THE SCHUR INDEX IN THE THEORY OF GROUP REPRESENTATIONS

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1. INTRODUCTION

Let KG denote the group algebra of the finite group G over a field K of characteristic 0. By a KG-module M we always mean a left G-module M which is a finite-dimensional vector space over K, such that the identity element of G acts as identity operator on M. Once a K-basis of M is specified, for each $g \in G$ the linear transformation $m \to g \cdot m$ $(m \in M)$ is represented by a matrix T(g) with entries in K. We shall say that M *affords* the *K-representation* T of G. Two K-representations T_1, T_2 of G are called *K-equivalent* (notation: $T_1 \underset{K}{\sim} T_2$) if there exists a non-singular matrix S with entries in K such that

$$S^{-1} T_1(g) S = T_2(g) \qquad (g \in G).$$

Each KG-module affords a set of mutually K-equivalent K-representations of G.

Suppose that L is a field containing K, and let U be an L-representation of G. We say that U is *realizable in* K if there exists a K-representation T such that $U \underset{L}{\sim} T$. If U is afforded by the LG-module M*, it is clear that U is realizable in K if and only if there exists a KG-module M such that

$$M^* \cong L \otimes_K M \qquad \text{as LG-modules.}$$

(For an exposition of tensor products, see Bourbaki [2]. The concept of extension of the ground field may also be found in van der Waerden [6, pp. 134-136, pp. 158-163, pp. 193-197].)

Let K* be the algebraic closure of K. The question of determining in which fields a given K*-representation is realizable leads naturally to the introduction of the Schur index, as follows:

Let U be an irreducible K*-representation of G with character ζ (defined by: $\zeta(g)$ = trace of U(g) $(g \in G)$), and let $K(\zeta)$ denote the field obtained by adjoining to K all of the values $\{\zeta(g): g \in G\}$. If F is an extension of K in which U is realizable, then certainly $F \supset K(\zeta)$. Define the *Schur index of* U *with respect to* K as

$$m_K(U) = \operatorname{Min}(F : K(\zeta)), \tag{1}$$

the minimum being taken over all fields F in which U is realizable.

The Schur index has the following basic properties (see Brauer [4], Schur [5], Witt [7]):

THEOREM 1. *Let* U *be an irreducible* K*-*representation of* G *with character* ζ. *There exists a finite extension* F *of* K *in which* U *is realizable, such that*

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$(F : K(\zeta)) = m_K(U)$. *For any finite extension* L *of* K *in which* U *is realizable, we have* $m_K(U) \mid (L : K(\zeta))$, *where the notation* "$a \mid b$" *means* "a *divides* b." *Finally,* $m_K(U)$ *is the minimal value of* m *such that* mU *is realizable in* $K(\zeta)$.

THEOREM 2. *Each irreducible* K*-representation* T *of* G *is completely reducible over* K* *into a certain number of inequivalent irreducible* K*-representations* $U_1, \cdots, U_k$, *each occurring with the same multiplicity* m *given by*

$$m = m_K(U_i) \qquad (1 \leq i \leq k).$$

If ζ_i *is the character of* U_i, *then in fact*

$$k = (K(\zeta_i) : K) \qquad (1 \leq i \leq k). \tag{2}$$

Furthermore, $U_1, \cdots, U_k$ *may be taken to be algebraically conjugate over* K. *Conversely, each irreducible* K*-representation* U *of* G *occurs in the splitting up over* K* *of some irreducible* K*-representation* T *of* G, *and* T *is determined uniquely (up to* K*-equivalence) by* U.

THEOREM 3. *Let* T *be an irreducible* K*-representation of* G, *and extend* T *(by linearity) to a* K*-representation of* KG. *Set*

$$A = \{T(x) : x \in KG\}. \tag{3}$$

Then A *is a simple algebra over* K, *and we may write* $A = \Delta_n$, *where* Δ_n *denotes a full matrix ring over the skewfield* Δ. *If* P *is the center of* Δ, *then (with the notation of Theorem* 2*)*

$$P \cong K(\zeta_i), \qquad (\Delta : P) = \{m_K(U_i)\}^2 \qquad (1 \leq i \leq k). \tag{4}$$

During the fifty years since it was first introduced, the Schur index has become a useful tool in the theory of group representations. Despite this fact, there do not seem to be available any self-contained proofs of the above theorems by modern techniques. Schur's proof of Theorems 1 and 2 is substantially correct, but has some minor gaps. Witt [7] has pointed out Theorem 3, and has proved parts of Theorems 1 and 2.

It is our purpose to give complete proofs of the above theorems by making use of the rudiments of the theory of splitting fields for division algebras. The concept of Schur index seems to be best understood in this context. We shall assume familiarity with the theory of semi-simple rings (see [1], [3], [6]), as well as with the orthogonality relations for the characters of the irreducible K*-representations of G (see [6, 183-189]).

2. DIVISION ALGEBRAS

By a *division algebra* we shall mean a skewfield Δ which is a finite-dimensional vector space over its center P. We say that a field F *splits* Δ (or is a *splitting field* for Δ) if $F \supset P$ and

$$F \otimes_P \Delta \cong F_t \qquad \text{for some } t, \tag{5}$$

where as usual F_t denotes the algebra of all t-by-t matrices over F. From (5) we obtain

$$F \otimes_P \Delta_n \cong F \otimes_P \Delta \otimes_P P_n \cong F_{tn} \quad \text{for each } n.$$

Conversely, if $F \supset P$ and $F \otimes_P \Delta_n \cong F_{tn}$ for some t and some n, then the uniqueness part of Wedderburn's Theorem [1, p. 32] implies that F splits Δ.

We shall state without proof the basic theorem on splitting fields.

THEOREM 4 ([1, pp. 76-77], [3, pp. 119-120], [6, p. 205]). *There exists an integer* m, *called the* index *of* Δ, *such that* $(\Delta : P) = m^2$. *If* F *is any maximal subfield of* Δ, *then* F *splits* Δ, *and* $(F : P) = m$. *Further, for any finite extension* L *of* P *which splits* Δ, *we have* $m \mid (L : P)$.

If F is any splitting field for Δ, a dimension argument shows at once that

$$F \otimes_P \Delta_n \cong F_{mn}$$

for each n. If N denotes a minimal left ideal in $F \otimes_P \Delta_n$, the above implies that $(N : F) = mn$, that every minimal left ideal in $F \otimes_P \Delta_n$ is isomorphic to N (as $F \otimes_P \Delta_n$-module), and that every left ideal in $F \otimes_P \Delta_n$ can be expressed as a direct sum of minimal left ideals.

On the other hand, let M be a minimal left ideal in Δ_n. Then

$$(M : \Delta) = n, \qquad (M : P) = n(\Delta : P) = nm^2.$$

Now $F \otimes_P M$ is a left ideal in $F \otimes_P \Delta_n$, and

$$(F \otimes_P M : F) = (M : P) = nm^2.$$

Therefore $F \otimes_P M$ is a direct sum of m minimal left ideals in $F \otimes_P \Delta_n$.

The above may be paraphrased in terms of representations as follows: let T be the irreducible P-representation of Δ_n afforded by M, and U the irreducible F-representation of Δ_n afforded by N. Then we have proved that

$$T \underset{F}{\sim} \begin{pmatrix} U & & 0 \\ & \ddots & \\ 0 & & U \end{pmatrix},$$

where m U's occur. In other words, mU is realizable in P. Furthermore, since T is the unique irreducible P-representation of Δ_n (unique up to P-equivalence), we have established

COROLLARY. $m'U$ *is realizable in* P *if and only if* $m \mid m'$.

We note further that if L is any extension of F, then

$$L \otimes_P \Delta_n \cong L \otimes_F (F \otimes_P \Delta_n) \cong L_{mn},$$

so that L splits Δ, and $L \otimes_F N$ is also a minimal left ideal in $L \otimes_F (F \otimes_P \Delta_n)$. This shows that the irreducible F-representation U remains irreducible under all extensions of the field F; it is customary to describe this by saying that U is *absolutely irreducible*.

3. DECOMPOSITION OF THE GROUP ALGEBRA

Since K has characteristic 0, it follows from Maschke's Theorem [6, p. 179] that the algebra KG is semi-simple. Let

$$KG = \sum_{j=1}^{r} KG \cdot \varepsilon_j \tag{6}$$

be the decomposition of KG into simple components (that is, into minimal 2-sided ideals), where correspondingly

$$1 = \varepsilon_1 + \cdots + \varepsilon_r \tag{7}$$

is the unique decomposition of the unit element 1 of KG into the sum of a maximal number of mutually orthogonal central idempotents. Each component $KG \cdot \varepsilon_j$ is a simple algebra whose center is a finite extension of K. There are exactly r inequivalent irreducible K-representations of G, afforded by minimal left ideals in the simple components of KG.

Likewise, let

$$K^*G = \sum_{i=1}^{s} K^*G \cdot e_i \qquad (1 = e_1 + \cdots + e_s) \tag{8}$$

give the decomposition of K^*G into simple components. Since also

$$K^*G = \sum_{j=1}^{r} K^*G \cdot \varepsilon_j ,$$

and since it is easily proved that every decomposition of K^*G into a direct sum of 2-sided ideals is obtainable from (8) by grouping summands, it follows at once that each ε_j is a sum of certain of the $\{e_i\}$, and that each e_i is a summand of exactly one ε_j. (This proves the last statement in Theorem 2.)

Since $K^*G \cdot e_i$ is simple, we may write $K^*G \cdot e_i \cong \Omega_{z_i}$ for some z_i, where Ω is a division algebra whose center contains K^*. However, K^* is algebraically closed, whence $\Omega = K^*$. Therefore

$$K^*G \cdot e_i \cong K^*_{z_i} \qquad \text{for some } z_i \ (1 \leq i \leq s). \tag{9}$$

For each i, choose from $K^*G \cdot e_i$ a full set of z_i^2 matrix units. In expressing each matrix unit as a linear combination of the elements of G with coefficients from K^*, only finitely many coefficients occur. Hence there exists a finite extension L of K such that LG contains the $z_1^2 + \cdots + z_s^2$ matrix units. Therefore

(10) $$LG = \sum_{i=1}^{s} LG \cdot e_i$$

is the decomposition of LG into simple components, and

(11) $$LG \cdot e_i \cong L_{z_i} \qquad (1 \leq i \leq s).$$

This implies that the inequivalent irreducible K*-representations of G are already realizable in L. If these representations are denoted by $U_1, \cdots, U_s$, then U_i is afforded by a minimal left ideal in $LG \cdot e_i$, and is absolutely irreducible. Let ζ_i be the character of U_i ($i \leq i \leq s$). Since ζ_i vanishes on $LG \cdot e_j$ for $j \neq i$, we see that $\zeta_i = \zeta_j$ if and only if $i = j$. Further, if c(LG) denotes the center of LG, then (10) and (11) imply that $(c(LG) : L) = s$.

On the other hand, let $\mathfrak{C}_1, \cdots, \mathfrak{C}_t$ denote the conjugate classes of G, and set

$$C_i = \sum_{g \in \mathfrak{C}_i} g \qquad (1 \leq i \leq t).$$

It is easy to show that $C_1, \cdots, C_t$ form an L-basis for c(LG), whence $t = s$.

Now let g_j denote any element of the class $\mathfrak{C}_j$. From the orthogonality relations one readily obtains [6, p. 189]

(12) $$e_i = z_i(G:1)^{-1} \sum_{j=1}^{s} \zeta_i(g_j^{-1})C_j \qquad (1 \leq i \leq s).$$

If we set $K_i = K(\zeta_i)$, then $e_i \in c(K_iG)$, and $(K_iG)e_i$ is a 2-sided ideal in K_iG. From

(13) $$L(K_iG)e_i = LG \cdot e_i \cong L_{z_i}$$

it follows that $(K_iG)e_i$ is in fact a simple component of K_iG. We may therefore write

(14) $$(K_iG)e_i \cong (\Delta_i)_{n_i},$$

where Δ_i is a division algebra over K_i. Now we have

$$c(\Delta_i) \cong c((\Delta_i)_{n_i}) \cong c(K_iG \cdot e_i) = c(LG \cdot e_i) \cap (K_iG)e_i = Le_i \cap (K_iG)e_i = K_ie_i.$$

Hence we may identify K_i with $c(\Delta_i)$, and write

(15) $$(\Delta_i : K_i) = m_i^2,$$

where m_i is the index of Δ_i. Furthermore, (13) and (14) imply that L is a splitting field for Δ_i, so that by Section 2 we have

(16) $$L \otimes_{K_i} (\Delta_i)_{n_i} \cong L_{m_in_i},$$

which shows that

$$z_i = m_i n_i. \tag{17}$$

LEMMA. *Let* F *be a finite extension of* K *such that* $F \subset L$. *Then* U_i *is realizable in* F *if and only if* F *splits* Δ_i.

Proof. We may restrict our attention to those fields F which contain K_i. From $c(K_i G \cdot e_i) = K_i e_i$, we deduce that $FG \cdot e_i$ is a simple algebra with center Fe_i. Therefore

$$FG \cdot e_i \cong \Lambda \otimes_F F_\nu \quad \text{for some } \nu,$$

where Λ is a division algebra with center F. Let

$$a = \text{index of } \Lambda = \sqrt{(\Lambda : F)}.$$

By definition, U_i is the L-representation afforded by a minimal left ideal in $L(FG \cdot e_i)$, and L is a splitting field for Λ. By the corollary to Theorem 4 it follows that a is the smallest integer such that aU_i is realizable in F. Hence U_i is realizable in F if and only if $a = 1$, that is, $\Lambda = F$. But $\Lambda = F$ if and only if F splits Δ_i, since

$$\Lambda \otimes_F F_\nu \cong F \otimes_{K_i} (\Delta_i)_{n_i}. \qquad \text{Q.E.D.}$$

From this lemma and Theorem 4, it follows that

$$m_i = \operatorname{Min}(F : K_i),$$

the minimum being taken over all fields F over K in which U_i is realizable. Hence

$$m_i = m_{K_i}(U_i),$$

and there exist such fields F for which $(F : K_i) = m_i$. The corollary to Theorem 4 shows further that m_i is the smallest integer such that $m_i U_i$ is realizable in K. This completes the proof of Theorem 1.

We may remark finally that $z_i \mid (G:1)$ (see [6, p. 189]), and therefore we conclude from (17) that all Schur indices divide $(G:1)$.

4. CONJUGATE REPRESENTATIONS

Choose L (as in the preceding section) to be a finite normal extension of K such that (10) and (11) hold, and let $\mathfrak{G}$ denote the Galois group of L over K. If X is any matrix with entries in L, and $\sigma \in \mathfrak{G}$, define X^σ to be the matrix obtained from X by applying σ to each entry of X. If U is any L-representation of G, and $\sigma \in \mathfrak{G}$, then

$$g \to \{U(g)\}^\sigma \qquad (g \in G),$$

defines another L-representation U^σ, called a *conjugate* of U (with respect to K). Conjugate representations have conjugate characters. Conversely, if U, V are irreducible L-representations of G, with characters ζ, η respectively, and if $\eta = \zeta^{\sigma}$ for some $\sigma \in \mathfrak{G}$, then clearly $V \underset{L}{\cong} U^{\sigma}$.

Any $\sigma \in \mathfrak{G}$ can be extended to an automorphism of LG by means of

$$(\sum \alpha_g\, g)^{\sigma} = \sum \alpha_g^{\sigma} g \qquad (\{\alpha_g\} \in L).$$

Each such automorphism leaves KG fixed. Further, an element of LG which is unchanged by all automorphisms from $\mathfrak{G}$ must lie in KG. Let $\sigma \in \mathfrak{G}$ be applied to equation (9); then we see that σ just permutes $e_1, \cdots, e_s$ among themselves. Further, if

$$e_i = z_i(G:1)^{-1} \sum_{j=1}^{s} \zeta_i(g_j^{-1})\, C_j,$$

then

$$e_i^{\sigma} = z_i(G:1)^{-1} \sum_{j=1}^{s} \zeta_i^{\sigma}(g_j^{-1})\, C_j.$$

Therefore $e_j = e_i^{\sigma}$ if and only if $\zeta_j = \zeta_i^{\sigma}$. We refer to e_i and e_i^{σ} as *conjugate* idempotents. If e_i and e_j are conjugate idempotents, say $e_j = e_i^{\sigma}$, then also σ gives a KG-isomorphism of $LG \cdot e_i$ onto $LG \cdot e_j$, which shows that $z_i = z_j$, $m_i = m_j$, $K_i \cong K_j$ (K-isomorphism), and that we may take $U_j = U_i^{\sigma}$.

Let us concentrate on the idempotent e_1, say, and let

$$\mathfrak{H} = \{\sigma \in \mathfrak{G} : e_1^{\sigma} = e_1\}.$$

If $k = (\mathfrak{G} : \mathfrak{H})$, then e_1 has precisely k distinct conjugates in LG, say $e_1, \cdots, e_k$. Now $\sigma \in \mathfrak{G}$ if and only if $\zeta_1^{\sigma} = \zeta_1$, that is, if and only if σ acts as the identity on K_1. Therefore $\mathfrak{H}$ is the Galois group of L over K_1, and so $k = (K_1 : K)$. Furthermore, $e_1 + \cdots + e_k$ is invariant under $\mathfrak{G}$, hence must be a central idempotent in KG. It must be some ε_j, since in the decomposition of any ε_j into a sum of e_i's, together with each e_i must occur all of its conjugates. Thus

$$e_1 + \cdots + e_k = \varepsilon_1, \qquad \text{say,}$$

which implies (by virtue of (10)) that

(18) $$L(KG)\varepsilon_1 = LG(e_1 + \cdots + e_k) = LG \cdot e_1 \oplus \cdots \oplus LG \cdot e_k.$$

If we write

(19) $$KG \cdot \varepsilon_1 \cong \Delta_n,$$

where Δ is a simple algebra whose center contains K, then comparison of dimensions of the two sides of (18) gives

(20) $$n^2(\Delta : K) = z_1^2 + \cdots + z_k^2 = km_1^2 n_1^2.$$

Next, let d_1 be a primitive idempotent in $K_1 G \cdot e_1$, that is, an idempotent such that $K_1 G \cdot d_1$ is a minimal left ideal in $K_1 G$. If $\sigma \in \mathfrak{G}$ carries e_1 into e_i, let it map d_1 onto d_i; then d_i is a primitive idempotent in $K_i G \cdot e_i$, and $d_1, \cdots, d_k$ are distinct conjugates over K. If we set

$$\delta = d_1 + \cdots + d_k, \tag{21}$$

the reasoning used in the preceding paragraph shows that δ is a primitive idempotent in $KG \cdot \varepsilon_1$. From (21) we obtain

$$L(KG)\delta = LG(d_1 + \cdots + d_k) = L(K_1 G \cdot d_1) \oplus \cdots \oplus L(K_k G \cdot d_k).$$

Now $K_i G \cdot d_i$ affords the K_i-representation $m_i U_i$, since each minimal left ideal in $(K_i G)e_i$ splits into m_i minimal left ideals over L. If T is the irreducible K-representation afforded by $(KG)\delta$, the above implies that

$$T \underset{L}{\sim} m_1 U_1 + \cdots + m_k U_k.$$

Furthermore, $U_1, \cdots, U_k$ are conjugate irreducible K*-representations of G, and

$$m_1 = \cdots = m_k = m_K(U_1) = \cdots = m_K(U_k).$$

We have already shown that $k = (K_i : K)$ $(1 \leq i \leq k)$, and therefore the proof of Theorem 2 is complete.

Keeping the above notation, extend T to a K-representation of KG. Then T is faithful on $KG \cdot \varepsilon_1$, but annihilates the other simple components of KG, so that the algebra A defined by (3) is in fact K-isomorphic to $KG \cdot \varepsilon_1$. The notation of Theorem 3 is thus consistent with that of equation (19). Setting $P = c(\Delta)$, we shall show that $P \cong K_i$ $(1 \leq i \leq k)$. We have

$$c(\Delta) \cong c(KG \cdot \varepsilon_1) = KG \cap c(LG \cdot \varepsilon_1)$$
$$= KG \cap c(LG \cdot e_1 \oplus \cdots \oplus LG \cdot e_k) = KG \cap (Le_1 \oplus \cdots \oplus Le_k).$$

However, if $\alpha_1, \cdots, \alpha_k \in L$ are such that $\alpha_1 e_1 + \cdots + \alpha_k e_k \in KG$, then it is clear that $\alpha_1^\sigma = \alpha_1$ for all $\sigma \in \mathfrak{H}$, whence $\alpha_1 \in K_1$. It follows readily that $\alpha_1, \cdots, \alpha_k$ are conjugate with respect to K, and that $\alpha_1 e_1 + \cdots + \alpha_k e_k \rightarrow \alpha_1$ maps

$$KG \cap (Le_1 + \cdots + Le_k)$$

isomorphically onto K_1. This shows that $P \cong K_1$ (K-isomorphism). Therefore

$$(\Delta : K) = (\Delta : P)(P : K) = k(\Delta : P). \tag{22}$$

From (21) we see that the number n of minimal left ideals into which $KG \cdot \varepsilon_1$ splits is the same as the number n_1 of minimal left ideals into which $K_1 G \cdot e_1$ splits. The substitution in (20) then yields

$$n^2 k(\Delta : P) = k m_1^2 n_1^2,$$

whence

$$(\Delta : P) = m_1^2 = \{m_K(U_i)\}^2 \qquad (1 \leq i \leq k).$$

This completes the proof of Theorem 3.

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