EXTENSIONS AND REFINEMENTS OF SOME PROPERTIES OF SUMS INVOLVING PELL NUMBERS

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Abstract. Falcón Santana and Díaz-Barrero [*Missouri Journal of Mathematical Sciences*, **18.1**, pp. 33–40, 2006] proved that the sum of the first $4n+1$ Pell numbers is a perfect square for all $n \geq 0$. They also established two divisibility properties for sums of Pell numbers with odd index. In this paper, the sum of the first n Pell numbers is characterized in terms of squares of Pell numbers for any $n \geq 0$. Additional divisibility properties for sums of Pell numbers with odd index are also presented, and divisibility properties for sums of Pell numbers with even index are derived.

1. Introduction

The Pell numbers are an integer sequence defined recursively by $P_0 = 0$, $P_1 = 1$, and $P_n = 2P_{n-1} + P_{n-2}$ for all $n \geq 2$. Whereas the Fibonacci numbers are associated with the golden ratio, $\frac{1+\sqrt{5}}{2}$, the Pell numbers are associated with the so-called silver ratio, $1 + \sqrt{2}$. Pell numbers arise in many areas of mathematics. For example, successive convergents to the continued fraction expansion of $\sqrt{2}$ take the form

$$\frac{P_n + P_{n+1}}{P_{n+1}}$$

for each $n \geq 0$ [8, sequences A000129 and A001333]. Moreover, all solutions to the Pell equations $x^2 - 2y^2 = \pm 1$ [3, Section 14.5, Theorem 244] and all square triangular numbers [1, pp. 16-17] can be characterized in terms of Pell numbers. To construct right triangles with integer length sides that are nearly isosceles (the lengths of the legs differ by 1), the lengths of the sides must be given by

$$a = 2P_nP_{n+1}, \quad b = P_{n+1}^2 - P_n^2, \quad \text{and} \quad c = P_{2n+1},$$

for some $n \geq 1$ [5, p. 195]. Pell numbers also have applications in certain combinatorial enumeration problems [2] and [7].

Falcón Santana and Díaz-Barrero [6] examined the sequence

$$S_n = \sum_{k=0}^{n} P_k,$$

and proved that S_{4n+1} is always a perfect square; in particular,

$$S_{4n+1} = (P_{2n} + P_{2n+1})^2.$$

They also established two divisibility results involving sums of Pell numbers of odd index. Benjamin, Plott, and Sellers [2] reconsidered the identities obtained in [6] by interpreting the Pell numbers as enumerators of tilings of a board of length n using white squares, black squares and gray dominoes. In so doing, they obtained proofs of each result from a purely combinatorial viewpoint.

Our goal is to provide extensions and refinements of the results obtained by Falcón Santana and Díaz-Barrero and Benjamin, Plott, and Sellers. In the next section, we obtain a closed form representation for the entire sequence S_n, as well as for the sums

$$\sum_{k=0}^{n} P_{2k+1} \quad \text{and} \quad \sum_{k=0}^{n} P_{2k}.$$

Several Pell number identities needed to derive our main results are developed in Section 3. Then in Section 4, the sum of the first n Pell numbers is characterized in terms of squares of Pell numbers. Additional divisibility results involving sums of Pell numbers of odd index and divisibility properties for sums of Pell numbers of even index are also presented.

2. Sums of Pell Numbers

We start by presenting explicit formulas for the three sums

$$S_n = \sum_{k=0}^{n} P_k, \quad \sum_{k=0}^{n} P_{2k+1}, \quad \text{and} \quad \sum_{k=0}^{n} P_{2k}.$$

Rewrite the Pell number recurrence as

$$2P_k = P_{k+1} - P_{k-1}. \tag{1}$$

Summing both sides of this expression from $k = 1$ to $k = n$ gives

$$\begin{aligned} 2\sum_{k=1}^{n} P_k &= \sum_{k=1}^{n} (P_{k+1} - P_{k-1}) \\ &= (P_2 - P_0) + (P_3 - P_1) + (P_4 - P_2) + \cdots + \\ &\qquad (P_n - P_{n-2}) + (P_{n+1} - P_{n-1}) \\ &= P_{n+1} + P_n - 1. \end{aligned}$$

Because $P_0 = 0$ it follows that

$$S_n = \sum_{k=0}^{n} P_k = \sum_{k=1}^{n} P_k = \frac{1}{2}(P_{n+1} + P_n - 1). \tag{2}$$

If we now replace the subscript k in (1) by $2k+1$ and sum from $k=0$ to $k=n$, we find

$$\begin{aligned} 2\sum_{k=0}^{n} P_{2k+1} &= \sum_{k=0}^{n}(P_{2k+2} - P_{2k}) \\ &= (P_2 - P_0) + (P_4 - P_2) + (P_6 - P_4) + \cdots + (P_{2n+2} - P_{2n}) \\ &= P_{2n+2}. \end{aligned}$$

Thus,

$$\sum_{k=0}^{n} P_{2k+1} = \frac{1}{2} P_{2n+2}. \tag{3}$$

Finally, replace the subscript k in (1) by $2k$ and sum from $k=1$ to $k=n$. The right-hand side again telescopes, leaving

$$2\sum_{k=1}^{n} P_{2k} = P_{2n+1} - 1.$$

Because $P_0 = 0$,

$$\sum_{k=0}^{n} P_{2k} = \sum_{k=1}^{n} P_{2k} = \frac{1}{2}(P_{2n+1} - 1). \tag{4}$$

3. Some Pell Number Identities

To proceed further with our analysis of the sums (2), (3), and (4), we need several Pell number identities. The first two can be derived from the Pell number analogue of Cassini's identity [4, equation (30),p. 249]:

$$P_{n-1}P_{n+1} - P_n^2 = (-1)^n.$$

Replacing P_{n+1} by $2P_n + P_{n-1}$ from the Pell number recurrence and rearranging terms yields

$$2P_{n-1}P_n = P_n^2 - P_{n-1}^2 + (-1)^n. \tag{5}$$

Adding $P_n^2 + P_{n-1}^2$ to both sides of (5) and factoring the resulting left-hand side then gives

$$(P_{n-1} + P_n)^2 = 2P_n^2 + (-1)^n. \tag{6}$$

To establish the remaining identities, we will make use of the relation

$$P_{r+s} = P_r P_{s+1} + P_{r-1} P_s$$

[4, equation (28) on page 249]. Set $r = s = n$ to obtain

$$\begin{aligned} P_{2n} &= P_n P_{n+1} + P_{n-1} P_n = P_n(P_{n+1} + P_{n-1}) \\ &= 2P_n(P_n + P_{n-1}). \end{aligned} \tag{7}$$

Next, set $r = 2n+1$ and $s = 2n$. Then

$$P_{4n+1} = P_{2n+1}^2 + P_{2n}^2.$$

From (5), with n replaced by $2n+1$,

$$P_{2n+1}^2 = 2P_{2n}P_{2n+1} + P_{2n}^2 + 1,$$

so

$$P_{4n+1} = 2P_{2n}P_{2n+1} + 2P_{2n}^2 + 1 = 2P_{2n}(P_{2n+1} + P_{2n}) + 1. \tag{8}$$

Finally, set $r = 2n+2$ and $s = 2n+1$. Then

$$\begin{aligned} P_{4n+3} &= P_{2n+2}^2 + P_{2n+1}^2 \\ &= P_{2n+2}^2 + P_{2n}(2P_{2n+1} + P_{2n}) + 1 \\ &= P_{2n+2}(P_{2n+2} + P_{2n}) + 1 \\ &= 2P_{2n+2}(P_{2n+1} + P_{2n}) + 1. \end{aligned} \tag{9}$$

4. Main Results

Our first objective is to express each term in S_n in terms of squares of Pell numbers. From (2), (7), (8), and (5)

$$\begin{aligned} S_{4m} &= \frac{P_{4m+1} + P_{4m} - 1}{2} \\ &= \frac{2P_{2m}(P_{2m+1} + P_{2m}) + 1 + 2P_{2m}(P_{2m} + P_{2m-1}) - 1}{2} \\ &= 2P_{2m+1}P_{2m} \\ &= P_{2m+1}^2 - P_{2m}^2 - 1. \end{aligned} \tag{10}$$

For the case $n = 4m+1$, (2), (7), and (8) yield

$$\begin{aligned} S_{4m+1} &= \frac{P_{4m+2} + P_{4m+1} - 1}{2} \\ &= \frac{2P_{2m+1}(P_{2m+1} + P_{2m}) + 2P_{2m}(P_{2m+1} + P_{2m}) + 1 - 1}{2} \\ &= (P_{2m+1} + P_{2m})^2, \end{aligned} \tag{11}$$

thus reproducing the result found in [2] and [6]. When $n = 4m+2$, we find

$$\begin{aligned} S_{4m+2} &= \frac{P_{4m+3} + P_{4m+2} - 1}{2} \\ &= \frac{2P_{2m+2}(P_{2m+1} + P_{2m}) + 1 + 2P_{2m+1}(P_{2m+1} + P_{2m}) - 1}{2} \\ &= (P_{2m+2} + P_{2m+1})(P_{2m+1} + P_{2m}). \end{aligned}$$

Rearranging the terms in the Pell number recurrence relation, it follows that $P_{2m+1} + P_{2m} = P_{2m+2} - P_{2m+1}$; hence,

$$S_{4m+2} = P_{2m+2}^2 - P_{2m+1}^2. \tag{12}$$

Finally, when $n = 4m + 3$,

$$\begin{aligned} S_{4m+3} &= \frac{2P_{2m+2}(P_{2m+2} + P_{2m+1}) + 2P_{2m+2}(P_{2m+1} + P_{2m}) + 1 - 1}{2} \\ &= 2P_{2m+2}^2. \end{aligned} \tag{13}$$

Before summarizing our findings, we note that S_{4m+1} and S_{4m+3} have alternate representations in terms of squares of Pell numbers. In particular,

$$1 + S_{4m+1} = 1 + (P_{2m+1} + P_{2m})^2 = 2P_{2m+1}^2$$

by (6) with $n = 2m + 1$, and

$$1 + S_{4m+3} = 1 + 2P_{2m+2}^2 = (P_{2m+2} + P_{2m+1})^2$$

by (6) with $n = 2m + 2$. Thus,

$$S_{4m+1} = 2P_{2m+1}^2 - 1; \text{ and} \tag{14}$$

$$S_{4m+3} = (P_{2m+2} + P_{2m+1})^2 - 1. \tag{15}$$

Now, combining (10)–(15), we have the following theorem.

Theorem 1. *If* n *is even, then*

$$S_n = \sum_{k=0}^{n} P_k = P_{1+n/2}^2 - P_{n/2}^2 - \epsilon_n,$$

where

$$\epsilon_n = \begin{cases} 1, & n \equiv 0(mod\ 4) \\ 0, & n \equiv 2(mod\ 4) \end{cases}.$$

If n *is odd, then* $S_n = \left(P_{(n+1)/2} + P_{(n-1)/2}\right)^2 - \delta_n$, *where*

$$\delta_n = \begin{cases} 0, & n \equiv 1(mod\ 4) \\ 1, & n \equiv 3(mod\ 4) \end{cases}.$$

Alternately, if n *is odd, then* $S_n = 2P_{(n+1)/2}^2 - \hat{\delta}_n$, *where* $\hat{\delta}_n = 1 - \delta_n$.

Next, consider sums of Pell numbers with odd index only. Falcón Santana and Díaz-Barrero [6] showed that

$$P_{2n+1} \left| \sum_{k=0}^{2n} P_{2k+1} \right. \quad \text{and} \quad P_{2n} \left| \sum_{k=1}^{2n} P_{2k-1} \right. . \tag{16}$$

Benjamin, Plott, and Sellers [2] combined these two results into the single statement

$$P_{n+1} \left| \sum_{k=0}^{n} P_{2k+1} \right. . \qquad (17)$$

From (3) and (7),

$$\sum_{k=0}^{n} P_{2k+1} = \frac{1}{2} P_{2n+2} = P_{n+1}(P_{n+1} + P_n). \qquad (18)$$

Formulas (16) and (17) follow immediately from (18), as does the following divisibility property.

Theorem 2. *For all* $n \geq 0$, $(P_{n+1} + P_n) \left| \sum_{k=0}^{n} P_{2k+1} \right.$.

Another new divisibility property is given by the following theorem.

Theorem 3. *For all* $m \geq 1$, $P_m \left| \sum_{k=0}^{2m-1} P_{2k+1} \right.$ *and* $(P_m + P_{m-1}) \left| \sum_{k=0}^{2m-1} P_{2k+1} \right.$.

Proof. From (18) and (7),

$$\sum_{k=0}^{2m-1} P_{2k+1} = P_{2m}(P_{2m} + P_{2m-1}) = 2P_m(P_m + P_{m-1})(P_{2m} + P_{2m-1}).$$

□

We now move on to sums of Pell numbers with even index. If $n = 2m$, then (4) and (8) imply

$$\sum_{k=0}^{n} P_{2k} = \frac{1}{2}(P_{4m+1} - 1) = P_{2m}(P_{2m} + P_{2m+1}) = P_n(P_n + P_{n+1}); \qquad (19)$$

on the other hand, if $n = 2m + 1$, then (4) and (9) imply

$$\sum_{k=0}^{n} P_{2k} = \frac{1}{2}(P_{4m+3} - 1) = P_{2m+2}(P_{2m+1} + P_{2m}) = P_{n+1}(P_n + P_{n-1}). \qquad (20)$$

From (19) and (20) we obtain the following divisibility properties.

Theorem 4. *If* n *is even, then*

$$P_n \left| \sum_{k=0}^{n} P_{2k} \right. \quad \text{and} \quad (P_n + P_{n+1}) \left| \sum_{k=0}^{n} P_{2k} \right. .$$

On the other hand, if n is odd, then

$$P_{n+1} \left| \sum_{k=0}^{n} P_{2k} \quad \textit{and} \quad (P_n + P_{n-1}) \right| \sum_{k=0}^{n} P_{2k} \ .$$

Moreover, combining (19) and (20) with (7) yields the following theorem.

Theorem 5. *If n is even, then*

$$P_{n/2} \left| \sum_{k=0}^{n} P_{2k} \quad \textit{and} \quad (P_{n/2} + P_{n/2-1}) \right| \sum_{k=0}^{n} P_{2k} \ .$$

On the other hand, if n is odd, then

$$P_{(n+1)/2} \left| \sum_{k=0}^{n} P_{2k} \quad \textit{and} \quad (P_{(n+1)/2} + P_{(n-1)/2}) \right| \sum_{k=0}^{n} P_{2k} \ .$$

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