ISOMORPHISMS OF THE ENDOMORPHISM RING OF A FREE MODULE OVER A PRINCIPAL LEFT IDEAL DOMAIN

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1. INTRODUCTION

The problem considered here is a special case of the general problem of the extent to which a module (F, A) is determined by its endomorphism ring E(F, A). We consider the special case of free modules A, over principal left ideal domains F (not necessarily commutative rings without proper zero divisors, in which each left ideal is principal). Let (F, A) and (G, B) be two such modules (of arbitrary rank). We are able to show (Theorem 4.1) that any isomorphism of E(F, A) and E(G, B) is induced by a semilinear transformation of (F, A) upon (G, B). The theorem is proved by modifying the methods used in [3] by Baer, who studied the problem when F and G are division rings. Use is made of the results in [9] on the lattice of submodules of free modules, and of the fact that the subring $E_0(F, A)$ of endomorphisms of finite rank determines the behavior of the entire ring E(F, A).

Other studies of the isomorphisms of endomorphism rings of modules appear in [1], [2], and [7]. The modules considered in these papers are (essentially) torsion modules A over (commutative) complete discrete valuation rings F. These rings are far more restricted than our rings of scalars. Our modules are torsion-free, and they are restricted by the requirement of the existence of a basis.

A related problem for groups is considered in [8]. The modules (F, A) studied there are free, of finite rank, over principal ideal domains in which each left and right ideal is principal. The authors determine all automorphisms of the unit group of E(F, A), in case A has rank at least three over F.

2. DEFINITIONS AND PRELIMINARIES

Throughout the paper, F and G will denote rings with identities, and (F, A) will indicate a unitary left F-module A. The set of all linear functionals on A (F-homomorphisms of A into F) forms a right F-module called the *adjoint module*, and it is denoted by (A*, F). We shall denote by E(F, A) the ring of all F-endomorphisms of A. The elements of E(F, A) shall operate on the elements of A from the right. If $x \in A$ and $y \in A^*$, the effect of the homomorphism y on the element x will be denoted by (x, y).

The ring F (not necessarily commutative) will be called a *principal left ideal domain* if it is a ring without proper zero divisors, and if moreover each left ideal is a principal left ideal. Any torsion-free module (F, A) over such a ring may be assigned a unique *rank* r(A), which is the cardinal number of any of its maximal linearly independent subsets. The proof of this fact for torsion-free groups in [5, pp. 29-33] applies here, if one replaces the arguments involving commutativity of

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the integers by the fact that any two non-zero elements of F have a non-zero (least) common left multiple [6, p. 30]. It follows of course that we may assign a unique rank r(S) to each submodule S of A. Since each linearly independent set may be extended to a maximal one, it follows that if $S \subset Q$ are submodules of A, then $r(S) \leq r(Q)$. Similarly, since any non-empty set contains a maximal linearly independent subset, the rank of each submodule S does not exceed the cardinal number of any set of generators of S.

Now, if (F, A) is actually a free module, then every submodule S of A is also free [4, Theorem 5.3, p. 13]. Since a basis of S is certainly a maximal linearly independent subset of S, and (F, A) is torsion-free, the cardinality of any basis of S coincides with r(S). To each $\sigma \in E(F, A)$ we may also assign a unique rank, $r(\sigma) = r(A\sigma)$, the rank of its range submodule. Now let $E_0(F, A)$ denote the set of all σ in E(F, A) of finite rank. Then we have

LEMMA 2.1. $E_0(F, A)$ *is a two-sided ideal of* E(F, A).

Proof. The preceding remarks imply that $r(S + Q) \leq r(S) + r(Q)$ for submodules S and Q, and this tells us that E_0 is closed under addition. Now let $\sigma \in E_0$, and $\tau \in E$. Then $A(\tau\sigma) = (A\tau)\sigma \subset A\sigma$, so that $r(\tau\sigma) \leq r(\sigma)$. If we put $A\sigma = S$, then $A(\sigma\tau) = S\tau$ and $r(S\tau) \leq r(S)$, since $S\tau$ has a set of generators with cardinal number r(S).

The next Lemma actually proves a small portion of the main result.

LEMMA 2.2. *Let* (F, A) *and* (G, B) *be free modules over principal left ideal domains, such that* E(F, A) *and* E(G, B) *are isomorphic. Then* r(A) = r(B).

Proof. Let $\{b_\alpha\}$ be a basis for (F, A), and let e_α be the endomorphism which projects A on Fb_α and annihilates the complementary summands. Then it is easy to verify that $e_\alpha^2 = e_\alpha$ and $e_\alpha e_\beta = 0$ if $\alpha \neq \beta$. Let $e_\alpha \rightarrow f_\alpha$ under the isomorphism, so that $f_\alpha^2 = f_\alpha$ and $f_\alpha f_\beta = 0$ if $\alpha \neq \beta$. It can now easily be seen that $H = \Sigma_\alpha Bf_\alpha$ is a direct sum of (non-zero) submodules of B, so that $r(H) \geq r(A)$.

Since $H \leq B$, it follows that $r(B) \geq r(A)$. Since the roles of (F, A) and (G, B) can be interchanged, we have r(A) = r(B).

LEMMA 2.3. *Let* α *be an isomorphism of* E(F, A) *on* E(G, B), *where* (F, A) *and* (G, B) *are free over principal left ideal domains. Then* α *induces an isomorphism of* $E_0(F, A)$ *on* $E_0(G, B)$.

Proof. It clearly suffices to prove that the image of an endomorphism of finite rank is an endomorphism of finite rank. Let $A\sigma = S$, where $r(S) = n < \infty$. Let $\{b_\gamma\}$ be a fixed basis for A, and let $s_1, s_2, \cdots, s_n$ be a basis for S. Since each s_i is expressible in terms of a finite number of the b_γ, there exists a finite set of the b_γ [call them $b_1, b_2, \cdots, b_k$] such that

$$S \subset \sum_{i=1}^{k} Fb_i = Q,$$

where $r(Q) = k < \infty$. Since Q is a direct summand, we may define the endomorphism e which is the identity on Q and which annihilates the complementary summand, so that $Q = Ae$. It follows easily that $\sigma e = \sigma$, so that $\sigma^\alpha e^\alpha = \sigma^\alpha$. Let $\sigma^\alpha = \tau$ and $e^\alpha = f$, so that $\tau f = \tau$, where f is an idempotent in E(G, B). Since $B\tau = B(\tau f) = (B\tau)f \subset Bf$, the proof will be complete if $r(Bf) < \infty$. But under the isomorphism α, eE(F, A)e is mapped isomorphically onto fE(G, B)f. It follows by [5, p. 214(a)] that eE(F, A)e is isomorphic to E(F, Ae) and that fE(G, B)f is isomorphic to E(G, Bf). Hence

E(F, Ae) is isomorphic to E(G, Bf), so that $r(Bf) = r(Ae) = k < \infty$ by the preceding Lemma.

LEMMA 2.4. *Let* (F, A) *be free, over a principal left ideal domain, and let* ϕ *be an automorphism of* E(F, A) *which leaves the elements of* $E_0(F, A)$ *elementwise fixed. Then* ϕ *is the identity.*

Proof. Let $\{b_\alpha\}$ be a fixed basis of A, and e_α the projection on Fb_α which annihilates the remaining basis elements. Let $\sigma \in E(F, A)$. The following statements are obviously equivalent: $\sigma = \sigma^\phi$, $b_\alpha \sigma = b_\alpha \sigma^\phi$ for all α, $b_\alpha(e_\alpha \sigma) = b_\alpha(e_\alpha \sigma^\phi)$ for all α, $e_\alpha \sigma = e_\alpha \sigma^\phi$ for all α. Clearly $e_\alpha \in E_0(F, A)$ for all α. Since E_0 is a two-sided ideal, $e_\alpha \sigma \in E_0$ also. Hence $e_\alpha^\phi = e_\alpha$ and $(e_\alpha \sigma)^\phi = e_\alpha \sigma$. Hence we have (for each α) $e_\alpha \sigma^\phi = e_\alpha^\phi \sigma^\phi = (e_\alpha \sigma)^\phi = e_\alpha \sigma$, and the proof is complete.

3. THE RING $E_0(F, A)$

If S is a subset of A, then its annihilator in A* is the submodule S of elements $f \in A^*$ for which $(S, f) = 0$. If $T \subset A^*$, we similarly define T', the annihilator of T in A. A submodule Q of A or A* is called closed if $Q'' = (Q')' = Q$. If P is any subset of $E_0(R, A)$, then $\mathfrak{R}(P)$ denotes the totality of σ in E_0 for which $P\sigma = 0$. Similarly $\mathfrak{L}(P)$ denotes the left annihilator of P. Also, N(P) is the totality of x in A for which $xP = 0$, and AP is the set of elements ap for $a \in A$ and $p \in P$. We refer to the ideals $\mathfrak{R}(P)$ $[\mathfrak{L}(P)]$ as right [left] annihilators. If S is any subset of A, then R(S) is the totality of ρ in E_0 for which $S\rho = 0$, and L(S) is the set of α in E_0 for which $A\alpha \subset S$.

In this section, (F, A) is a free module of arbitrary rank, over a principal left ideal domain F, and $E_0(F, A)$ is always the ring under consideration.

LEMMA 3.1. (a) $AL(S) = S$, *for all submodules* S *of* A.

(b) $N[R(S)] = S$ *if and only if* S *is a closed submodule of* A.

(c) *If the submodule* S *is a direct summand, it is closed.*

(d) $R(S) = \mathfrak{R}[L(S)]$ *for all submodules* S *of* A.

(e) $L(S) = \mathfrak{L}[R(S)]$ *for closed submodules* S.

(f) $\mathfrak{L}(J) = L[N(J)]$ *for any subset* J *of* E.

(g) $R[N(J)] = J$, *if* J *is a right annihilator.*

(h) *To each right annihilator* J *there corresponds a unique closed submodule* S *such that* $J = R(S)$, *and the mapping* $S \to R(S)$ *is a lattice anti-isomorphism of the lattice of closed submodules of* A *and the lattice of right annihilators of* E_0.

Proof. All the statements in the lemma are proved in [9] for the ring E(F, A). The proofs go over unchanged, since all the endomorphisms used in the proofs are easily seen to be of finite rank.

LEMMA 3.2. *Every left ideal* H *of* $E_0(F, A)$ *has the form* L(S) *for a unique submodule* S, *and* $H = L(AH)$.

Proof. This is proved in [9] for the case in which A has finite rank. Only minor modifications are necessary. The proof of Lemma 19 and the inclusion $H \subset L(S)$ remain unchanged. Now let $\rho \in L(S)$, so that $A\rho \subset S$. Let $s_1, s_2, \cdots, s_k$ be a basis for $A\rho$, so that $s_i \in S$. Then there exist $f_1, f_2, \cdots, f_k$ in A* such that $x\rho = \Sigma_i (x, f_i)s_i$. The remainder of the proof is the same as in [9].

LEMMA 3.3 *Let* $A = P \oplus H$, *where* P *is a cyclic submodule. Then there exists an isomorphism* γ *of the ring* F *onto the subring* $R(H) \cap L(P)$ *of the ring* $E_0(F, A)$, *and an isomorphism* α *of the additive group* A *upon the subring* R(H) *of* $E_0(F, A)$ *such that*

(a) $(xa)^\alpha = x^\gamma a^\alpha$ *for* $x \in F$, $a \in A$ (*so that* (γ, α) *is a semilinear transformation*),

(b) $S^\alpha = L(S) \cap R(H)$ *for every submodule* S *of* A.

Proof. We can follow the proof of Proposition 5, p. 176 of [3] for the case of the ring E(F, A) with F a division ring. No changes are necessary, since the endomorphisms x^γ and a^α constructed there are of finite rank, and the only special property of F that is required is that (F, A) be torsion-free.

LEMMA 3.4. *If* σ *is a one-to-one semilinear transformation of the module* (F, A) *onto the module* (G, B), *then* $\sigma^{-1} L(S)\sigma = L(S^\sigma)$ *for every submodule* S *of* A.

Proof. The mapping $\eta \to \sigma^{-1}\eta\sigma$ is an isomorphism of E(F, A) onto E(G, B). Hence, by Lemma 2.3, $\eta \in E_0(F, A)$ if and only if $\sigma^{-1}\eta\sigma \in E_0(F, B)$. With this in mind we may now use the proof in Lemma 3, p. 185 of [3].

We now prove an important uniqueness result. The argument is a modification of a proof of Baer [2, Lemma 3.3, p. 199].

LEMMA 3.5. *Let* r(A) *be at least two. If* ϕ *is an automorphism of* $E_0(F, A)$ *which leaves each left ideal invariant, then* ϕ *is the identity.*

Proof. Let S be any submodule of A. Then since $R(S) = \mathfrak{R}[L(S)]$,

$$R(S)^\phi = \mathfrak{R}[L(S)^\phi] = \mathfrak{R}[L(S)] = R(S).$$

Now let $a \in A$, $\alpha \in E_0(F, A)$. Then the following statements are equivalent:

$$a\alpha = 0, \quad \alpha \in R(a) = R(Fa), \quad \alpha^\phi \in R(Fa), \quad a\alpha^\phi = 0.$$

Similarly, α maps A into the submodule S if and only if α^ϕ maps A into S.

Let B be a basis for A, and let $b \in B$, $g \in A$. Let $\alpha(b, g)$ be the unique element in $E_0(F, A)$ which maps b onto g and annihilates the remaining basis elements. Since $A\alpha(b, g) \subset Fg$, it follows that $A\alpha^\phi(b, g) \subset Fg$. If $g \neq 0$, $b\alpha(b, g) \neq 0$, so that $b\alpha^\phi(b, g) \neq 0$ and $b\alpha^\phi(b, g) = e(b, g)g$, where $e(b, g) \in F$ and $e(b, g) = 0$ if, and only if $g = 0$. Also, $\alpha^\phi(b, g)$ annihilates the remaining basis elements. From $\alpha^2(b, b) = \alpha(b, b)$ it follows that $e^2(b, b) = e(b, b)$, and since F has no proper zero divisors, $e(b, b) = 1$. This gives $\alpha^\phi(b, b) = \alpha(b, b)$. From

$$\alpha(b, b + g) = \alpha(b, b) + \alpha(b, g)$$

we infer that $\alpha^\phi(b, b + g) = \alpha^\phi(b, b) + \alpha^\phi(b, g)$. Suppose now that b and g are linearly independent over F. Then, since

$$b\alpha^\phi(b, b + g) = b[\alpha^\phi(b, b) + \alpha^\phi(b, g)] = b + e(b, g)g$$

and

$$[e(b, b + g)b + e(b, b + g)g] = e(b, b + g)(b + g) = b\alpha^\phi(b, b + g),$$

we have $e(b, b + g) = 1$ and $e(b, b + g) = e(b, g)$. Hence $e(b, g) = 1$ and $\alpha^{\phi}(b, g) = \alpha(b, g)$.

Suppose now that b and g are dependent. Since $r(A) \geq 2$, we can find $w \in B$, $w \neq b$. Then w and g must be independent. For suppose the contrary. Then, because the module A is torsion-free, there exist x, y, u, v in F, all different from zero, such that

$$xw = yg \quad \text{and} \quad vb = ug.$$

Let γ be the (non-zero) least common left multiple of y and u, so that $\gamma = \gamma_1 y$ and $\gamma = \gamma_2 u$. Then $(\gamma_1 x)w = (\gamma_2 v)b$, so that w and b are dependent, a contradiction.

But since $\alpha(b, g) = \alpha(b, w) \cdot \alpha(w, g)$,

$$\alpha^{\phi}(b, g) = \alpha^{\phi}(b, w) \cdot \alpha^{\phi}(w, g) = \alpha(b, w) \cdot \alpha(w, g) = \alpha(b, g).$$

Hence $\alpha^{\phi}(b, g) = \alpha(b, g)$ for any $b \in B$, $g \in A$. Now we may repeat the last paragraph of the aforementioned proof to show that ϕ is the identity.

4. THE ISOMORPHISM THEOREM

In this section, (F, A) and (G, B) are free modules of arbitrary rank over principal left ideal domains.

LEMMA 4.1. *Let* σ *be an isomorphism of* $E_0(F, A)$ *upon* $E_0(G, B)$. *If* S *is any submodule of* A, *define* $S^{\sigma^*} = B[L(S)]^{\sigma}$. *Then*

(a) σ^* *is an isomorphism of the lattice of submodules of* A *upon the lattice of submodules of* B.

(b) *for each submodule* S *of* A, $L(S^{\sigma^*}) = L(S)^{\sigma}$;

(c) *if* S *is a closed submodule of* A, $S^{\sigma^*} = N[R(S)]^{\sigma}$ *and* $R(S^{\sigma^*}) = [R(S)]^{\sigma}$.

Proof. (a) The mapping $S \to L(S)$ is an isomorphism of the lattice of submodules of A and the lattice of left ideals of E_0, by Lemmas 3.1(a) and 3.2. The map $L(S) \to L(S)^{\sigma}$ is a lattice isomorphism of the left ideals of $E_0(F, A)$ and $E_0(G, B)$ induced by the ring isomorphism. Lemmas 3.1 and 3.2 applied to (G, B) and $E_0(G, B)$ say that the map $L(S)^{\sigma} \to BL(S)^{\sigma}$ is an isomorphism of the left ideal lattice of $E_0(G, B)$ and the submodule lattice of B. The product of these three mappings is the desired isomorphism.

(b) $L(S)^{\sigma} = L\{BL(S)^{\sigma}\}$, since $H = L(BH)$ for left ideals of $E_0(G, B)$, by Lemma 3.2. But $L\{BL(S)^{\sigma}\} = L(S^{\sigma^*})$ by definition.

(c) Using the fact that $\mathfrak{L}(J^{\sigma}) = [\mathfrak{L}(J)]^{\sigma}$ and (a), (f), and (e) of Lemma 3.1, we have

$$N[R(S)]^{\sigma} = BL\{N[R(S)]^{\sigma}\} = B\,\mathfrak{L}[R(S)]^{\sigma} = B\{\mathfrak{L}[R(S)]\}^{\sigma} = BL(S)^{\sigma} = S^{\sigma^*}.$$

Since R(S) is a right annihilator (Lemma 3.1(h)), the same is true of $R(S)^{\sigma}$, and therefore $R(S^{\sigma^*}) = R\{N[R(S)]^{\sigma}\} = R(S)^{\sigma}$, by (g) of Lemma 3.1.

LEMMA 4.2. *Suppose* E(F, A) *and* E(G, B) *are isomorphic and* $A = P \oplus H$, *where* P *is a cyclic submodule. Then we can find submodules* P* *and* H* *of* B *such that* $B = P^* \oplus H^*$, *where* P* *is cyclic.*

Proof. By Lemma 2.3, the given isomorphism σ induces an isomorphism σ of $E_0(F, A)$ onto $E_0(G, B)$. Define $P^* = P^{\sigma^*}$ and $H^* = H^{\sigma^*}$, using the notation of the preceding Lemma. Since $A = P \oplus H$, it follows from (a) of the preceding Lemma that $B = P^* \oplus H^*$. Now let e be the unique endomorphism in $E_0(F, A)$ which is the identity on P and annihilates H, so that P = Ae. It can be verified exactly as in Proposition 1, p. 178 of [3], that $L(P) = E_0 e$. By (b) of the preceding Lemma, $L(P^*) = L(P)^\sigma$. Thus $L(P^*) = [E_0(F, A) \cdot e]^\sigma = E_0(G, B)f$, where f is the image of e under σ.

Thus $P^* = BL(P^*) = BE_0 f = Bf$. ($BE_0 = B$, since if $a \in B$, we can take for b_1 a fixed basis element of B, and E_0 contains the endomorphism under which $b_1 \to a$ and which annihilates the remaining basis elements).

But in the proof of Lemma 2.3 we showed that if e is idempotent in E(F, A) and $e^\sigma = f$, where σ is an isomorphism of E(F, A) and E(G, B), then Ae and Bf have the same number of basis elements. Hence $P^* = Bf$ is cyclic.

We are now in a position to follow the arguments of Baer [3, p. 186].

THEOREM 4.1. *Let* (F, A) *and* (G, B) *be free modules of arbitrary rank over principal left ideal domains. If* σ *is an isomorphism of* E(F, A) *upon* E(G, B), *there exists a one-to-one semilinear transformation* ω *of* (F, A) *upon* (G, B) *such that* $\eta^\sigma = \omega^{-1}\eta\omega$ *for each* $\eta \in E(F, A)$.

Proof. By Lemma 2.2, r(A) = r(B). If r(A) = 1, then E(F, A) and E(G, B) are isomorphic to F and G, respectively, and the Theorem is easily seen to be true. Assume therefore that $r(A) = r(B) \geq 2$. The isomorphism σ induces an isomorphism (which we also call σ) of $E_0(F, A)$ onto $E_0(G, B)$. By the preceding Lemma, we can write $B = P^* \oplus H^*$, where P* is cyclic. By Lemma 3.3, there exists a semilinear transformation τ of (F, A) upon $[R(H) \cap L(P), R(H)]$ such that $S^\tau = L(S) \cap R(H)$ for each submodule S of A. Similarly there exists a semilinear transformation γ of $[R(H^*) \cap L(P^*), R(H^*)]$ upon (G, B) such that $T^{\gamma^{-1}} = L(T) \cap R(H^*)$ for each submodule T of B. By Lemma 4.1 (b), $L(P^*) = L(P)^\sigma$, and by (c) of the same Lemma, $R(H^*) = R(H)^\sigma$, since H, being a direct summand, is closed. Thus σ effects a semilinear transformation of $[R(H) \cap L(P), R(H)]$ upon $[R(H^*) \cap L(P^*), R(H^*)]$. Hence $\omega = \tau\sigma\gamma$ is a semilinear transformation of (F, A) upon (G, B). Now, if S is any submodule of A, then

$$S^{\tau\sigma} = [L(S) \cap R(H)]^\sigma = L(S)^\sigma \cap R(H)^\sigma = L(S^{\sigma^*}) \cap R(H^*) = (S^{\sigma^*})^{\gamma^{-1}}.$$

Hence $S^\omega = S^{\sigma^*}$ for every submodule S of A.

The semilinear transformation ω induces an isomorphism (also called ω) of E(F, A) upon E(G, B) defined by $\eta^\omega = \omega^{-1}\eta\omega$ for each $\eta \in E(F, A)$. Let the restriction of this ring isomorphism ω to $E_0(F, A)$ also be denoted by ω. Then $L(S)^\omega = \omega^{-1}L(S)\omega = L(S^\omega)$, by Lemma 3.4. Hence

$$L(S)^\sigma = L(S^{\sigma^*}) = L(S^\omega) = L(S)^\omega$$

for each submodule S of A. Let $\alpha = \sigma\omega^{-1}$. Then α is an automorphism of E(F, A) which leaves invariant all the left ideals of $E_0(F, A)$. By Lemma 3.5, α is the identity on $E_0(F, A)$. By Lemma 2.4, it is the identity on all E(F, A), so that $\sigma = \omega$ and $\eta^\sigma = \omega^{-1}\eta\omega$ for all $\eta \in E(F, A)$.

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