, \mathcal{T}/Y, \mathcal{I}_Y)$, Y is α-open and tau*-closed in $\tilde{X}$. Then F is $\mathcal{I}^*_{\mathbf{g}}$-closed in $\tilde{X}$.*

Proof. Let $F \subset U$ and U be **g**-open in $\tilde{X}$. Then $F*(\mathcal{I}_Y, \mathcal{T}\setminus Y) = F^*(\mathcal{I}, \mathcal{T}) \cap Y \subset U \cap Y$. Then $Y \subset U \cup (\tilde{X} - F^*(\mathcal{I}, \mathcal{T}))$. Since Y is $\mathcal{T}^*$-closed, $Y^* \subset Y$. Therefore, $F^* \subset Y^* \subset Y \subset U \cup (\tilde{X} - F^*(\mathcal{T}, \mathcal{I}))$. This implies $F^* \subset U$ and hence, $Cl^*(F) \subset U$. So F is $\mathcal{I}^*_{\mathbf{g}}$-closed in $\tilde{X}$. □

Definition 4.21. $\{F_\alpha/\alpha \in \Omega\}$ is said to be a locally finite (resp. locally countable) family of sets in $(\tilde{X}, \mathcal{T}, \mathcal{I})$ if for every $\tilde{x} \in \tilde{X}$, there exists an open set U in $\tilde{X}$ containing $\tilde{x}$ that intersects only a finite (resp. countable) number of members $F_{\alpha_1}, \ldots, F_{\alpha_n}$ (resp. $F_{\alpha_i}, i = 1, \ldots \infty$) of $\{F_\alpha/\alpha \in \Omega\}$.

Theorem 4.22. *Let $(\tilde{X}, \mathcal{T}, \mathcal{I})$ be an ideal space which is *-finitely additive, and let $\{F_\alpha/\alpha \in \Omega\}$ be a locally finite family of sets in $(\tilde{X}, \mathcal{T}, \mathcal{I})$. Then $\left[\bigcup_{\alpha\in\Omega} F_\alpha\right]^* = \bigcup_{\alpha\in\Omega}(F_\alpha)^*$.*

Proof. $F_\alpha \subset \bigcup F_\alpha$ implies $F^*_\alpha \subset (\bigcup F_\alpha)^*$. Therefore, $\left[\bigcup_{\alpha\in\Omega} F_\alpha\right]^* \supseteq \bigcup_{\alpha\in\Omega}(F_\alpha)^*$. On the other hand, if $x \in \bigcup_{\alpha\in\Omega}(F_\alpha)^*$, then there exists an open set V containing x that intersects only a finite number of members $F_{\alpha_1}, \ldots, F_{\alpha_n}$. Let U be a semi-open set containing x. Then $V \cap U$ is a semi-open set containing x, which implies $(V \cap U) \cap \bigcup_{\alpha\in\Omega}(F_\alpha) \notin \mathcal{I}$. That is, $\left[(V \cap U) \cap \bigcup_{\alpha\neq\alpha_i}(F_\alpha)\right] \cup \left[(V \cap U) \cap \bigcup_{i=1}^n (F_\alpha)\right] \notin \mathcal{I}$. This implies $\{\emptyset\} \cup \left((V \cap U) \cap \bigcup_{i=1}^n (F_\alpha)\right) \notin \mathcal{I}$ and this implies $U \cap \left[\bigcup_{i=1}^n (F_\alpha)\right] \notin \mathcal{I}$. Therefore, $x \in \left(\bigcup_{i=1}^n (F_\alpha)\right)^* = \bigcup_{i=1}^n (F_\alpha)^* \subseteq \bigcup_{\alpha\in\Omega}(F_\alpha)^*$.
Therefore, $\left(\bigcup_{\alpha\in\Omega} F_\alpha\right)^* \subseteq \bigcup_{\alpha\in\Omega}(F_\alpha)^*$. From 1 and 2, the result follows:

$$\left(\bigcup_{\alpha\in\Omega} F_\alpha\right)^* = \bigcup_{\alpha\in\Omega}(F_\alpha)^*.$$

□

Theorem 4.23. *Let the ideal space $(\tilde{X}, \mathcal{T}, \mathcal{I})$ be *-finitely additive and let $\{F_\alpha/\alpha \in \Omega\}$ be a locally finite family of sets in $(\tilde{X}, \mathcal{T}, \mathcal{I})$. If each F_α is $\mathcal{I}^*_{\mathbf{g}}$-closed, then $\bigcup_{\alpha\in\Omega} F_\alpha$ is $\mathcal{I}^*_{\mathbf{g}}$-closed in $\tilde{X}$.*

Proof. Let $\bigcup_{\alpha\in\Omega} F_\alpha \subset V$ and V be **g**-open in $\tilde{X}$. Then $F_\alpha \subseteq V$ for all $\alpha \in \omega$ implies $Cl^*(F_\alpha) \subseteq V$ for all $\alpha \in \omega$. By Theorem 4.22, $Cl^*(\bigcup_{\alpha\in\Omega} F_\alpha) =$

$\bigcup_{\alpha\in\Omega} Cl^*(F_\alpha) \subseteq V$.
Therefore, $\bigcup_{\alpha\in\Omega} F_\alpha$ is $\mathcal{I}_{\mathbf{g}}^*$-closed. □

Theorem 4.24. *Let the ideal space $(\tilde{X}, \mathcal{T}, \mathcal{I})$ be *-countably additive and let $\{F_\alpha/\alpha \in \Omega\}$ be a locally finite family of sets in $(\tilde{X}, \mathcal{T}, \mathcal{I})$. If each F_α is $\mathcal{I}_{\mathbf{g}}^*$-closed, then $\bigcup_{\alpha\in\Omega} F_\alpha$ is $\mathcal{I}_{\mathbf{g}}^*$-closed in $\tilde{X}$.*

Proof. The proof is similar to the proof of Theorem 4.23. □

5. Conclusion and Future Scope

We developed the theory of ideal topology by constructing some new classes of operators and sets in order to study some characterizations and basic properties of these classes of operators and sets.

Our studies may lead to further research along the following lines.

(a) The introduction of $\mathcal{I}_{\mathbf{g}}^*$-compact and $\mathcal{I}_{\mathbf{g}}^*$-connected functions may be studied. Some characterizations and basic properties of these classes of functions can be investigated. We could develop relations between these classes and the classes of $\mathcal{I}_{\mathbf{g}}^*$-continuous functions.

(b) Give a deeper analysis of decomposition of continuing via idealization and investigate properties of $\mathcal{I}_{\mathbf{g}}^*$-closed sets and strong $\mathcal{I}_{\mathbf{g}}^*$-closed sets.

(c) Develop a new strong continuous functions via $\mathcal{I}_{\mathbf{g}}^*$-closed sets called contra $\mathcal{I}_{\mathbf{g}}^*$-continuous functions and strongly contra-e-$\mathcal{I}$-continuous functions. Also, explore weaker notions of sets and functions and weakly $\mathcal{I}_{\mathbf{g}}^*$-continuous functions.

(d) Study the relationships between the new notions of maximal $\mathcal{I}_{\mathbf{g}}^*$-closed sets, $\mathcal{I}_g^*$-compact spaces and $\mathcal{I}_g^*$-connected spaces via ideal topological spaces.

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