ON THE EXPRESSION OF GENERALIZED INVERSES OF PERTURBED BOUNDED LINEAR OPERATORS

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Abstract. Let X and Y be two Hilbert spaces or Banach spaces, and let $T\colon X \to Y$ be a bounded linear operator with closed range. Let $\tilde{T} = T + \delta T$ with $\|\delta T\|\|T^\dagger\| < 1$. We give some equivalent conditions for the generalized inverse of $\tilde{T}$ to have the simplest expression $\tilde{T}^\dagger = (I + T^\dagger \delta T)^{-1} T^\dagger = T^\dagger (I + \delta T T^\dagger)^{-1}$.

1. Introduction. Let X and Y be two Banach spaces, let $L(X,Y)$ denote the Banach space of all bounded linear operators $T\colon X \to Y$ with the operator norm $\|T\| = \sup\{\|Tx\| : \|x\| = 1\}$, and let $LC(X,Y)$ be the subspace of all $T \in L(X,Y)$ with closed range. For $T \in LC(X,Y)$, if there exist two projections (bounded linear operators which are idempotents) P of X onto $N(T)$ along $N(T)^c$ and Q of Y onto $R(T)$ along $R(T)^c$, where $N(T)^c$ or $R(T)^c$ is a *topological complement* of the null space $N(T)$ of T in X or the range $R(T)$ of T in Y, respectively, then the *generalized inverse* $T^\dagger \in LC(Y,X)$ of T with respect to P, Q is uniquely determined by the four equalities

$$TT^\dagger T = T,\ T^\dagger T T^\dagger = T^\dagger,\ T^\dagger T = I - P,\ TT^\dagger = Q. \tag{1}$$

Since the generalized inverse of T also depends on the choice of the projections P and Q (i.e., the choice of $N(T)^c$ and $R(T)^c$ as topological complements of $N(T)$ and $R(T)$, respectively), we sometimes write $T^\dagger$ as $T^\dagger_{P,Q}$ to emphasize the dependence. When X, Y are Hilbert spaces, we further require that the projections P, Q be *orthogonal*, i.e., $N(T)^c = N(T)^\perp$ and $R(T)^c = R(T)^\perp$, so that $(T^\dagger T)^* = T^\dagger T$and $(TT^\dagger)^* = TT^\dagger$. In this special case (1) becomes

$$TT^\dagger T = T,\ T^\dagger T T^\dagger = T^\dagger,\ (T^\dagger T)^* = T^\dagger T,\ (TT^\dagger)^* = TT^\dagger, \tag{2}$$

and $T^\dagger$, which depends only on T, always exists and is usually called the *Moore-Penrose inverse* of T. See [1] and [7] for more details.

In [5] it was shown that in the Hilbert space case, if $T \in LC(X,Y)$ is one-to-one with $R(\delta T) \subset R(T)$ or onto with $N(T) \subset N(\delta T)$, then

$$\tilde{T}^\dagger = (I + T^\dagger \delta T)^{-1} T^\dagger = T^\dagger (I + \delta T T^\dagger)^{-1} \tag{3}$$

if $\|T^\dagger\|\|\delta T\| < 1$. It was further shown in [8] that if $R(\delta T) \subset R(T)$ and $N(T) \subset N(\delta T)$, then (3) is true when $\|T^\dagger\|\|\delta T\| < 1$. The expression (3) is the simplest possible for the generalized inverse of the perturbed operator. Moreover (3) implies that

$$\|\tilde{T}^\dagger\| \leq \frac{\|T^\dagger\|}{1 - \|T^\dagger\|\|\delta T\|}, \quad \frac{\|\tilde{T}^\dagger - T^\dagger\|}{\|T^\dagger\|} \leq \frac{\|T^\dagger\|\|\delta T\|}{1 - \|T^\dagger\|\|\delta T\|},$$

which gives the best possible estimates for $\tilde{T}^\dagger$. In this paper we give a unified approach to generalize the above results and prove that in the Hilbert space case (3) is valid if and only if the perturbed operator $\tilde{T}$ has the same range and null space as T, and we will also give several equivalent conditions for (3) to be true in the more general Banach space setting.

The problem of the expression of the generalized inverse of the perturbed operator has been investigated in recent papers [3, 4, 5, 8]. In the case of Banach spaces a classical result on the expression of the generalized inverse of the perturbed operator is Theorem 3.9 in [7], and more results have been obtained in [3]. Basically such results say that for the so-called *stable perturbation* [3] or more generally the perturbation such that $(I + \delta T T^\dagger)^{-1}\tilde{T}$ maps $N(T)$ into $R(T)$ [7], the expression (3) is true. Our equivalent conditions to (3) will extend the previous results. In particular we will show that if $\|T^\dagger\|\|\delta T\| < 1$, then (3) is true if and only if $\tilde{T}$ is a stable perturbation of T, and the two sufficient conditions above in [3] and [7] are actually equivalent. However, in the Hilbert space case, we do not expect the simple expression (3) under the stable perturbation because of the orthogonality requirement. Some complicated expressions for the stable perturbation and the more special range preserving or null space preserving perturbation have been obtained in [4, 8].

2. Preliminary Results. Let X, Y be Banach spaces and $T \in LC(X, Y)$. We assume that $T^\dagger$ exists with the corresponding topological decompositions $X = N(T) \oplus N(T)^c$ and $Y = R(T) \oplus R(T)^c$. Let $\tilde{T} = T + \delta T$. Throughout the paper we assume that the perturbation δT is small enough such that $\|T^\dagger\|\|\delta T\| < 1$. Then, by the Neumann Lemma [6], both $(I + T^\dagger \delta T)^{-1}$and $(I + \delta T T^\dagger)^{-1}$ exist and

$$(I + T^\dagger \delta T)^{-1} T^\dagger = T^\dagger (I + \delta T T^\dagger)^{-1}. \tag{4}$$

In this paper we always denote

$$U \equiv (I + T^\dagger \delta T)^{-1}(I - T^\dagger T), \;\; V \equiv (I - TT^\dagger)(I + \delta T T^\dagger)^{-1}.$$

The following three lemmas are not only useful in the next section, but also give general properties of U and V.

Lemma 2.1.

(i) U is the projection of X onto $(I+T^\dagger\delta T)^{-1}N(T)$ along $N(T)^c$. In particular, if X, Y are Hilbert spaces, then U is the projection of X onto $(I+T^\dagger\delta T)^{-1}N(T)$ along $N(T)^\perp$.

(ii) V is the projection of Y onto $R(T)^c$ along $(I+\delta TT^\dagger)R(T)$. In particular, if X, Y are Hilbert spaces, then V is the projection of Y onto $R(T)^\perp$ along $(I+\delta TT^\dagger)R(T)$.

Proof. (i) By (1), $(I-T^\dagger T)(I+T^\dagger\delta T) = I-T^\dagger T$. So $(I-T^\dagger T)(I+T^\dagger\delta T)^{-1} = I-T^\dagger T$. It follows that

$$\begin{aligned} U^2 &= (I+T^\dagger\delta T)^{-1}(I-T^\dagger T)(I+T^\dagger\delta T)^{-1}(I-T^\dagger T) \\ &= (I+T^\dagger\delta T)^{-1}(I-T^\dagger T) = U. \end{aligned}$$

Thus, U is the projection of X onto $R(U) = (I+T^\dagger\delta T)^{-1}R(I-T^\dagger T) = (I+T^\dagger\delta T)^{-1}N(T)$ along $N(U) = N(I-T^\dagger T) = N(T)^c$.

(ii) Since $(I+\delta TT^\dagger)(I-TT^\dagger) = I-TT^\dagger$, we have $(I+\delta TT^\dagger)^{-1}(I-TT^\dagger) = I-TT^\dagger$. So $V^2 = V$. It is obvious that $R(V) = R(T)^c$ and $N(V) = (I+\delta TT^\dagger)R(T)$.

Remark 2.1. Even in the case of Hilbert spaces, U or V may not be orthogonal projections.

Lemma 2.2.

(i) $N(\tilde{T}) \subset R(U) = (I+T^\dagger\delta T)^{-1}N(T)$.

(ii) $R(\tilde{T}) \supset N(V) = (I+\delta TT^\dagger)R(T)$.

Proof. Since

$$U = (I+T^\dagger\delta T)^{-1}[I + T^\dagger\delta T - T^\dagger\delta T - T^\dagger T] = I - (I+T^\dagger\delta T)^{-1}T^\dagger\tilde{T}, \qquad (5)$$

$\tilde{T}x = 0$ implies $x = Ux \in R(U)$. Hence, (i) is true. (ii) is from the fact that

$$V = [I + \delta TT^\dagger - \delta TT^\dagger - TT^\dagger](I + \delta TT^\dagger)^{-1} = I - \tilde{T}T^\dagger(I + \delta TT^\dagger)^{-1}. \tag{6}$$

<u>Lemma 2.3</u>. $\tilde{T}U = V\tilde{T}$. Hence,

(i) $R(\tilde{T}U) = R(V\tilde{T}) \subset R(V) = R(T)^c$ and

(ii) $N(V\tilde{T}) = N(\tilde{T}U) \supset N(U) = N(T)^c$.

<u>Proof</u>. By (4), (5), and (6), we have

$$\begin{aligned}\tilde{T}U &= \tilde{T}[I - (I + T^\dagger\delta T)^{-1}T^\dagger\tilde{T}] = \tilde{T}[I - T^\dagger(I + \delta TT^\dagger)^{-1}\tilde{T}] \\ &= [I - \tilde{T}T^\dagger(I + \delta TT^\dagger)^{-1}]\tilde{T} = V\tilde{T}.\end{aligned}$$

(i) and (ii) follow immediately.

<u>Remark 2.2</u>. The projection U was first studied in [2] in the Hilbert space case and was also investigated in [3] when X, Y are Banach spaces.

In the next section we will show that when X and Y are Banach spaces, (3) is satisfied if and only if either $N(\tilde{T}) \supset R(U)$ or $R(\tilde{T}) \subset N(V)$.

3. Main Results. Now we give some equivalent conditions to the expression (3) for the generalized inverse of $\tilde{T}$. We first present the result for Hilbert spaces.

<u>Theorem 3.1</u>. Let X, Y be Hilbert spaces. A necessary and sufficient condition for (3) to be true is that $R(\tilde{T}) = R(T)$ and $N(\tilde{T}) = N(T)$.

<u>Proof</u>. Suppose that $R(\tilde{T}) = R(T)$ and $N(\tilde{T}) = N(T)$. Then, since $R(\delta T) \subset R(T)$ and $N(\delta T) \supset N(T)$,

$$\tilde{T} = T(I + T^\dagger\delta T) = (I + \delta TT^\dagger)T. \tag{7}$$

Let A denote either one of the two expressions in (4). Then

$$\begin{aligned}\tilde{T}A &= T(I + T^\dagger\delta T)(I + T^\dagger\delta T)^{-1}T^\dagger = TT^\dagger, \\ A\tilde{T} &= T^\dagger(I + \delta TT^\dagger)^{-1}(I + \delta TT^\dagger)T = T^\dagger T, \\ \tilde{T}A\tilde{T} &= TT^\dagger T(I + T^\dagger\delta T) = T(I + T^\dagger\delta T) = \tilde{T}, \\ A\tilde{T}A &= (I + T^\dagger\delta T)^{-1}T^\dagger TT^\dagger = (I + T^\dagger\delta T)^{-1}T^\dagger = A.\end{aligned}$$

Hence, $\tilde{T}^\dagger = A$ by (2), i.e., (3) holds.

Now suppose that (3) gives the expression of $\tilde{T}^\dagger$. Then, by (5),

$$\tilde{T}^\dagger \tilde{T} = (I + T^\dagger \delta T)^{-1} T^\dagger \tilde{T} = I - U. \tag{8}$$

From Lemma 2.1 (i), $I - U$ is a projection of X onto $N(T)^\perp$. Since $\tilde{T}^\dagger \tilde{T}$ is the orthogonal projection of X onto $N(\tilde{T})^\perp$ along $N(\tilde{T})$, we must have $N(\tilde{T}) = N(T)$. On the other hand, from (6),

$$\tilde{T}\tilde{T}^\dagger = \tilde{T} T^\dagger (I + \delta T T^\dagger)^{-1} = I - V. \tag{9}$$

By Lemma 2.1 (ii), $I - V$ is a projection of Y onto $R(T)$. Since $\tilde{T}\tilde{T}^\dagger$ is the orthogonal projection of Y onto $R(\tilde{T})$ along $R(\tilde{T})^\perp$, $R(\tilde{T}) = R(T)$.

In the remainder of this section we assume that X, Y are Banach spaces without mentioning it explicitly.

Proposition 3.1. $R(U) = N(\tilde{T})$ if and only if $\tilde{T}U = 0$, if and only if $V\tilde{T} = 0$, and if and only if $N(V) = R(\tilde{T})$.

Proof. By Lemma 2.2 (i), $R(U) = N(\tilde{T})$ if and only if $R(U) \subset N(\tilde{T})$ if and only if $\tilde{T}U = 0$. By Lemma 2.2 (ii), $N(V) = R(\tilde{T})$ if and only if $N(V) \supset R(\tilde{T})$ if and only if $V\tilde{T} = 0$. Since $\tilde{T}U = V\tilde{T}$ from Lemma 2.3, $\tilde{T}U = 0$ if and only if $V\tilde{T} = 0$.

Now we are able to give several equivalent conditions to (3).

Theorem 3.2. Let X, Y be Banach spaces, let $T \in LC(X, Y)$, and let $\tilde{T} = T + \delta T$. If $\|T^\dagger\|\|\delta T\| < 1$, then the following are equivalent:

(i) $\tilde{T}^\dagger = (I + T^\dagger \delta T)^{-1} T^\dagger = T^\dagger (I + \delta T T^\dagger)^{-1}$.
(ii) $R(U) = N(\tilde{T})$, i.e., $(I + T^\dagger \delta T)^{-1} N(T) = N(\tilde{T})$.
(iii) $\tilde{T}U = 0$.
(iv) $\tilde{T}(I + T^\dagger \delta T)^{-1}$ maps $N(T)$ to 0.
(v) $N(V) = R(\tilde{T})$, i.e.,$(I + \delta T T^\dagger) R(T) = R(\tilde{T})$.
(vi) $V\tilde{T} = 0$.
(vii) $(I + \delta T T^\dagger)^{-1} \tilde{T}$ maps $N(T)$ into $R(T)$.

Moreover in any case $\tilde{T}^\dagger$ is with respect to the topological decompositions

$$X = N(\tilde{T}) \oplus N(T)^c, \;\; Y = R(\tilde{T}) \oplus R(T)^c. \tag{10}$$

Proof. Because of Proposition 3.1, it is enough to prove the equivalence of (i) and (ii). Suppose that (ii) is true. Let A be the expressions in (4). Then, as shown in (8) and (9), $A\tilde{T} = I - U$ and so $A\tilde{T}$ is the projection of X onto $N(U) = N(T)^c$ along $R(U) = (I + T^\dagger \delta T)^{-1} N(T)$ by Lemma 2.1 (i), and $\tilde{T}A = I - V$ and so $\tilde{T}A$ is the projection of Y onto $N(V) = (I + \delta T T^\dagger) R(T)$ along $R(V) = R(T)^c$ by Lemma 2.1 (ii). From the assumption that $R(U) = N(\tilde{T})$, we see that $A\tilde{T}$ is the projection of X onto $N(\tilde{T})^c = N(T)^c$ along $N(\tilde{T})$. Since $T^\dagger T T^\dagger = T^\dagger$,

$$UA = (I + T^\dagger \delta T)^{-1} (I - T^\dagger T) T^\dagger (I + \delta T T^\dagger)^{-1} = 0,$$

from which it follows that

$$A\tilde{T}A = (I - U)A = A - UA = A. \tag{11}$$

On the other hand, $\tilde{T}U = 0$ since $R(U) = N(\tilde{T})$, so

$$\tilde{T}A\tilde{T} = \tilde{T}(I - U) = \tilde{T} - \tilde{T}U = \tilde{T}. \tag{12}$$

Lastly, since $R(\tilde{T}A) \supset R(\tilde{T})$ because of (12), $R(\tilde{T}A) = R(\tilde{T})$ since $R(\tilde{T}A) \subset R(\tilde{T})$. Hence, $\tilde{T}A$ is the projection of Y onto $R(\tilde{T})$ along $R(\tilde{T})^c = R(T)^c$. Therefore, $\tilde{T}^\dagger = A$, i.e., (i) is satisfied.

Now suppose that (i) is true. Then $\tilde{T}^\dagger \tilde{T} = I - U$, which, by Lemma 2.1 (i), is the projection of X onto $N(T)^c$ along $R(U)$. Since $\tilde{T}^\dagger \tilde{T}$ is the projection of X onto $N(\tilde{T})^c$ along $N(\tilde{T})$, we have (ii).

Remark 3.1. That (ii) implies (i) is also from Theorem 3.9 in [7] and Proposition 3.2 of [3].

Remark 3.2. (vii) is the sufficient condition used in [7] to guarantee (i), and the equivalence of (ii) and (vii) was also proved in [3].

In [3] it was shown that if $\tilde{T}$ is a stable perturbation of T, i.e.,

$$R(\tilde{T}) \cap R(T)^c = \{0\}, \tag{13}$$

then (3) is true. The next theorem indicates that it is also necessary.

Theorem 3.3. Under the same assumption as in Theorem 3.2, (3) is valid if and only if

(i) $\tilde{T}$ is a stable perturbation of T, or

(ii) $X = N(\tilde{T}) + N(T)^c$.

Proof. Suppose that (3) is true, then (i) and (ii) follow from (1). If (i) is assumed, i.e., (13) is satisfied, then, since $R(\tilde{T}U) \subset R(\tilde{T})$, from Lemma 2.3 (i),

$$R(\tilde{T}U) \subset R(\tilde{T}) \cap R(T)^c = \{0\},$$

i.e., $\tilde{T}U = 0$. Hence, by Theorem 3.2, (3) is true. Now suppose that (ii) is true. Then, since $N(V\tilde{T}) \supset N(\tilde{T})$, by Lemma 2.3 (ii),

$$N(V\tilde{T}) \supset N(\tilde{T}) + N(T)^c = X,$$

i.e., $V\tilde{T} = 0$. So we have (3) by Theorem 3.2.

Remark 3.3. Therefore (10) is also a necessary and sufficient condition to the expression (3).

To end this section we present some sufficient conditions for (3) to be true, in which the sufficiency of (i) (iii) and (iv) was also proved in [3].

Proposition 3.2. Any of the following implies (3).

(i) $\dim N(\tilde{T}) = \dim N(T) < \infty$.
(ii) $\dim R(\tilde{T}) = \dim R(T) < \infty$.
(iii) $N(\tilde{T}) = N(T)$.
(iv) $R(\tilde{T}) = R(T)$.

Proof. We need the fact that both $(I + T^\dagger \delta T)^{-1}$ and $(I + \delta T T^\dagger)$ are isomorphisms. Since $N(\tilde{T}) \subset R(U) = (I + T^\dagger \delta T)^{-1} N(T)$ by Lemma 2.2 (i), (i) implies that $N(\tilde{T}) = R(U)$, so (3) holds by Theorem 3.2 (ii). Similarly using Lemma 2.2 (ii), we see that (ii) implies that $R(\tilde{T}) = N(V)$ which implies (3) by Theorem 3.2 (v). Lastly the sufficiency of (iii) and (iv) are obvious from Theorem 3.3.

Corollary 3.1. If X and Y are finite dimensional normed spaces, then (3) is true if and only if Rank $\tilde{T}$ = Rank T.

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References

1. S. L. Campbell and C. D. Meyer, Jr., *Generalized Inverses of Linear Transformations,* Pitman, New York, 1979.

2. G. Chen, M. Wei, and Y. Xue, "Perturbation Analysis of the Least Squares Solution in Hilbert Spaces," *Linear Alg. Appl.,* 244 (1996), 69–81.

3. G. Chen and Y. Xue, "Perturbation Analysis for the Operator Equation $Tx = b$ in Banach Spaces," *J. Math. Anal. Appl.,* 212 (1997), 107–125.

4. G. Chen and Y. Xue, "The Expression of the Generalized Inverse of the Perturbed Operator Under Type I Perturbation in Hilbert Spaces," *Linear Alg. Appl.,* 285 (1998), 1–6.

5. J. Ding and L. Huang, "On the Perturbation of the Least Squares Solutions in Hilbert Spaces," *Linear Alg. Appl.,* 212/213 (1994), 487–500.

6. T. Kato, *Perturbation Theory for Linear Operators,* Springer-Verlag, New York, 1984.

7. M. Z. Nashed, "Perturbations and Approximations for Generalized Inverses and Linear Operator Equations," *Generalized Inverses and Applications,* M. Z. Nashed (editor), Academic Press, New York, 325–396, 1976.

8. Y. Wei and J. Ding, "Representations for Moore-Penrose Inverses in Hilbert Spaces," *Applied Math. Letters* (to appear).

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