NONZERO INJECTIVE COVERS OF MODULES

Richard Belshoff and Jinzhong Xu

Abstract. We show that if R is a ring such that every nonzero left R-module has a nonzero injective cover, then R is left Artinian. The converse is not true. If R is commutative, then the properties are equivalent.

1. Background. A *projective cover* of a left R–module M (first introduced by H. Bass [2] in the early 1960s) can be characterized as a pair (P, f), where P is a projective left R-module and $f: P \to M$ is a linear map satisfying the following two properties:

(i) if P' is projective and $g: P' \to M$ is a linear map, then there exists a linear map $h: P' \to P$ such that $f \circ h = g$.
(ii) any linear map $h: P \to P$ such that $f \circ h = f$ is an automorphism of P.

The dual notion of an *injective envelope* of an R-module M consists of an injective left R–module E and a linear map $f: M \to E$ such that

(i) if E' is injective and $g: M \to E'$ is a linear map, then there exists a linear map $h: E \to E'$ such that $h \circ f = g$, and
(ii) any linear map $h: E \to E$ such that $h \circ f = f$ is an automorphism of E.

Using these categorical definitions of projective cover and injective envelope as motivation, other types of covers and envelopes can be defined. This was initiated by E. Enochs in [4]. An *injective cover* of a left R-module M consists of an injective module E and a linear map $f: E \to M$ satisfying the following two properties:

(i) if E' is injective and $g: E' \to M$ is a linear map, then there exists a linear map $h: E' \to E$ such that $f \circ h = g$.
(ii) any linear map $h: E \to E$ such that $f \circ h = f$ is an automorphism of E.

It was shown in [4] that a ring R is left Noetherian if and only if every left R-module has an injective cover. However, we find that injective covers of many nonzero modules are zero. For example, the injective cover of $\mathbb{Z}$, or any finitely generated $\mathbb{Z}$-module, is zero. Therefore, we are interested in studying a nonzero map $\phi: E \to M$, where $M \neq 0$ is a left R-module, and E is an injective module. Using the result cited above, it is easy to see that for a left Noetherian ring R, if every nonzero module M has the property that there is a nonzero map $E \to M$ where E is injective, then M has a nonzero injective cover.

In this paper, R will denote an associative ring with identity, and all modules will be unital left R-module. We denote by $l(x) = \{r \in R \mid rx = 0\}$ the left annihilator of $x \in R$, and if R is commutative we use the notation $\mathrm{Ann}(x)$.

2. Perfect Rings and Injective Modules. Perfect rings were originally defined and studied using projective and flat modules. If R is any ring, then by Theorem P of [2], R is left perfect if and only if R satisfies any one of the following equivalent conditions:

- every module has a projective cover.
- the Jacobson radical $J = J(R)$ is T-nilpotent and R/J is Artinian
- every flat module is projective.

The results in this section provide a new way of viewing perfect rings by using injective modules.

Theorem 1. Let $J = J(R)$ be the Jacobson radical of a ring R and suppose that for every $a \in R$, the set of left annihilator ideals $\{l(a^n) \mid n \geq 1\}$ satisfies the ascending chain condition (ACC). If for every nonzero left R-module M there is a nonzero map $E \to M$ where E is injective, then J is a nil-ideal.

Proof. By Zorn's Lemma, there is a maximal nil-ideal N contained in J. We claim that $N = J$. If $J/N \neq 0$, then there is a nonzero map $\phi\colon E \to J/N$ where E is injective. Define $\mathrm{Tr}_{J/N}(E)$ to be the sum of all the images $h(E)$, where $h\colon E \to J/N$ is a linear map. That is, $\mathrm{Tr}_{J/N}(E) = \{\sum h(E) \mid h \in \mathrm{Hom}_R(E, J/N)\}$. Certainly $\mathrm{Tr}_{J/N}(E)$ is a submodule of J/N. Thus, $\mathrm{Tr}_{J/N}(E) = N_1/N$ for some ideal N_1 containing N, and $N_1/N \neq \overline{0}$.

For any $a \in N_1$, there is an integer n such that $l(a^n) = l(a^{2n})$. Let $\overline{c} = \overline{a}^n = a^n + N$. Then $\overline{c} \in \mathrm{Tr}_{J/N}(E)$ and so there is an integer t so that $\overline{c} = h_1(e_1) + \cdots + h_t(e_t)$, for some linear maps $h_i\colon E \to J/N$, and some elements $e_i \in E$, for $1 \leq i \leq t$. If we let $e^* = e_1 + \cdots + e_t$, let $E^* = E \oplus \cdots \oplus E$ denote t copies of E, and let $h = \oplus_{i=1}^t h_i$, then $h\colon E^* \to J/N$, $e^* \in E^*$ and $h(e^*) = \overline{c}$. Define $f\colon Rc^2 \to E^*$ by $f(rc^2) = rce^*$. Then we have the diagram

$$\begin{tikzcd}
0 \arrow[r] & Rc^2 \arrow[r] \arrow[d, "f"'] & R \arrow[dl, dotted, "g"] \\
& E^* &
\end{tikzcd}$$

and since E^* is injective, there is a map g which extends f. Let $e^* = g(1) \in E^*$. Then

$$ce^* = f(c^2) = g(c^2) = c^2 g(1) = c^2 e_1^*.$$

Now $h(ce^*) = h(c^2 e_1^*)$ implies that $\overline{c}^2(1 - h(e_1^*)) = \overline{0} \in J/N \subset R/N$. It follows that $\overline{c}^2 = \overline{0} \in R/N$, so $a^{2n} = c^2 \in N$ which means that a is nilpotent in R. Therefore, N_1 is a nil-ideal of R, and $N \subset N_1$.

Corollary 1. If R is left self-injective and for every $a \in R$ the set of left annihilator ideals $\{l(a^n) \mid n \geq 1\}$ has ACC, then $J(R)$ is nil and $R/J(R)$ is regular.

Proof. If R is left self-injective, then every nonzero cyclic left R-module is a homomorphic image of an injective module, hence by Theorem 1, $J(R)$ is nil and by Anderson-Fuller, [1] $R/J(R)$ is regular.

Recall that a ring R is said to be left perfect if and only if every left R-module has a projective cover. This is the case if and only if $J(R)$ is left T-nilpotent and $R/J(R)$ is Artinian semi-simple. See [2].

Theorem 2. If R satisfies the following conditions, then R is a left perfect ring.

(a) For every nonzero R-module M there is a nonzero map $E \to M$ where E is injective.
(b) For any sequence $\{a_n \mid n \geq 1\}$ in R, the set of left annihilator ideals $\{l(a_1 \cdots a_n) \mid n \geq 1\}$ has ACC.
(c) The union of any ascending chain of left T-nilpotent ideals is left T-nilpotent.

Proof. We will first show that the Jacobson radical J is left T-nilpotent. By (c) and Zorn's Lemma, there is a maximal left T-nilpotent ideal $N \subset J$. If $J \neq N$, i.e. if $J/N \neq 0$, then by hypothesis (a) we have a nonzero map $\phi\colon E \to J/N$ where E is injective. Set $\mathrm{Tr}_{J/N}(E) = L/N$, where L is an ideal of R containing N. By (b), for any sequence $\{a_n \in L \mid n \geq 1\}$, there is an integer n_1 such that $l(a_1 \cdots a_{n_1 - 1}) = l(a_1 \cdots a_{n_1})$. Now $\overline{a_{n_1}} = a_{n_1} + N \in L/N$. By definition of $\mathrm{Tr}_{J/N}(E)$, there is some integer t so that $\overline{a_{n_1}} = h_1(e_1) + \cdots + h_t(e_t)$, where $h_i\colon E \to J/N$ is a linear map, and $e_i \in E$ for $i = 1, \ldots, t$. Let E^* be the direct sum of t copies of E, let $h = \oplus_{i=1}^t h_i$, and let $e^* = e_1 \oplus \cdots \oplus e_t$. Then $h\colon E^* \to J/N$ and $h(e^*) = \overline{a_{n_1}}$. Now define $f\colon Ra_1\cdots a_{n_1-1}a_{n_1}\to E^*$ by $f(ra_1\cdots a_{n_1-1}a_{n_1})=ra_1\cdots a_{n_1-1}e^*$, and let g be a map which extends f. So we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Ra_1\cdots a_{n_1-1}a_{n_1} \arrow[r] \arrow[d, "f"'] & R \arrow[dl, dotted, "g"] \\
& E^* &
\end{tikzcd}$$

If we let $e_1^*=g(1)\in E^*$, then

$$\begin{aligned}
&a_1\cdots a_{n_1-1}e^*=f(a_1\cdots a_{n_1})=a_1\cdots a_{n_1}e_1^*,\\
&h(a_1\cdots a_{n_1-1}e^*)=h(a_1\cdots a_{n_1}e_1^*),\\
&a_1\cdots a_{n-1}\overline{a_{n_1}}=a_1\cdots a_{n_1}h(e_1^*),\\
&\overline{a_1\cdots a_{n_1}}=\overline{a_1\cdots a_{n_1}}h(e_1^*).
\end{aligned}$$

So $\overline{a_1\cdots a_{n_1}}(\overline{1}-h(e_1^*))=\overline{0}$ implies that $a_1\cdots a_{n_1}\in N$. Set $b_1=a_1\cdots a_{n_1}\in N$.

Similarly, for the sequence $\{a_k\mid k\geq n_1+1\}$, we can find an integer n_2 such that $a_{n_1+1}\cdots a_{n_2}\in N$. Set $b_2=a_{n_1+1}\cdots a_{n_2}$. Consequently we get a sequence $\{b_t\mid t\geq 1\}\subset N$, where $b_t=a_{n_t+1}\cdots a_{n_{t+1}}$ with $n_0=0$. Since N is left T-nilpotent, for some integer s we have $b_1b_2\cdots b_s=0$, and then $a_1\cdots a_{n_{s+1}}=0$. Hence L is left T-nilpotent. It follows that $L=N$ is left T-nilpotent, and $\mathrm{Tr}_{J/N}(E)=L/N$ is zero. Therefore, $J=N$ is left T-nilpotent.

Next we will prove that R/J is Artinian semi-simple. First of all, any set of orthogonal idempotent elements of R/J can be lifted to a set of orthogonal idempotent elements of R. The set of left annihilator ideals $\{l(a_1\cdots a_n)\mid n\geq 1\}$ has ACC by assumption, and R/J has an indecomposable decomposition as a left R-module. Let $\overline{R}=R/J=N_1\oplus\cdots\oplus N_t$, where each N_i is an indecomposable submodule of R/J. Now we will show that N_i is a simple R/J-module.

Let H be a nonzero submodule of N_i. Since N_i is indecomposable, in order to prove $H=N_i$ we only need to show that H contains a nonzero direct summand of N_i. By the correspondence between idempotents and direct summands, we only need to construct a special idempotent element w in R/J such that it creates a direct summand of N_i.

By (a), there is an injective module E and a map $\phi\colon E \to H$ with $\phi \neq 0$. Set $\mathrm{Tr}_H(E) = I/J \subset H$, and $I/J \neq 0$. For any $a \in I$, by (b) there is an integer n such that $l(a^n) = l(a^{n+2})$. Now $\overline{a} = a + J \in \mathrm{Tr}_H(E)$, so there is an integer t such that $\overline{a} = h_1(e_1) \oplus \cdots \oplus h_t(e_t)$ for some linear maps $h_i\colon E \to H$ and for some $e_i \in E$. If we let E^* be the direct sum of t copies of E, let $e^* = e_1 \oplus \cdots \oplus e_t$ and let $h = \oplus_{i=1}^t h_i$, then $h\colon E^* \to H$ and $h(e^*) = \overline{a}$. Now define $f\colon Ra^{n+2} \to E^*$ by $f(ra^{n+2}) = ra^n e^*$, and let g be a map which extends f. We have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Ra^{n+2} \arrow[r] \arrow[d, "f"'] & R \arrow[dl, dotted, "g"] \\ & E^* & \end{tikzcd}$$

and if we set $g(1) = e_1^* \in E^*$, then

$$a^n e^* = f(a^{n+2}) = g(a^{n+2}) = a^{n+2} e_1^*.$$

Since $h(E^*)$ is contained in $\mathrm{Tr}_H(E) = I/J$, if we set $\overline{b} = h(e_1^*) \in I/J$, then since $h(e^*) = \overline{a}$,

$$\overline{a}^{n+1} = \overline{a}^{n+2}\overline{b}.$$

We can also find an integer m and an element $\overline{c} \in I/J$ such that $\overline{b}^m = \overline{b}^{m+1}\overline{c}$. Consequently,

$$\overline{a}^{(n+1)+m+(n+1)+1}\overline{b}^{m+(n+1)+1} = \overline{a}^{n+1},$$

$$\overline{a}^{2(m+(n+1)+1)}\overline{b}^{m+(n+1)+1} = \overline{a}^{(n+1)+m+1},$$

$$\overline{b}^{m+(m+n+1)+1}\overline{c}^{m+n+1+1} = \overline{b}^{m},$$

$$\overline{b}^{2(m+(n+1)+1)}\overline{c}^{m+(n+1)+1} = \overline{b}^{m+(n+1)+1}.$$

Set $x = \overline{a}^{n+1+m+1}$, $y = \overline{b}^{m+n+1+1}$ and $z = \overline{c}^{m+n+1+1}$. Then $x^2y = x$ and $y^2z = y$. And because $z - x \in I/J$, there is an integer k and an element $d \in I/J$ such that $(z-x)^{k+1}d = (z-x)^k$. From these three relations

$$\begin{aligned} &x^2y = x, \\ &y^2z = y, \\ &(z-x)^{k+1}d = (z-x)^k, \end{aligned}$$

we can prove that $x = xyx$. (See the Appendix.)

Since $x \neq 0$, if we set $w = yx$, then $w \in I/J \subset H$ and $w^2 = yxyx = yx = w$. Hence, $\overline{R} = \overline{R}w \oplus K$, where $\overline{R}w \subset N_i$, and $N_i = \overline{R}w \oplus (K \cap N_i)$. This implies that $N_i = H$, since N_i is indecomposable, and then R/J is Artinian semi-simple.

Corollary 2. If every nonzero left R-module has a nonzero injective cover, then R is left Artinian.

Proof. If every left R-module has an injective cover, then R is left Noetherian, and it follows that the hypotheses (a), (b) and (c) in Theorem 2 are all satisfied. Hence, R is left Artinian.

The following example shows that the converses to Theorem 2 and Corollary 2 are not true. We will give an example of a left Artinian ring R for which hypothesis (a) of Theorem 2 is not satisfied.

Let F be a field, let

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and let

$$R = Fe_{11} + Fe_{12} + Fe_{22} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \;\middle|\; a, b, c \in F \right\}.$$

We know that R is left Artinian. Now we will show that the hypothesis (a) is not satisfied by R.

Since $R = Re_{11} \oplus (Re_{12} + Re_{22})$, Re_{11} is a projective simple module of R. If there is a nonzero map $E \to Re_{11}$ with E injective, then Re_{11} is injective.

However, the homomorphism $f\colon Re_{12} \to Re_{11}$ given by $f(re_{12}) = re_{11}$, can not be extended. For if we consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & Re_{12} \arrow[r] \arrow[d, "f"'] & R \arrow[dl, dotted, "g"] \\
& Re_{11} &
\end{tikzcd}$$

where $g \mid Re_{12} = f$, if

$$g(1) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in Re_{11},$$

then $f(e_{12}) \neq 0$ but

$$g(e_{12}) = e_{12}g(1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Theorem 3. If R is commutative, then the following statements are equivalent.

1. R is perfect.

2. (a) For every nonzero R-module M there is a nonzero linear map $E \to M$ where E is injective.

(b) For any sequence $a_1, a_2, \ldots$ in R, the set of annihilator ideals $\{\mathrm{Ann}(a_1 \ldots a_n) \mid n \geq 1\}$ has ACC.

Proof. (1) $\Longrightarrow$ (2). Because R has DCC on principal ideals [2], there is an integer n such that $Ra_1 \cdots a_n = Ra_1 \cdots a_{n+k}$, for all $k \geq 1$. This implies that $\mathrm{Ann}(a_1 \cdots a_n) = \mathrm{Ann}(a_1 \cdots a_{n+k})$ for $k \geq 1$. Hence, (b) is satisfied. For (a), consider the decomposition $R = R_1 \oplus \cdots \oplus R_t$, where each R_i is a local ring with maximal ideal m_i. For any nonzero R_i-module M_i, the simple module $S_i = R_i/m_i$ can be embedded in M_i. Let $E_i(M_i)$ be the injective envelope of M_i as an R_i-module. Then we have a nonzero map $E_i(M_i) \to E_i(M_i)/m_iE_i(M_i) \to S_i \hookrightarrow M_i$.

Since R is commutative, $E_i(M_i)$ is R-injective. Therefore, for any nonzero R-module M, we have $M = R_1M \oplus \cdots \oplus R_tM$, where each $M_i = R_iM$ is an R_i-module. Now it is easy to see that there is a nonzero linear map $E \to M$ where E is injective.

(2) $\Longrightarrow$ (1). By Theorem 1, J is nil. For any sequence $\{a_n \mid n \geq 1\}$ in J, there is an integer n such that $\mathrm{Ann}(a_1 \cdots a_{n+1}) = \mathrm{Ann}(a_1 \cdots a_n)$. Since a_{n+1} is nilpotent, $a_{n+1}^k = 0$ and $a_{n+1}^{k-1}a_1 \cdots a_n a_{n+1} = 0$. Then $a_{n+1}^{k-1} \in \mathrm{Ann}(a_1 \cdots a_n)$, and so $a_{n+1}^{k-1}a_1 \cdots a_n = 0$. Continuing in this way we eventually get that $a_{n+1}^2 a_1 \cdots a_n = 0$, which implies that $a_{n+1} \in \mathrm{Ann}(a_1 \cdots a_n)$. This shows that J is T-nilpotent. Then by the proof of Theorem 2, R is perfect.

Corollary 3. If R is commutative, then R is Artinian if and only if every nonzero R-module has a nonzero injective cover.

3. Appendix. The purpose of this appendix is to prove the following proposition. The main idea of the proof is contained in the article by M. F. Dischinger [3].

Proposition 1. If x, y, z, d are elements of R, and if

(i) $x^2y = x$,

(ii) $y^2z = y$,

(iii) $(z-x)^{k+1}d = (z-x)^k$ for some positive integer k,

then $xyx = x$.

Proof. Using (i) and (ii) it is easy to see that

$$xz = x^2, \tag{1}$$

$$xyz = x. \tag{2}$$

The first equation follows because $xz = x^2yz = x(x^2y)yz = xx^2y = xx$. The second equation follows by a similar calculation: $xyz = x^2yyz = x^2y = x$.

Using equation (1), we have $(z-x)^2 = z^2 - zx - xz + x^2 = z^2 - zx$ and so

$$(z-x)^2 = z(z-x).$$

Then an induction argument shows that for any integer $n \geq 1$,

$$(z-x)^n = z^{n-1}(z-x). \tag{3}$$

Now use this and equations (1) and (2) to see that

$$xy(z-x)^2 = xyz(z-x) = x(z-x) = xz - x^2 = 0. \tag{4}$$

Also, by hypothesis (iii), there is some positive integer k so that

$$(z-x)^k = (z-x)^{k+1}d = (z-x)^k(z-x)d = z^{k-1}(z-x)^2d.$$

Using hypothesis (ii) as the basis for another induction argument, it follows that $y^n z^{n-1} = y$ for every positive integer n. Hence by (3),

$$y^n(z-x)^n = y^n z^{n-1}(z-x) = y(z-x) \tag{5}$$

for every positive integer n. Now using equation (3), hypothesis (iii) and the equation above, we have

$$y(z-x)^2d = y^k z^{k-1}(z-x)^2d = y^k(z-x)^{k+1}d = y^k(z-x)^k = y(z-x) \tag{6}$$

for some positive integer k. By equation (5) and this equation,

$$y(z-x)d = y^2(z-x)^2d = y^2(z-x).$$

By equations (4) and (6)

$$xy(z-x) = xy(z-x)^2d = 0.$$

Therefore, $xyz = xyx$. Now since $x = xyz$, we conclude that $xyx = x$.

References

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Richard Belshoff
Department of Mathematics
Southwest Missouri State University
Springfield, MO 65804
email: belshoff@math.smsu.edu

Jinzhong Xu
Department of Mathematics
University of Kentucky
Lexington, KY 40506
email: abc@ms.uky.edu