A RESULT IN THE GEOMETRY OF NUMBERS

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1. Let N denote the set of rational integers, and R the set of real numbers. Consider the function m defined on the non-negative real numbers by

$$m(c) = \max\{\min\{|\alpha - u|\,|\beta - u|;\ u \in N\};\ \alpha, \beta \in R,\ |\alpha - \beta| = 2c\}.$$

THEOREM 1. *The function* m *defined above has the following values:*

I. $$m(c) = \frac{1}{4} - c^2 \quad \text{if } 0 \le c \le \frac{1}{\sqrt{8}}.$$

II. $$m(c) = c^2 \quad \text{if } \frac{1}{\sqrt{8}} \le c \le \frac{1}{2}.$$

III. *For any positive integer* j,

$$m(c) = \begin{cases} c^2 - \frac{(j-1)^2}{4} & \text{if } j \le 2c \le \sqrt{j^2+1}, \\ \frac{(j+1)^2}{4} - c^2 & \text{if } \sqrt{j^2+1} \le 2c \le \sqrt{j^2+j+1/2}, \\ c^2 - \frac{j^2}{4} & \text{if } \sqrt{j^2+j+1/2} \le 2c \le j+1. \end{cases}$$

The problem of evaluating the function m was suggested to us by Professor Ivan Niven. In the ninth series of Earl Raymond Hedrick Lectures [Michigan State University, August 29 and 30, 1960, as yet unpublished], Professor Niven proved the following two lemmas:

LEMMA A. *If* β *and* α *are real numbers lying between the same pair of consecutive integers, then there exists an integer* u *such that*

$$|\beta - u|\,|\alpha - u| \le 1/4 \quad \text{and} \quad |\beta - u| < 1.$$

LEMMA B. *If* β *and* α *are real numbers with at least one integer between them, then there exists an integer* u *such that*

$$|\beta - u|\,|\alpha - u| \le \frac{|\beta - \alpha|}{2} \quad \text{and} \quad |\beta - u| < 1.$$

Using these two lemmas, Professor Niven constructed a very simple proof of a classical theorem of Minkowski [see 1; p. 48, Theorem IIA]. Whereas this is the major importance of these two lemmas, they also yield the result that

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$$m(c) \leq \begin{cases} 1/4 & \text{if } c \leq 1/2, \\ c & \text{if } c > 1/2, \end{cases}$$

and they prompt further investigation of the function m.

In Section 2 we prove Theorem 1. In Section 3, we use the methods of Section 2 to prove the following theorem.

THEOREM 2. *If α and β are any two real numbers, then there exists an integer* u *such that* $|\beta - u| < 1$ *and*

$$|\beta - u|\,|\alpha - u| \begin{cases} \leq 1/4 & \textit{if } |\alpha - \beta| < 1/2, \\ < |\alpha - \beta|/2 & \textit{if } |\alpha - \beta| \geq 1/2. \end{cases}$$

Note that of Theorem 1, Theorem 2, and Lemmas A and B, none implies or is implied by any other. However, the same proof which Professor Niven uses to show that Lemmas A and B imply the Minkowski theorem may also be used to show that Theorem 2 implies it.

2. We now turn to the establishment of Theorem 1. For ease in proof we reformulate the problem in the following way: For $b, c \in R$, $c \geq 0$, and $n \in N$, let $g(n, b, c)$ be the product of the distances from n to the ends of the interval of length 2c with center at b. Then

$$g(n, b, c) = |n - (b + c)|\,|n - (b - c)| = |(n - b)^2 - c^2|.$$

Let $f(b, c) = \min_{n \in N} g(n, b, c)$. Then

$$m(c) = \max_{b \in R} f(b, c).$$

We first prove a preliminary lemma.

LEMMA. *If* k *is a rational number of the form* m/2, *where* m *is a positive integer, and if* $b \in R$, *then there exist integers* n_0 *and* n_1 *such that*

$$(k - 1/2)^2 \leq (n_0 - b)^2 \leq k^2 \leq (n_1 - b)^2 \leq (k + 1/2)^2.$$

Proof. There exists an integer n_2 such that $k + b - 1 < n_2 \leq k + b$. Either

$$k + b - 1 < n_2 \leq k + b - 1/2 \quad \text{or} \quad k + b - 1/2 < n_2 \leq k + b.$$

If the former inequalities hold, take $n_0 = n_2 - 2k + 1$; if the latter, take $n_0 = n_2$. In either case, $(k - 1/2)^2 \leq (n_0 - b)^2 \leq k^2$.

Also, there exists an integer n_3 such that $k + b - 1/2 < n_3 \leq k + b + 1/2$. Let

$$n_1 = n_3 - 2k \quad \text{if } k + b - 1/2 < n_3 \leq k + b,$$

$$n_1 = n_3 \quad \text{if } k + b < n_3 \leq k + b + 1/2,$$

As before, in either case, $k^2 \leq (n_1 - b)^2 \leq (k + 1/2)^2$.

We remark in passing that this lemma is the best of its kind, in the sense that if k is any other rational number, there exists a real number b for which no two such integers n_0 and n_1 exist.

We now show that m actually has the values previously stated. The argument, as might be presumed from the statement of the theorem, is broken into five parts. The technique in each case is the same. For a given b in R, we obtain an upper bound for $g(n, b, c)$, for some n in N; this upper bound is also an upper bound for $f(b, c)$. We then exhibit a value of b for which f attains this bound. Hence we have the value of m.

I. Suppose $0 \leq c \leq 1/\sqrt{8}$. For $b \in R$, there exists an integer n_0 such that $b - 1/2 < n_0 \leq b + 1/2$. Now $2c^2 - 1/4 \leq 0 \leq (n_0 - b)^2 \leq 1/4$, so that

$$c^2 - 1/4 \leq (n_0 - b)^2 - c^2 \leq 1/4 - c^2.$$

Hence

$$g(n_0, b, c) = |(n_0 - b)^2 - c^2| \leq 1/4 - c^2$$

and

$$f(b, c) = \min_{n \in N} g(n, b, c) \leq 1/4 - c^2.$$

Now note that $g(1, 1/2, c) = g(0, 1/2, c) = 1/4 - c^2$. Also, if $n \neq 0, 1$, then $(n - 1/2)^2 > 1/4 > c^2$, so that

$$g(n, 1/2, c) = (n - 1/2)^2 - c^2 \geq 1/4 - c^2.$$

Thus $f(1/2, c) = 1/4 - c^2$. This together with the above gives us

$$m(c) = \max_{b \in R} f(b, c) = 1/4 - c^2.$$

II. Suppose that $1/\sqrt{8} \leq c \leq 1/2$. As before, for $b \in R$ there exists an integer n_0 such that $b - 1/2 < n_0 \leq b + 1/2$. In this case, $0 \leq (n_0 - b)^2 \leq 1/4 \leq 2c^2$, so that

$$-c^2 \leq (n_0 - b)^2 - c^2 \leq c^2.$$

Hence $g(n_0, b, c) = |(n_0 - b)^2 - c^2| \leq c^2$, and therefore $f(b, c) \leq c^2$.

Now clearly $g(0, 0, c) = c^2$. If $n \neq 0$, then $n^2 \geq 1 > 2c^2$, so that

$$g(n, 0, c) = n^2 - c^2 > c^2.$$

Thus $f(0, c) = c^2$, and we also have $m(c) = c^2$.

III. Let j be any positive integer, and suppose $j \leq 2c < j + 1$. We now consider three cases depending on where $2c$ lies in this interval.

(a) Suppose first that $j \leq 2c \leq \sqrt{j^2 + 1}$. Letting $k = (j - 1)/2$, we see that this is equivalent to

$$(k + 1/2)^2 \leq c^2 \leq k^2 + k + 1/2.$$

For any b in R, by the lemma there exists n_1 in N such that

$$k^2 \le (n_1 - b)^2 \le (k + 1/2)^2 .$$

(Note that if $k = 0$, the above inequality is satisfied if we take n_1 to be the integer nearest to b.) By subtracting c^2 throughout, we obtain

$$k^2 - c^2 \le (n_1 - b)^2 - c^2 \le (k + 1/2)^2 - c^2 .$$

Hence

$$g(n_1, b, c) \le \max\{|k^2 - c^2|, |(k + 1/2)^2 - c^2|\} .$$

But $c^2 \ge (k + 1/2)^2$, so that

$$|(k + 1/2)^2 - c^2| = c^2 - (k + 1/2)^2 \le k^2 + k + 1/2 - k^2 - k - 1/4 = 1/4$$
$$\le k^2 + k + 1/4 - k^2 \le c^2 - k^2 .$$

Thus $g(n_1, b, c) \le c^2 - k^2$, hence $f(b, c) \le c^2 - k^2$.

In order to obtain a value of b for which $f(b, c) = c^2 - k^2$, we consider the two situations where k is an integer and where k is an integer plus 1/2. If $k \in N$, then

$$g(\pm k, 0, c) = |k^2 - c^2| = c^2 - k^2 .$$

If $n^2 \le k^2$, then $c^2 - n^2 \ge c^2 - k^2$, hence $g(n, 0, c) \ge c^2 - k^2$. If $n^2 > k^2$, then $n^2 \ge (k + 1)^2$ and

$$(n - c)^2 \ge (k + 1)^2 - (k^2 + k + 1/2) = k^2 + k + 1/2 - k^2 \ge c^2 - k^2 .$$

Hence, if $k \in N$, then $f(0, c) = c^2 - k^2$.

If $k \notin N$, then $k \pm 1/2 \in N$ and

$$g(k + 1/2, 1/2, c) = g(-k + 1/2, 1/2, c) = |k^2 - c^2| = c^2 - k^2 .$$

If $n \ne k + 1/2, -k + 1/2$, then $(n - 1/2)^2 \ne k^2$. If $(n - 1/2)^2 < k^2$, then

$$c^2 - (n - 1/2)^2 > c^2 - k^2 ,$$

hence $g(n, 1/2, c) > c^2 - k^2$. If $(n - 1/2)^2 \ge (k + 1)^2$, then

$$(n - 1/2)^2 - c^2 \ge (k + 1)^2 - (k^2 + k + 1/2) = k^2 + k + 1/2 - k^2 \ge c^2 - k^2 ,$$

so that $f(1/2, c) = c^2 - k^2$. Thus, regardless of whether $k \in N$ or $k + 1/2 \in N$, there exists $b \in R$ such that $f(b, c) = c^2 - k^2$. Thus $m(c) = c^2 - k^2 = c^2 - (j - 1)^2/4$, as was to be proved.

(b) Suppose next that $\sqrt{j^2 + 1} \le 2c \le \sqrt{j^2 + j + 1/2}$. If we let $k = (j + 1)/2$, then

$$k^2 - k + 1/2 \le c^2 \le k^2 - k/2 + 1/8 .$$

If $b \in R$, then by the lemma there exists $n_0 \in N$ such that

$$(k - 1/2)^2 - c^2 \le (n_0 - b)^2 - c^2 \le k^2 - c^2,$$

so that $g(n_0, b, c) \le \max\{|k^2 - c^2|, |(k - 1/2)^2 - c^2|\}$. Since

$$0 < 1/4 = k^2 - k + 1/2 - (k - 1/2)^2 \le c^2 - (k - 1/2)^2 \le k^2 - k/2 + 1/8 - (k - 1/2)^2$$

$$= k^2 - (k^2 - k/2 + 1/8) \le k^2 - c^2,$$

it follows that $g(n_0, b, c) \le k^2 - c^2$. Hence $f(b, c) \le k^2 - c^2$.

By the same method as that used in (a), we can show that if $k \in N$, then

$$f(0, c) = \min_{n \in N} g(n, 0, c) = g(\pm k, 0, c) = k^2 - c^2,$$

while if $k + 1/2 \in N$, then $f(1/2, c) = g(k + 1/2, 1/2, c) = k^2 - c^2$. Thus the bound for $f(b, c)$ is attained and

$$m(c) = k^2 - c^2 = (j + 1)^2/4 - c^2.$$

(c) Finally, suppose $\sqrt{j^2 + j + 1/2} \le 2c \le j + 1$. Again letting $k = (j + 1)/2$, we find

$$k^2 - k/2 + 1/8 \le c^2 \le k^2.$$

If $b \in R$, then by the lemma there exists, as in part (b), an integer n_0 such that $g(n_0, b, c) \le \max\{|k^2 - c^2|, |(k - 1/2)^2 - c^2|\}$. But now

$$|k^2 - c^2| = k^2 - c^2 \le k^2 - (k^2 - k/2 + 1/8) = k^2 - k/2 + 1/8 - (k - 1/2)^2 \le c^2 - (k - 1/2)^2,$$

so that $g(n_0, b, c) \le c^2 - (k - 1/2)^2$. Thus $f(b, c) \le c^2 - (k - 1/2)^2$.

Again as in part (a), we can show that if $k \in N$, then

$$f(1/2, c) = g(k, 1/2, c) = c^2 - (k - 1/2)^2,$$

while if $k + 1/2 \in N$, then

$$f(0, c) = g(k - 1/2, 0, c) = c^2 - (k - 1/2)^2.$$

Hence in every case

$$m(c) = c^2 - (k - 1/2)^2 = c^2 - j^2/4.$$

This completes the proof of Theorem 1.

3. Using the notation of Section 2, we prove Theorem 2 by establishing the following equivalent statement.

THEOREM 2'. *If* $b, c \in R$, $c \ge 0$, *then there exist integers* n_1 *and* n_2 *such that*

(1) $$|n_1 - (b + c)| < 1, \quad |n_2 - (b - c)| < 1,$$

and

(2) $$\text{for } i = 1, 2, \quad g(n_i, b, c) \begin{cases} \leq 1/4 & \text{if } c < 1/2, \\ < c & \text{if } c \geq 1/2. \end{cases}$$

Proof. If $c = 0$, we may clearly take $n_1 = n_2$ to be the integer nearest to b. Suppose then that $0 < c < 1/2$. For $b \in R$, there exists an integer n_0 such that $-1/2 < n_0 - b \leq 1/2$. Then

$$-1 < -1/2 - c < n_0 - b - c \leq 1/2 - c < 1/2$$

and

$$-1/2 < -1/2 + c < n_0 - b + c \leq 1/2 + c < 1.$$

Hence $|n_0 - (b + c)| < 1$ and $|n_0 - (b - c)| < 1$. Also, $(n_0 - b)^2 \leq 1/4$, whence

$$-1/4 < -c^2 \leq (n_0 - b)^2 - c^2 \leq 1/4 - c^2 < 1/4$$

and $g(n_0, b, c) = |(n_0 - b)^2 - c^2| < 1/4$.

Now suppose $c \geq 1/2$. If $b + c \in N$, take $n_1 = b + c$. Then $|n_1 - (b + c)| = 0$ and $g(n_1, b, c) = 0$. If $b + c \notin N$, there exists $n_0 \in N$ such that $b + c - 1 < n_0 < b + c$ and hence $|n_0 - (b + c)| < 1$ and $|(n_0 + 1) - (b + c)| < 1$. Now let $\varepsilon = b + c - n_0$. Then

$$g(n_0, b, c) = |(n_0 - b)^2 - c^2| = |(c - \varepsilon)^2 - c^2| = 2c\varepsilon - \varepsilon^2 \geq \varepsilon - \varepsilon^2 > 0,$$

since $0 < \varepsilon < 1$. If $2c\varepsilon - \varepsilon^2 \geq c$, then $c(2\varepsilon - 1) = 2c\varepsilon - c \geq \varepsilon^2 > 0$, so that $2\varepsilon - 1 > 0$ since $c > 0$. Hence

$$g(n_0 + 1, b, c) = |(n_0 + 1 - b)^2 - c^2| = |(c - \varepsilon + 1)^2 - c^2|$$
$$= 2c - (2c\varepsilon - \varepsilon^2) - (2\varepsilon - 1) < 2c - c - 0 = c.$$

Thus take n_1 to be n_0 or $n_0 + 1$, whichever is appropriate.

To find n_2: If $b - c \in N$, take $n_2 = b - c$. Otherwise, there exists $n_0 \in N$ such that $b - c < n_0 < b - c + 1$. Hence $|n_0 - (b - c)| < 1$ and $|(n_0 - 1) - (b - c)| < 1$. Now let $\varepsilon = n_0 - (b - c)$. Then, as above, $g(n_0, b, c) = 2c\varepsilon - \varepsilon^2$. Again, if $2c\varepsilon - \varepsilon^2 \geq c$, then

$$g(n_0 - 1, b, c) = |(n_0 - 1 - b)^2 - c^2| = |(\varepsilon - c - 1)^2 - c^2| = |(c - \varepsilon + 1)^2 - c^2| < c,$$

as above. Therefore n_2 is the appropriate choice of n_0 or $n_0 - 1$.

We remark that, by the proof, equality holds in this theorem only when $c = 0$ and b is equal to an integer plus one half.

REFERENCE

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