ONE-PARAMETER SUBGROUPS IN SEMIGROUPS IN THE PLANE

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1. INTRODUCTION

Let S be a topological semigroup (with identity) that is embedded in the plane E. Suppose that P is a one-parameter subgroup in S containing the identity (that is, suppose P is algebraically and topologically isomorphic to the multiplicative group of positive real numbers). In this note, our interest centers on two questions: (1) What is the nature of an orbit Px for $x \in S$ if Px is neither a point nor a simple closed curve? (2) What is the nature of the boundary of P? We show, without any restriction on S, that the answer to the first question is that Px is homeomorphic to P. Assuming that S is closed, we show that the boundary of P (if not empty) is either a point (which is a zero for P) or the circle group. If S = E, then that stage in the proof of each result which makes use of the main lemma can be made somewhat simpler, because if the local cross section theorem is applied to the plane, it yields a section which is an arc (see [6]). Since an orbit can cross a section at most once, it becomes an easy matter to construct the simple closed curves needed in order to show that if an orbit enters a certain region and if it must get out, then it must "cross in the opposite direction." Alternately, if it is known that a certain orbit *can not* cross in the opposite direction, then one end of it must be bounded, and one can find an idempotent. We make these ideas precise and use them in the proof of Theorem 1 and (by way of Lemma 1) in the proof of Theorem 2. They are precisely the ideas used in the proof of Theorem 3.7 of [7] and Lemma 2.1 of [6]. Perhaps the most striking illustration of the difference between the case S = E and the general case is obtained by comparison of the figure in [7] with our figure.

So far, the only applications we have of the case $S \neq E$ are slightly technical. One yields Corollary 1.1. For another, observe that the nature of the boundary of P seems to rule out any "reasonable" semigroup structure for a set T which is the union of the curve $y = \sin 1/x$ $(0 < x < 1)$ with an open arc in its boundary. For the closure of T, such a structure has long been ruled out. However, even in the case S = E, our results appear to have gone unnoticed hitherto.

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2. THE MAIN LEMMA

The first lemma is purely topological. It is likely to seem obvious, but the proof is not entirely trivial. As was indicated in the Introduction, this lemma makes it possible to carry out arguments like those in [6] and [7], where the local cross section theorem does not yield enough information. All of the results of this paper depend on it; and as we intimate in the final remark, the results of Mostert in [6] on the nature of the boundary of G can be derived from it.

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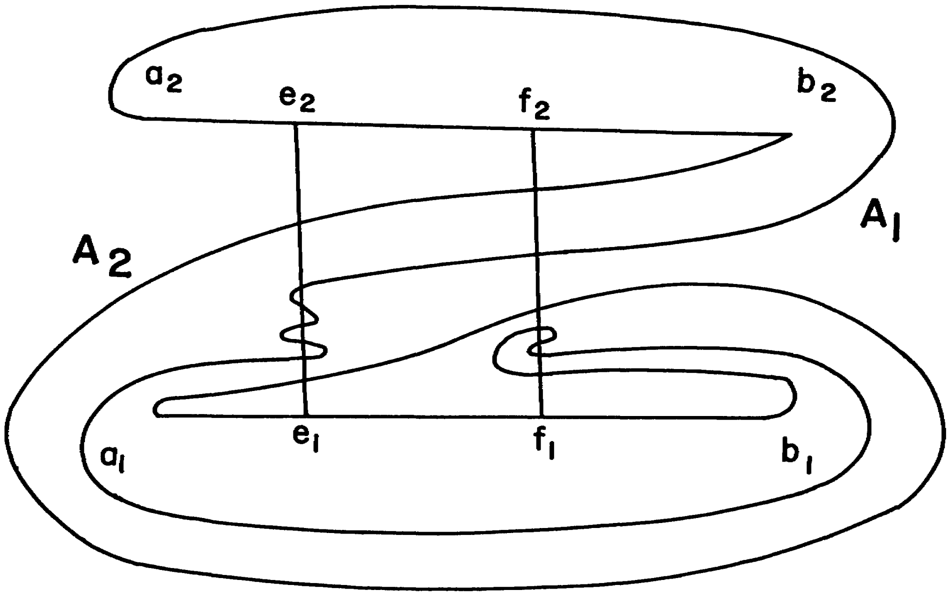
In order to simplify the statement of the main lemma, we introduce two fairly natural terms. It is sufficiently general for our purposes to define these terms for a rectangle. Suppose abcd is a rectangle with vertices a, b, c and d. Suppose that the two sides with vertex a are $\overline{ab}$ and $\overline{ac}$. Order the sides $\overline{ab}$ and $\overline{cd}$ by agreeing that $a < b$ and $c < d$. The sides $\overline{ac}$ and $\overline{bd}$ are not ordered. An arc A will be said to *cross* the rectangle if it has one end point on $\overline{ac}$, the other on $\overline{bd}$ and if, except for these end points, it is contained inside the rectangle. If A is an arc together with an order, we say that A crosses the rectangle in the *direction opposite* to that on $\overline{ab}$ and $\overline{cd}$ provided that A crosses the rectangle and the end point of A on $\overline{ac}$ is greater than the end point of A on $\overline{bd}$.

MAIN LEMMA. *Suppose that* $\overline{a_1 b_1}$ *and* $\overline{a_2 b_2}$ *are straight line segments containing points* e_1, f_1 *and* e_2, f_2, *respectively, as indicated in the figure. Order* $\overline{a_1 b_1}$ *and* $\overline{a_2 b_2}$ *so that* $a_i < b_i$ $(i = 1, 2)$. *Suppose that* A_1 *is an arc from* b_1 *to* a_2 *and* A_2 *is an arc from* b_2 *to* a_1 *such that* $\overline{a_1 b_1} \cup A_1 \cup \overline{a_2 b_2} \cup A_2$ *is a simple closed curve. Then there exists a sub-arc of either* A_1 *or* A_2 *that crosses* $e_1 f_1 e_2 f_2$ *in the opposite direction.*



Proof. We only sketch a proof. The missing details are long and tedious but standard.

Let Z denote the rectangle $e_1 f_1 e_2 f_2$. We prove that if A_1 contains no arc that crosses Z in the reverse direction, then A_2 must contain such an arc. In a series of steps, we essentially reduce the problem to the case where A_1 does not intersect Z. The first step amounts to removing points of $A_1 \cup A_2$ that are inside or on Z and that are on sub-arcs of $A_1 \cup A_2$ that do not cross Z at all. In case there exist no such points of A_1 (that is, in case $A_1 \cap Z = \emptyset$), a direct proof of the lemma is easy to give, or one can go to the last step in this proof. There is no essential difference between the way we handle the cases where A_1 has points inside Z and where A_1 has points on, but not inside, Z. Hence we assume that A_1 has points inside Z.

Now, only a finite number of sub-arcs of A_1 or A_2 can cross Z. Hence, there exists a positive number ε_1 such that no crossing of A_1 or A_2 intersects the ε_1-neighborhood of any other such crossing. Next consider the components of the intersection of $A_1 \cup A_2$ with the inside of Z. If p is on such a component that does not cross Z, we say that p is on a *loop* of $A_1 \cup A_2$. The set of points that are on some loop with both end points on $\overline{e_1 e_2}$ has some positive distance ε_2 from $\overline{f_1 f_2}$. Similarly, the set of points that are on some loop with both end points on $\overline{f_1 f_2}$ has some positive distance ε_3 from $\overline{e_1 e_2}$. Let α_i be a homeomorphism from the unit interval I onto A_i $(i = 1, 2)$. Let d denote the distance between points in the plane. Then there exists an $\varepsilon_4 > 0$ such that $d(\alpha_1(t), \alpha_2(t)) > \varepsilon_4$ for all t in I. Choose $\varepsilon < \frac{1}{2}\min(\varepsilon_1, \cdots, \varepsilon_4)$. There are piece-wise linear homeomorphisms β_1, β_2 on I such that $d(\beta_i(t), \alpha_i(t)) < \varepsilon$ for $t \in I$ and $\beta_i(0) = \alpha_i(0)$ while $\beta_i(1) = \alpha_i(1)$. Let $B_i = \beta_i(I)$ $(i = 1, 2)$. Then B_i is a polygonal approximation to A_i $(i = 1, 2)$ such that any crossing of B_i in the opposite direction corresponds to such a crossing of A_i.

It is now possible to remove not only all of the loops of $B_1 \cup B_2$ but also all points of $(B_1 \cup B_2) \cap Z$ that are not the end points of a crossing of B_1 or of B_2. In other words, there exist two other arcs C_i $(i = 1, 2)$ such that (1) C_i has the same end points as B_i (which are the same as those of A_i); (2) C_i is ordered so that the least and greatest points of C_i are the same as those of B_i; (3) if a sub-arc of C_i crosses Z, then it is a sub-arc of B_i; and if such a sub-arc crosses Z in the opposite direction when regarded as a sub-arc of C_i, then it has this property as a sub-arc of B_i; and (4) if p is a point of $C_1 \cup C_2$ that lies inside or on Z, then it lies on a sub-arc of $C_1 \cup C_2$ that crosses Z.

We have assumed that A_1 contains no arc that crosses Z in the opposite direction; therefore, C_1 contains no such arc. The proof of the lemma will be complete if we show that C_2 contains such an arc. If no sub-arc of C_1 crosses Z, then $C_1 \cap Z = \emptyset$, and we can proceed immediately to the last step in the proof. Thus let us assume that at least one sub-arc of C_1 crosses Z. Then there exists a sub-arc D of C_1 that crosses Z and has the property that if p denotes the end point of D on $\overline{e_1 e_2}$ and q denotes the end point of D on $\overline{f_1 f_2}$, then C_1 does not intersect $\overline{e_1 p}$ or $\overline{f_1 q}$. Under our assumption, $p < q$.

Let S_1 denote the simple closed curve formed by uniting the arc $\overline{f_1 b_1}$, the part of C_1 from b_1 to q, and $\overline{q f_1}$. Let S_2 denote the simple closed curve formed by uniting $\overline{e_1 b_1}$, the part of C_1 from b_1 to p, and $\overline{p e_1}$. Let E denote the arc $D \cup \overline{q f_1} \cup \overline{f_1 b_1}$, and let F denote the sub-arc of $C \cup \overline{a_2 b_2}$ from q to b_2. It is not difficult to show that $\overline{a_1 f_1}$ and F abut on E from opposite sides. Hence, a sub-arc of $\overline{a_1 f_1}$ from a_1 and a sub-arc of F from q lie on opposite sides of S_1. But if a sub-arc of $\overline{a_1 f_1}$ is on one side of S_1, then $\overline{a_1 f_1}$ is on this side; F has the same property. Thus a_1 and b_2 are on opposite sides of S_1. Hence, there exists a largest point f of C_2 on S_1, and this point must lie on $\overline{f_1 f_2}$. An argument similar to that above shows that f and a_1 must lie on opposite sides of S_2. Therefore there exists a first point e of C_2 that is greater than f, that lies on S_2, and that lies in $\overline{e_1 e_2}$. The arc of C_2 from f to e crosses Z in the opposite direction, and by the choice of C_2 there exists a sub-arc of A_2 that crosses Z in the opposite direction. This completes the proof of the lemma.

3. SEMIGROUPS EMBEDDED IN THE PLANE

The remaining results concern semigroups that are embedded in the plane. Invariably, S denotes such a semigroup with identity 1, and P denotes a one-parameter subgroup in S that contains the identity. That is, P is a sub-semigroup of S, containing 1 and topologically and algebraically isomorphic to the multiplicative group of all positive real numbers. A *line* is a subset of S that is homeomorphic to the ordinary real line. Of course, Px is the set of all px for $p \in P$, in other words, it is the *orbit* of x. Similarly, if A, B ⊂ S, then AB denotes the set of products ab for $a \in A$, $b \in B$.

THEOREM 1. *For each* $x \in S$, *either* $Px = x$, Px *is a simple closed curve, or* Px *is a line.*

Proof. If there exists a t in P ($t \neq 1$) such that $tx = x$, then Px is homeomorphic to the factor group of P modulo a closed subgroup different from the identity. Hence, without regard to the embedding of S, we see that in this case Px is a point or a simple closed curve. Thus, it remains for us to prove that if the (continuous) map $t \to tx$ is one-to-one, then it is a homeomorphism.

If this is false, there exists a sequence $\{t_n\}$ ($t_n \in P$) such that $t_n x \to x$ but $t_n \not\to 1$. There is no loss in generality in supposing that the order on P and the sequence are chosen so that t_n is monotone increasing. We can even assume that for some a, b ∈ P, $a < 1 < b$ and $b/a < t_n/t_m$ if $n > m$. Let $A = [a, b]$, and let $K = \{t_n x\} \cup \{x\}$. Then K is compact; and, with the above restrictions on t_n, the map $(t, k) \to tk$ from $A \times K$ to AK is one-to-one and hence a homeomorphism. It follows that the set consisting of Ax and the arcs $A_n = At_n x$ is an equicontinuous collection; therefore, we can assume that each one is a straight line segment [1]. Furthermore, $at_n x \to ax$ and $bt_n x \to bx$. To simplify statements, we suppose that Ax lies along the X-axis with x at the origin. Let N be the perpendicular bisector of the segment [ax, x]. Denote the Y-axis by Y. We suppose that each segment A_n intersects both N and Y. Let e_n denote the intersection of A_n and N, let f_n denote the intersection of A_n and Y, and let c denote the intersection of N and Ax. In the order inherited from P, $at_n x < e_n < f_n < bt_n x$. Fix the positive integer n. Now $[b, t_n]x$ is an arc from bx to $at_n x$, and ax obviously belongs to the closure of $(bt_n, +\infty)x$. Therefore it is possible to choose $ux \in (bt_n, +\infty)x$ sufficiently close to ax so that, if the straight line segment from ux to ax is joined with

$$Ax \cup ([b, at_n]x) \cup A_n \cup ([bt_n, u]x),$$

the result is a simple closed curve. Furthermore we can suppose that this segment does not intersect N. By the main lemma there exists an arc of the simple closed curve which crosses the "rectangle" with vertices e_n, f_n, c, and x in the direction opposite to that on $\overline{cx}$ and $\overline{e_n f_n}$. In virtue of the choice of ux, such an arc must be contained in either $[b, at_n]x$ or $[bt_n, u]x$, and by the definition of crossings, all its points except its end points are contained between N and Y. In other words, there exist sequences $\{v_n\}$ and $\{w_n\}$ with $v_n, w_n \in P$ such that $v_n > w_n$, $v_n x \in N$, $w_n x \in Y$, $(w_n, v_n)x$ is contained between N and Y, and $w_n x \to x$. Now the sequence $\{v_n/w_n\}$ does not have 1 as a limit point. For if it does, there is no loss in generality in supposing that $v_n/w_n \to 1$. Thus $(v_n/w_n)(w_n x) \to x$; namely, $v_n x \to x$. But this is impossible, since $v_n x \in N$. Hence there exists a $t \in P$ ($1 < t < b$) such that $v_n > tw_n$ for all n. Thus tw_n is contained between N and Y for all n. However, $tw_n x \to tx$, and since $tx \in (x, bx)$, this is a contradiction. The proof of the theorem is now complete.

There are many topologies on the group P_0 of positive real numbers that are properly contained in the ordinary topology and that turn P_0 into a topological semigroup (see the Appendix). According to the following corollary, none of these semigroups is embeddable in the plane.

COROLLARY 1.1. *Let* R *be a topological semigroup embedded in the plane. If there exists a continuous algebraic isomorphism from the group of positive real numbers (with the ordinary topology) to* R, *then it is a homeomorphism.*

The next corollary is a step in the direction of determining the possible boundaries of P.

COROLLARY 1.2. *If* $e \in S$ *is an idempotent in the boundary of* P *then either* $Pe = e$ *or* Pe *is the circle group.*

Proof. If $Pe \neq e$ and Pe is not a simple closed curve, then according to the theorem, Pe is a line. Therefore $Pe - \{e\}$ is the union of two components A and B. Denote the two components of $P - \{1\}$ by C and D. We may obviously suppose that $Ce = A$ and $De = B$. Now $e(B \cap C^-) \subset eC^- \subset (eC)^- = A^-$; therefore, since e is an identity for B, $B \cap C^- = \emptyset$. Similarly, $A \cap D^- = \emptyset$. Therefore $A \subset C^-$ and $B \subset D^-$. Now there exist elements $g \in C$, $h \in D$ such that $gh \in C$. Therefore, $ghe \in A$. However, $he \in B \subset D^-$, so that there exists a sequence $h_n \in D$ such that $h_n \to he$. Hence $gh_n e \to ghe$. Since h_n is unbounded in P, $gh_n \in D$ for all sufficiently large n. For such n, $gh_n e \in B$, so that $ghe \in B^-$. Hence $ghe \in A \cap B^-$, which is a contradiction. The corollary is now evident.

The following two lemmas are used to complete the description of the possible boundaries of one-parameter subgroups. In Lemma 1 we come as close as we can at present to saying that an orbit which is a line is embedded as a closed subset. In particular, we use it in the proof of Theorem 2 to show that a certain such orbit is closed. The proof of Lemma 1 is another application of the main lemma. We say that an orbit Px is *unbounded in both directions* if neither $P_+ x$ nor $P_- x$ is contained in a compact subset of the plane (here, P_+ denotes the set of t that are no less than 1, and P_- denotes the set of t that are no greater than 1).

LEMMA 1. *Suppose that* $y, z \in S$ *are such that* Py *is a line, unbounded in both directions, and* z *lies in the boundary of* Py. *Then* $Pz = z$.

Proof. Except for one detail, the proof is essentially the same as the proof (in Theorem 1) that if the map $t \to tx$ is one-to-one, then it is a homeomorphism. For if $Pz \neq z$, then Pz contains an arc. Hence there exist elements $a, b \in P$ $(a < 1 < b)$ and a monotone increasing sequence $\{t_n\} \in P$ such that (1) $[a, b]z$ is an arc; (2) $t_n y \to z$, and (3) $b/a < t_m/t_n$ if $m > n$. Thus the sequence of arcs $[a, b]t_n y$ converges to the arc $[a, b]z$. Therefore we may suppose that the arcs are straight line segments with $[a, b]z$ situated on the X-axis with z at the origin. We may also choose straight lines N and Y as in the proof of Theorem 1, and assume that each of the arcs $[a, b]t_n y$ crosses N and Y. As in the proof of that theorem, we seek a collection of sub-arcs of Py, each of which crosses the region between N and Y in the opposite direction to the direction of the arcs $[a, b]t_n y$. However, the method of obtaining these is somewhat different from that in the previous theorem.

For any pair of integers n and m, regard $[a, b]t_n y$ and $[a, b]t_m y$ as parallel segments (intersecting N and Y). To be definite, suppose $t_n < t_m$. Then $[bt_n, at_m]y$ is an arc of Py from $bt_n y$ to $at_m y$. Let

$$B = [a, b]t_n y \cup [bt_n, at_m]y \cup [a, b]t_m y.$$

Choose a square Q sufficiently large to encompass B. Since Py is unbounded in both directions, there exist elements $p, q \in P$ such that neither p nor q is inside Q and $B \subset [p, q]y$. Now join p and q by an arc C so that $C \cup [p, q]y$ is a simple closed curve and so that C misses Q. By the main lemma, there exists an arc of Q that crosses a certain rectangle with vertices on N and Y in the direction opposite to the direction of the arcs $[a, b]t_n y$ and $[a, b]t_m y$. By virtue of the choice of C, such an arc must be contained in either $[bt_n, at_m]y$ or $[bt_m, q]y$.

By using this result, it is possible to construct two sequences $\{v_n\}$ and $\{w_n\}$ ($v_n, w_n \in P$) such that $v_n y \in N$, $w_n y \in Y$, $v_n > w_n$, and $w_n y \to z$. From this point on, essentially the same reasoning that produced a contradiction in the proof of Theorem 1 again leads to a contradiction. We omit the details and conclude that $Pz = z$.

The following lemma is known, at least for topological rings, where it obviously really concerns semigroups. In any case, the proof is standard, and we omit it. A *right zero* is an element z such that $xz = z$ for all $x \in S$. A *left zero* is defined similarly.

LEMMA 2. *If* z *is a right zero for a topological semigroup* T *and* $K \subset T$ *is compact, then for every neighborhood* V *of* z *there exists a neighborhood* W *of* z *such that* $KW \subset V$.

An important consequence of this lemma is the fact that if K is a compact neighborhood of z, then there exists a neighborhood W of z such that $KW \subset K$. Therefore if $x \in K \cap W$, then $x^2 \in K$, and an argument by induction shows that $x^n \in K$ for all n. Let $\Gamma(x)$ denote the closure of the set of powers of x. Then $\Gamma(x)$ is a compact semigroup contained in K. Therefore $\Gamma(x)$, and hence K, contains an idempotent (this may of course be z, but for our purposes that is so much the better).

THEOREM 2. *Suppose that* S *is embedded as a closed subset of the plane. Let* F *denote the boundary of* P. *If* $F \neq \emptyset$, *then there exists an idempotent* $e \in F$, *and either* $F = \{e\}$ *or* F *is the circle group. In particular, if* P *contains a zero* 0 *in its boundary, then* $P^- = P \cup \{0\}$.

Proof. Suppose that F contains no idempotents. Then, for $y \in F$, Py is unbounded in both directions. For suppose $P_+ y \subset K$, with K compact. As is observed in [7, p. 386], if z is a limit point of $P_+ y$, then $Pz \subset (P_+ y)^-$. Thus $(Pz)^-$ is a compact semigroup, and F contains an idempotent. This is contrary to our assumption. A similar argument applies to $P_- y$. We conclude that unless F contains an idempotent, Py is unbounded in both directions, for all $y \in F$. But then Py is closed, for otherwise there exists a z in the boundary of Py. Therefore $z \in F$, so that Pz is unbounded in both directions. In particular, $Pz \neq z$. This is contrary to Lemma 1. Therefore Py is closed. Finally, $y^2 \in (Py)^-$, so that $y^2 \in Py$. But if $y^2 = py$, then y/p is an idempotent in F.

Thus, in any case, F contains an idempotent e. By Corollary 1.2, either $Pe = e$ or Pe is a simple closed curve.

First consider the case $Pe = e$. Here e is a zero for P^- which we now denote 0. There exist no other idempotents in F. For suppose that e' is another idempotent. Then Pe', being either a point or a simple closed surve, is closed. However, $0 = 0e' \in P^- e' \subset (Pe')^- = Pe'$, which is a contradiction, since $0 \neq e'$.

We shall show that if $0 \in (P_-)^-$, then $(P_-)^-$ is compact, and, hence,

$$(P_-)^- = P_- \cup \{0\}$$

[4]. It follows that 0 is not in $(P_+)^-$, since otherwise a similar argument shows that $(P_+)^- = P_+ \cup \{0\}$, so that P^- is a simple closed curve, which is impossible. Therefore, $(P_+)^- \subset P$. For suppose there exists an $x \in (P_+)^- - P$. Then it is easy to see that $Px \subset (P_+)^-$; consequently, $0 \in (P_+)^-$, which is a contradiction. Hence if we can show that $0 \in (P_-)^-$ implies $(P_-)^-$ is compact, we shall have proved that $P^- = P \cup \{0\}$.

Thus suppose $0 \in (P_-)^-$, and let K be a (compact) disc with center at 0. For $x \in P$, we let $(0, x)$ denote the set of $t \in P$ such that $t < x$. By Lemma 2, there exists a neighborhood W of 0 such that $st \in K$ whenever $s \in K \cap S$ and $t \in W \cap S$. Let $b \in K \cap W \cap P_-$. According to the discussion following Lemma 2, there is an idempotent in $\Gamma(b)$, which is therefore in F. Since 0 is the only idempotent in F, $0 \in \Gamma(b)$. Now the set of accumulation points of $\Gamma(b)$ is an ideal in $\Gamma(b)$. Since $\Gamma(b)$ is abelian, the set of accumulation points is a group, by a result of Koch's [5]. Therefore 0 is the only accumulation point of $\Gamma(b)$, and $b^n \to 0$.

If for some integer N, $(b^{N+1}, b^N) \subset K$, then, since $b \in W$, $b(b^{N+1}, b^N) \subset K$; and, by induction, $b^n(b^{N+1}, b^N) \subset K$ for all n. Therefore $(0, b^N) \subset K$. Thus

$$(P_-)^- = (0, b^N)^- \cup [b^N, 1],$$

so that $(P_-)^-$ is compact.

Hence we may assume that for every integer n, $(b^{n+1}, b^n) \not\subset K$. Then for each n there exists an $x_n \in (b^{n+1}, b^n)$ that belongs to the boundary of K. Now $x_n = p_n b^n$ for some $p_n \in P$ $(b \leq p_n \leq 1)$. There is no loss in generality in assuming that $p_n \to p$ for some $p \in P$. Therefore $p_n b^n \to 0$ since $b^n \to 0$. But then $x_n \to 0$, which is a contradiction, since every x_n belongs to the boundary of K and 0 is in the interior of K. We conclude that $(P_-)^-$ is compact and that if $Pe = e$ for any idempotent $e \in F$, then $F = \{e\}$.

It remains to consider the case where Pe is a simple closed curve for some idempotent $e \in F$. We may assume that $S = P^-$. Then Pe is an ideal in S, and P obviously lies outside Pe. Let D denote the (closed) disc bounded by Pe. Let E* denote the result of shrinking the plane E (which contains S) to a point, and let S* denote the image of S under this shrinking. Of course, E* is a plane, and it is easy to see that S* is a closed semigroup with identity and that P is essentially unchanged. Now D is a zero in the boundary of P in S*. By the results above, no other members of S* lie in the boundary of P. Relative to E, this says that in this case the boundary of P, is Pe. This completes the proof of the theorem.

For an application of the previous results, we turn to the case where S is the entire plane E. As usual, H(1) denotes the maximal subgroup in E, G denotes the component of the identity in H(1), and L denotes the boundary of G. We show that from every right zero in L there emanates a one-parameter subgroup of G (of course, a corresponding statement holds for left zeros). This proposition holds two-fold interest for us. First, we have used it in a fairly crucial way in [3] to find a rather simple condition that assures the presence of a copy of the entire semigroup of real numbers. This makes possible an extension of the results in [2]. Second, we feel that the proposition should be of some interest in connection with the unsettled question of the extent to which multiplication on G and L separately determine multiplication on $G \cup L$ (in this connection see [2] and [7]).

Suppose L contains a right zero w. Then L contains no simple closed curve, and G is topologically a plane [6]. Let K be a compact neighborhood of w. According to the discussion following Lemma 2, there exists an $x \in K \cap G$ and an idempotent $e \in L$ (and also in K) such that $e \in \Gamma(x)$. Now either of the possible groups on G is solvable, so by [8, Lemma 2] there exists a one-parameter subgroup $P \subset G$ such that $x^n \in P$ for all n. Hence $e \in P^-$. Since L contains no simple closed curve, e is a zero for P, and $P^- = P \cup \{e\}$. Thus in every neighborhood of a right zero there exists an idempotent that forms the boundary of a one-parameter subgroup of G.

We now make use of the possible multiplications on L as given in [6]. For one thing, when L contains a right zero z, either (1) L is a half-ray whose end point is a (two-sided) zero 0, and $x^2 = 0$ for all $x \in L$, or (2) L is a line and either (i) L contains a zero 0 or (ii) $L = zG$, and every element of L is a right zero.

In case (1), L contains no idempotents other than 0, therefore $z = 0$, and the argument above shows that there exists a one-parameter subgroup $P \subset G$ with $P^- = P \cup \{z\}$. In case (2, i), $z = 0$, and an examination of the possible multiplications for L shows that z is not a limit point of idempotents. Thus we have again the existence of a $P \subset G$ with $P^- = P \cup \{z\}$. Finally, consider the case (2, ii). Here every element of L is a right zero, so that the argument above yields the existence of a right zero z and a one-parameter subgroup $P \subset G$ such that $P^- = P \cup \{z\}$. Since every $w \in L$ can be written in the form $w = zg$ for some $g \in G$, and since $zg = g^{-1}zg$,

$$(g^{-1}Pg)^- = (g^{-1}Pg) \cup \{w\}.$$

We have thus proved the following theorem.

THEOREM 3. *If* z *is a right zero in* L, *then there exists a one-parameter subgroup* $P \subset G$ *such that* $P^- = P \cup \{z\}$.

Remark. In conclusion we mention that the fact that Px is a line whenever it is not a point or simple closed curve and the fact that Px must be closed under conditions such as occurred in the proof of Theorem 2 can be used to advantage to give a set of arguments, slightly different from those in [6], to determine the nature of L. However, these arguments are not much simpler than those in [6]; and, aside from the facts already mentioned, they do not, so far, seem to throw much additional light on the problem. We have accordingly omitted them.

APPENDIX

Professor Fort has given the following construction of a semigroup topology for the additive group R of all real numbers:

Let p be the sequence of integers defined so that $p_1 = 1$ and $p_n = 2n(1 + p_{n-1})$ for $n > 1$. For each strictly increasing sequence q of real numbers, let N(q) be the set of all numbers a such that, for some j,

$$a = p_{i_1} + \cdots + p_{i_j}, \text{ where } i_k \geq q_k \text{ if } 1 \leq k \leq j.$$

One now verifies the following: (1) if m is a positive integer and $q_1 > m$, then $N(q) \cap (m + N(q)) = \emptyset$; (2) if q is given and $r_i = q_{2i}$, then $N(r) + N(r) \subset N(q)$; and (3) if q, r, s are such sequences and $s \geq q$, $s \geq r$, then $N(q) \cap N(r) \supset N(s)$. For each $\varepsilon > 0$ and each sequence q, let

$$W(q, \varepsilon) = \{x \in R: |x| < \varepsilon \text{ or } |x - a| < \varepsilon \text{ for some } a \in N(q)\}.$$

By virtue of (3), these sets, together with their translates form the base for a topology T for R. By virtue of (2), T is a Hausdorff topology; and by virtue of (2), addition is simultaneously continuous in both variables. Obviously, T is included in the ordinary topology; and since the sequence p converges to zero in this topology, the inclusion is proper.

We now outline an argument which shows that Fort's procedure can be made to yield different semigroup topologies. First, if the sequences q are restricted to belong to any collection Q satisfying

(i) if $q \in Q$ and $r_i = q_{2i}$, then $r \in Q$, and

(ii) if $q, r \in Q$, then there exists an $s \in Q$ with $s \geq r$ and $s \geq q$,

then a topology T(Q) is obtained. Second, for any such Q, the infinite interval $(x, +\infty)$ is open in the topology T(Q). Therefore a homeomorphism h between two such topologies is a measurable function with respect to the ordinary topology. If it is, furthermore, a semigroup isomorphism, then it must have the form $h(x) = \alpha x$ for some $\alpha \in R$. Using this fact, we show that any semigroup isomorphism is necessarily the identity. We then prove that if Q and Q' are appropriately different, $T(Q) \neq T(Q')$. Thus our earlier assertion, that there are many semigroup topologies on R, is justified.

The following properties of the sequence p are easy to verify and generally useful in deriving properties about T(Q):

(I) For any integer j, $p_1 + \cdots + p_j < p_{j+1}$; in fact, $j(p_1 + \cdots + p_j) < p_{j+1}$.

(II) For any number α there exists an integer N such that if $i > j > N$, then $p_i > \alpha p_j$.

(III) A number has at most one representation as a sum of terms p_i.

Suppose $h(x) = \alpha x$, $\alpha \neq 1$. We shall show that h is not a homeomorphism between T(Q) and T(Q') for any Q, Q'. If $\alpha < 0$, then obviously αp does not converge to zero in any T(Q). We therefore assume $\alpha > 0$. Since $h^{-1}(x) = (1/\alpha)x$, we may suppose $\alpha > 1$. We show that in this case, too, αP does not converge to zero in T(Q).

As mentioned in (II), there exists an integer N such that if $i > j > N$, then $p_i > \alpha p_j$. Suppose, for some $i > N$,

$$(*)\qquad \begin{cases} \alpha p_i = \nu + p_{t_1} + \cdots + p_{t_m} \text{ for some } \nu \\ (0 \leq \nu \leq 1) \text{ and some finite sequence } t_1, \cdots, t_n. \end{cases}$$

Then $\alpha p_i \geq p_{t_m}$. Therefore $p_i \geq p_{t_m}$, since if $p_i < p_{t_m}$, then $\alpha p_i < p_{t_m}$, which is a contradiction. Also, if $p_i > p_{t_m}$, then by (I), $p_i > \nu + p_{t_1} + \cdots + p_{t_m} = \alpha p_i$; consequently, $p_i > \alpha p_i$. However, $\alpha > 1$, so that this is impossible. Thus, if $i > N$, equation (*) can hold only if $p_{t_m} = p_{t_i}$.

Using (II) again, we obtain a $k > i$ such that

$$(1 + p_1 + \cdots + p_{k-1})/p_k < 1/p_i.$$

Suppose that for such k,

$$(**)\qquad \begin{cases} \alpha p_k = \mu + p_{s_1} + \cdots + p_{s_n} \text{ for some } \mu \ (0 \leq \mu \leq 1) \\ \text{and some finite sequence } s_1, \cdots, s_n. \end{cases}$$

By the previous result, $p_k = p_{s_n}$. Therefore, by (*) and (**),

$$(\mu + p_{s_1} + \cdots + p_{s_n - 1})/p_k = (\nu + p_{t_1} + \cdots + p_{t_m - 1})/p_i .$$

Denote the left-hand member of this equation by x and the right-hand member by y. Then

$$x \leq (1 + p_1 + \cdots + p_{k-1})/p_k < 1/p_i \leq y ,$$

which is a contradiction.

Thus $|\alpha p_k - z| \geq 1$ for all $z \in N(q)$ and for any sequence q. Therefore we can choose a subsequence $\{p_{n_i}\}$ of p such that $|\alpha p_{n_i} - z| \geq 1$ for all $z \in N(q)$. Hence αp_{n_i} does not converge to zero with respect to T(Q). We conclude that the only homeomorphism between T(Q) and T(Q') that is also an isomorphism is the identity. In other words, (R, T(Q)) and (R, T(Q')) are isomorphic as topological semigroups only if T(Q) = T(Q').

Now suppose Q and Q' are collections of sequences satisfying (i) and (ii), and suppose Q contains a sequence q whose range is disjoint from the range of every member of Q'. (For example, let $q_1(i) = 2^i$, and let $q_2(i) = 3^i$. Let Q and Q' be the smallest collections of sequences that contain q_1 and q_2, respectively, and satisfy (i) and (ii)). Let $q' \in Q'$ be arbitrary. Let $a = p_i$, where $i = q_1'$. Then, of course, $a \in N(q')$. However, if $a \in W(q, 1/2)$, then $|a - z| < 1/2$ for some $z \in N(q)$. Since a and z are integers, a = z. But a and z are each expressed as a sum of terms of p, so that the only p_j appearing in the expression for z is p_i. But i is in the range of q', and the indices of terms in z are in the range of q, which is impossible. Therefore $a \notin W(q, 1/2)$; that is, $W(q, 1/2) \not\supset N(q')$ for all $q' \in Q'$. Hence W(q, 1/2) is not open in T(Q'), and $T(Q) \not\subset T(Q')$. In particular, $T(Q) \neq T(Q')$.

Evidently each T(Q) has a countable base, is regular, and is the countable union of compact, one-dimensional subspaces (closed intervals). Hence (R, T(Q)) is one-dimensional and, therefore, embeddable in Euclidean three-space.

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