RELATION ALGEBRAS AND PROJECTIVE GEOMETRIES

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1. INTRODUCTION AND SUMMARY

Extension of an idea of Jónsson [1] provides a method for deriving consequences in the algebraic theory of binary relations, as set forth by Tarski [5], from certain familiar facts of projective geometry.

An *algebra of relations* is a family of binary relations, or sets of ordered couples, over some domain D, that is closed under the Boolean operations, that contains the identity relation I on D, and with each relation X contains its converse X^{-1}, and with each pair of relations X and Y their (relative) product XY. Tarski [5] gave a list of axioms that are satisfied by every algebra of relations, and defined a *relation algebra* to be any abstract algebraic system satisfying these axioms. The author showed [3], by a rather complicated example, that not every relation algebra is isomorphic to an algebra of relations. Jónsson [1] showed how to construct relation algebras from a projective plane, which are not representable if the plane does not satisfy Desargues' theorem; he thus provided a more natural construction for relation algebras that are not isomorphic to any algebra of relations. In fact, Jónsson proved more; in particular, his algebras are all *integral*: $XY = 0$ implies $X = 0$ or $Y = 0$.

Jónsson's construction will be extended here to projective geometries of dimensions $d \geq 1$ and orders $n \geq 3$; the seemingly trivial case $d = 1$ turns out to yield very simple examples of relation algebras not isomorphic to any algebra of relations.

A problem of Jónsson and Tarski [2] presents the conjecture that every integral algebra of relations is isomorphic to a relation algebra whose elements are subsets of a group, with $I = \{1\}$ and the operations having their usual meanings. It will be shown that any relation algebra derived from a 'projective geometry' of dimension $d = 1$ and finite order $n \geq 3$ is integral, and that it is isomorphic to an algebra of relations precisely in case there exists a projective plane of order n, while it is isomorphic to an algebra of subsets of a group precisely in case n is a power of a prime. Thus the existence of a projective plane whose order is not a power of a prime would refute the conjecture of Jónsson and Tarski.

2. DEFINITIONS

We shall be concerned only with a subclass of relation algebras, for which we shall employ simply the word 'algebra.' An *algebra* A is a Boolean algebra equipped with a commutative and associative multiplication, with neutral element I, that distributes over union, and which satisfies the further condition

$$I \subset XY \text{ if and only if } X \cap Y \neq 0$$

(the symbol $\subset$ indicates inclusion in the wide sense).

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It follows directly that A is integral. For, if $X, Y \neq 0$, then $X \cap X \neq 0$ implies $I \subset XX$, whence $0 \neq Y = IY \subset (XX)Y = X(XY)$, and $XY \neq 0$. It follows also that I is an atom. First, $I \neq 0$, since $I = 0 \subset 00$ would imply $0 \cap 0 \neq 0$. Second, if $0 \neq J \subset I$, then $I \cap J \neq 0$ implies $I \subset IJ = J$, whence $J = I$. Further, every homomorphism ϕ between such algebras is a monomorphism. For, if not, some $\phi X = 0$ for $X \neq 0$, whence $\phi I \subset (\phi X)(\phi X) = 0$, and the neutral element ϕI of the image algebra would not be an atom.

A *representation* of an algebra A is a complete isomorphism Δ of A onto an algebra B of relations over some domain D. A second representation Δ' of A by an algebra B' over a domain D' is *equivalent* to Δ if there exists a one-to-one map of D onto D' carrying Δ into Δ'. We note that in any algebra of relations in the present restricted sense, each element is its own converse, that is, each relation is symmetric. For, if $(x, y) \in X$, then $(y, x) \in 1$, the universal relation on D, whence (y, x) lies either in X or in the complement X' of X; but $(y, x) \in X'$ would imply $(x, x) \in I \cap XX'$ while $I \cap XX' = 0$, since $X \cap X' = 0$ implies $I \not\subset XX'$, and I is an atom.

An algebra over a group Γ is an algebra B, in the present sense, whose elements are subsets of Γ and whose operations have the usual meanings. Such an algebra always has its 'regular representation' Π, under which $X \subset \Gamma$ goes into the relation

$$\Pi(X) = \{(g, h): gx = h \text{ for some } x \in X\},$$

with domain Γ. Any isomorphism Δ of an algebra A with an algebra B over a group thus induces a representation $\Pi\Delta$ of A, and we shall speak of Δ or $\Pi\Delta$ indifferently as a *representation of* A *over a group*.

A *geometry* G will be defined to consist of a set of *points*, together with certain subsets, called *lines*. For axioms we take the following:

(I) *there exists at least one line, and each line contains at least four points;*

(II) *each pair of distinct points* p *and* q *lies on a unique line,* $\overline{pq}$;

(III) *if* p, q, *and* r *are distinct points, and a line meets* $\overline{pq}$ *and* $\overline{pr}$ *in distinct points, then it meets* $\overline{qr}$.

It is clear how to define the *dimension* d of G, which we shall always assume is finite. It is well known that these axioms imply that every line has the same number $n + 1$ of points, where n, by definition, is the *order* of G. Axiom I implies that $d \geq 1$ and $n \geq 3$; the case $n = 2$ can be accommodated, as noted below, but is uninteresting. In the case $d = 1$, which is important in this context, G is nothing more than a set of $n + 1 \geq 4$ points, with a single line containing all of them.

3. ALGEBRAS DERIVED FROM GEOMETRIES

With each geometry G we associate an algebra A(G). The Boolean structure of A(G) is determined by defining it to be the algebra of all subsets of the set which consists of all points of G together with one further element I that is not a point of G. It will be convenient and harmless to write p and I for the atoms of A(G), rather than $\{p\}$ and $\{I\}$. If we stipulate that the multiplication in A(G) distribute over union, then, in order to define the multiplication, it will suffice to specify the product of each pair of atoms. Letting p, q, r, $\cdots$ always denote points of G, we do this by the following rules:

(i) $$II = I, \qquad Ip = pI = p, \qquad pp = p \cup I;$$

(ii) $$pq = \overline{pq} - p - q \qquad \text{for } p \neq q.$$

Clearly, for the verification that A(G) is an algebra in our sense, only associativity requires attention, and this only in showing that $(pq)r = p(qr)$. By commutativity, the case $p = q = r$ is immediate, and the remaining case for p, q, and r not all distinct reduces to that where $p = q$. Now

$$(pp)r = (p \cup I)r = (\overline{pr} - p - r) \cup r = \overline{pr} - p.$$

Using $n \geq 3$, we find that $p(pr) = p(\overline{pr} - p - r) = \overline{pr} - p$, since, for all $x \subset \overline{pr} - p$, choosing $t \neq p, r, x$, we have $t \subset \overline{pr} - p - r$, and $x \subset \overline{pt} = pt - p - t$. Next, if p, q, and r are distinct but collinear,

$$(pq)r = (\overline{pq} - p - q)r = \overline{pq} \cup I,$$

where each $x \subset \overline{pq} - r$ is obtained from some $\overline{sr} - s - r$ $(s \neq r, x)$, and the part $r \cup I$ of the right member is rr. The result now follows by symmetry. Finally, suppose p, q, and r are not collinear. Then $x \subset (pq)r$ implies $x \subset \overline{sr} - s - r$ for some $s \subset \overline{pq} - p - q$. Since x lies on $\overline{rs}$, which meets $\overline{pq}$ in s, distinct from p, x is not p. Thus $\overline{px}$, which meets $\overline{sr}$ in x and $\overline{sq}$ in p, must, by (III), meet $\overline{qr}$ in a point t, which cannot be any of p, q, r, or x, since no three of these points are collinear. Thus $t \subset \overline{qr} - q - r = qr$, and $x \subset \overline{pt} - p - t = pt \subset p(qr)$, showing that $(pq)r \subset p(qr)$, and equality follows by symmetry.

This completes the proof that A(G) is an algebra.

It may be of some interest to note that the algebras isomorphic to algebras A(G) have a simple intrinsic characterization: *an algebra* A *is isomorphic to* A(G), *for some geometry* G, *if and only if* (i) A *is complete and atomistic, with more than two atoms,* (ii) $pp = p \cup I$ *for each atom* $p \neq I$, *and* (iii) *the universal element is expressible as a finite product in the form* $1 = \prod(p_i \cup I)$, *where the* p_i *are atoms.*

The last condition is merely a paraphrase of the condition that G have finite dimension; with this observation, it is clear that each A(G) has the required properties. For the converse, suppose an algebra A with these properties is given. We shall make the set G of all atoms $p \neq I$ of A into a geometry by specifying as lines of G all $\overline{pq} = pq \cup p \cup q$ for $p \neq q$. Axiom III is the easiest to verify: suppose a line meets $\overline{pq}$ in q' and $\overline{pr}$ in r' (we may assume that the five points are distinct). Then $0 \neq p \subset qq' \cap rr'$, whence $I \subset qq'rr' = qrq'r'$, which in turn implies that $\overline{q'r'} \cap \overline{qr} \neq 0$. For Axiom II it will suffice to show that if p, q, and r are distinct, and $r \subset \overline{pq}$, then $\overline{pq} = \overline{pr}$. Since $r \subset \overline{pq}$ is equivalent to the hypothesis (symmetric in q and r) that $I \subset pqr$, it will suffice to show that $\overline{pq} \subset \overline{pr}$. For $s \subset pq$ $(s \neq p, q, r)$, it will suffice to show that $s \subset pr$. Now $r \subset pq$ and $s \subset pq$ implies $I \subset pqr \cap pqs$, hence $q \subset pr \cap ps$, and

$$I \subset (pr)(ps) = (pp)rs = (p \cup I)rs = prs \cup rs;$$

since I is an atom, and $r \cap s = 0$ implies $I \not\subset rs$, it follows that $I \subset prs$, whence $s \subset pr$. For Axiom I: By hypothesis, G contains $p \neq q$, and hence G contains a line. If $\overline{pq}$ is any line, then $p \neq q$, and $pq \neq 0$ must contain at least one atom r, which cannot be p or q, since, for example, $p \subset pq$ would imply $I \subset ppq = q \cup pq$ and either $q = I$ or $p = q$. Thus $\overline{pq}$ contains at least the three points p, q, and r.

If this were all, we should have $pq = r$ and $qr = p$, which would give $(pq)r = rr = r \cup I$, different from $p(qr) = pp = p \cup I$.

[The case of order $n = 2$ can be accommodated by replacing the condition $pp = p \cup I$ by $pp = I$; the trivial geometries of dimensions $d = -1$ and $d = 0$ correspond naturally to the trivial algebras with only one or two atoms.]

4. REPRESENTATIONS

Suppose that the geometry G can be embedded as a hyperplane in a geometry H of one higher dimension. Let D be the affine space $D = H - G$. With each point p of G, associate a relation $\Delta(p)$ on the domain D defined by

$$\Delta(p) = \{(x, y): x \neq y \text{ and } p \in \overline{xy}\},$$

and let $\Delta(I)$ be the identity on D. Then Δ has a unique extension to an *affine representation* of A(G), that is, to a completely additive homomorphism of A(G) as Boolean algebra onto a Boolean algebra of subsets of the Cartesian product $D \times D$.

THEOREM 1. *Each affine representation of* A(G) *is a representation, and each completely additive representation of* A(G) *is equivalent to some affine representation.*

To show that an affine representation is indeed a representation, it suffices to show that $\Delta(p)\Delta(q) = \Delta(pq)$. If $p = q$, two points x and y of D stand in the relation $\Delta(p)\Delta(q)$ precisely in case there exists $z \neq x, y$ in D such that both $\overline{xz}$ and $\overline{zy}$ meet G in p. This will be the case if and only if x and y lie on a common line passing through p, that is, if x and y stand in the relation $\Delta(p) \cup \Delta(I)$. If $p \neq q$, the relation holds precisely in case $\overline{xp}$ and $\overline{yq}$ meet in some point $z \neq x, y$. This implies that $x \neq y$, and that $\overline{xy}$ meets $\overline{zp}$ in x and $\overline{yq}$ in y, and hence meets $\overline{pq}$ in some point $r \subset \overline{pq} - p - q$, which gives $(x, y) \in \Delta(r) \subset \Delta(pq)$. Conversely, if $(x, y) \in \Delta(r)$, for some $r \subset \overline{pq}$, then from $r \subset \overline{xy} \cap \overline{pq}$ (x, y, p, q, r all distinct) it follows by Axiom III that $\overline{xp}$ and $\overline{yq}$ meet in some point z of D.

To prove that every completely additive representation is equivalent to some affine representation, we suppose that Δ is a representation of A(G) over some domain D, and we show that if D and G are disjoint, then the set H consisting of D together with the set of all points of G can be made into a geometry which contains G as hyperplane and for which the corresponding affine representation is exactly Δ.

Among the lines of H we must count the lines of G; as the remaining lines of H we take the sets

$$L(x, p) = \{x\} \cup \{p\} \cup \{y: (x, y) \in \Delta(p)\},$$

for all x in D and p in G.

H contains a line, since G does. Each line of G contains at least four points, by hypothesis, while for the lines $L(x, p)$ this follows from $\Delta(p) \subset \Delta(p)^2$. Thus Axiom I holds. Axiom II follows from the observation that $L(x, p) = L(y, q)$ precisely in the case where $p = q$ and either $x = y$ or $(x, y) \in \Delta(p)$; and that, since Δ is completely additive, each (x, y), for $x \neq y$, is contained in exactly one of the atoms $\Delta(p)$.

For Axiom III, it will suffice to show that if A, B, B', C, C' are five distinct points such that $\overline{BB'}$ and $\overline{CC'}$ meet in A, then $\overline{BC}$ and $\overline{B'C'}$ meet in some point A'.

If all five points lie in G, this conclusion follows from Axiom III for the geometry G. Otherwise, by symmetry, we assume that neither $\overline{CC'}$ nor $\overline{B'C'}$ lies in G. We denote by a, a', b, c, and d the intersections with G of those of the lines $\overline{BC}$, $\overline{B'C'}$, $\overline{BB'}$, $\overline{CC'}$, and $\overline{BC'}$ that do not lie in G.

We use repeatedly the following argument. If X, Y, and Z are noncollinear points of D, and x, y, and z are the intersections of $\overline{YZ}$, $\overline{XZ}$, and $\overline{XY}$ with G, then from

$$(X, Z) = (X, Y)(Y, Z) \in \Delta(y) \cap \Delta(z)\Delta(x),$$

it follows that $y \subset zx$ and the points x, y, and z are collinear. Applying this to various triangles of the configuration, and using the fact that collinearity has the usual transitivity properties in G, we are able to conclude that all the points a, a', b, c, and d actually present are collinear, whence the conclusion follows easily. The details fall into five cases.

Case 1. None of A, B, B', C, C' is on G. Consideration of the triangles ABC and AB'C' shows that the triples abc and a'bc are collinear. Then caa' is collinear, and $(B, B') \in \Delta(c) \subset \Delta(a)\Delta(a')$; that is, for some A' in D, $(B, A') \in \Delta(a)$ and $(A', B') \in \Delta(a')$. But then $\overline{BC} = L(B, a)$ and $\overline{B'C'} = L(B', a')$ meet in A'.

Case 2. A, but none of B, B', C, C', is on G. The triangles BCC' and BB'C' give acd and a'cd collinear, hence caa' collinear, and the conclusion follows as in Case 1.

Case 3. B, but none of A, B', C, C', is on G. The triangle AB'C' gives caa' collinear, and the conclusion follows as before.

Case 4. B and C' are on G, but none of A, C', B' is on G. The triangle AB'C' gives BCa' collinear, and $\overline{B'C'}$ and $\overline{BC}$ meet in a'.

Case 5. B and B', hence also A, are on G, but C, C' are not on G. Then, from the given collinearity of ABB' we have

$$(C, C') \in \Delta(A) \cap \Delta(B)\Delta(B'),$$

and there exists A' with $(C, A') \in \Delta(B)$, $(A', C') \in \Delta(B')$, that is, with A' on both $\overline{CB}$ and $\overline{C'B'}$.

COROLLARY 1.1. *If* $d > 2$, *or* G *is a Desarguesian plane, then* A(G) *is representable, and all completely additive representations of* A(G) *are equivalent.*

COROLLARY 1.2 (Jónsson [1]). *If* G *is a non-Desarguesian plane, then* A(G) *is not representable.*

COROLLARY 1.3. *If* G *is a line of order* n, *then* A(G) *is representable if and only if there exists a projective plane of order* n. *The number of equivalence classes of completely additive representations is the number of ways of embedding* G *in planes* H *that are not isomorphic under some isomorphism keeping* G *pointwise fixed.*

The case $d = 1$ and $n = 6$ yields an algebra A with eight atoms, the smallest known nonrepresentable relation algebra. Explicitly, A has atoms

$$I, p_0, p_1, \cdots, p_6,$$

with the multiplication table

$$II = I, \qquad Ip_i = p_i I = p_i, \qquad p_i p_i = p_i \cup I,$$

and, for $i \neq j$,

$$p_i p_j = \bigcup_{k \neq i,j} p_k .$$

5. REPRESENTATIONS OVER A GROUP

THEOREM 2. *Let* $\triangle$ *be an affine representation of* A(G) *over an affine space* D = H - G *such that the translation group* Γ *of* D *is transitive on* D. *Then* $\triangle$ *is equivalent to the representation* $\triangle'$ *of* A(G) *over the group* Γ *defined by mapping each atom* p *of* A(G), *consisting of a point* p *of* G, *into the set of all translations in* Γ, *in the direction* p.

To prove this, choose as 'origin' any point w of D. Since Γ acts transitively on D and only the identity has fixed points, the map Ω from Γ into D defined by $\Omega(g) = w^g$ is a one-to-one correspondence between Γ and D. Now $(g, h) \in \triangle'(p)$ means that, in Γ, $h = gk$ for some translation k in the direction p. And $(\Omega g, \Omega h) \in \triangle(p)$ means that, in D, w^g and w^h are distinct, and that the line determined by them in H meets G in p, hence that some translation k in the direction p carries w^g into $w^{gk} = w^h$. Since the two conditions are clearly equivalent, Ω carries $\Pi\triangle'$, corresponding to $\triangle'$ under the regular representation, into $\triangle$.

COROLLARY 2.1. *If* $d \geq 2$, *then every representation of* A(G) *is equivalent to a representation over a group.*

COROLLARY 2.2. *If* d = 1, *and* n *is infinite or is a power of a prime, then* A(G) *possesses representations over a group.*

We shall establish a converse to Corollary 2.2. Let G have dimension 1, and let $\triangle$ be an isomorphism from A(G) onto an algebra A over a group Γ. Define $\Gamma(p) = \triangle(p) \cup I$, for each atom $p \neq 1$. From the equation $\triangle(p) = \triangle(p)^{-1}$, noted in Section 3, and the relation $(\triangle(p) \cup I)^2 = \triangle(p) \cup I$, it follows that each $\Gamma(p)$ is a group, and, indeed, a proper subgroup of Γ. From the fact that the atoms of A(G) are I and the p, it follows first that Γ is the union of the $\Gamma(p)$, and second that, for $p \neq q$, $\Gamma(p)$ and $\Gamma(q)$ have trivial intersection. Moreover, for $p \neq q$, we have

$$\Gamma(p)\Gamma(q) = (\triangle(p) \cup I)(\triangle(q) \cup I) = \triangle(p)\triangle(q) \cup \triangle(p) \cup \triangle(q) \cup I$$

$$= \bigcup_{r \neq p,q} \triangle(r) \cup \triangle(p) \cup \triangle(q) \cup I = \Gamma.$$

LEMMA. *Suppose that a group* Γ *is the union of a family of proper subgroups* Γ_i *such that, for* $\Gamma_i \neq \Gamma_j$, $\Gamma_i \cap \Gamma_j = 1$ *while* $\Gamma_i \Gamma_j = \Gamma$. *Then* Γ *is abelian and is either without torsion or of prime exponent; moreover, all the subgroups* Γ_i *are isomorphic.*

The family must contain at least three subgroups. That the Γ_i are proper subgroups implies that there exist at least two. That $\Gamma_i \Gamma_j = \Gamma$ for $\Gamma_i \neq \Gamma_j$ implies that no Γ_i is trivial. Therefore a product of two nontrivial elements from Γ_i and Γ_j can belong to neither, and hence must lie in a third group of the family.

The subgroups Γ_i are normal in Γ. If this were not so, some conjugate $g^h = g^{-1}gh$ of an element $g \neq 1$ from some Γ_i would lie in a $\Gamma_j \neq \Gamma_i$. Since $\Gamma = \Gamma_i \Gamma_j$, $h = h_i h_j$ for some $h_i \in \Gamma_i$, $h_j \in \Gamma_j$. Then $g_i = g^{h_i}$ is in Γ_i, and $g_i^{h_j}$ is in Γ_j. But the latter implies that g_i is in Γ_j as well as in Γ_i, hence that $g_i = g^{h_i} = 1$, and that $g = 1$, contrary to assumption.

The group Γ is abelian. First, since the Γ_i are normal subgroups with pairwise trivial intersections, elements from different Γ_i commute. Given any g and h in Γ, g will lie in some Γ_i, and h can be written as a product $h = h_j h_k$ of elements from Γ_j and Γ_k distinct from Γ_i. Since h_j and h_k commute with g, their product h commutes with g.

Next we show that all the Γ_i are isomorphic. Given distinct Γ_i and Γ_j, choose a third Γ_k different from both. Since Γ is the product of the normal subgroups Γ_j and Γ_k, with trivial intersection, Γ is their direct product: $\Gamma = \Gamma_j \times \Gamma_k$. Let π be the canonical projection map from Γ onto Γ_j with kernel Γ_k. Since the subgroup Γ_i of Γ has trivial intersection with the kernel Γ_k of π, π maps Γ_i one-to-one into Γ_j. Since also

$$\Gamma_j = \pi\Gamma = \pi(\Gamma_i \times \Gamma_k) = \pi\Gamma_i ,$$

π maps Γ_i onto Γ_j. Thus the restriction of π to Γ_i is an isomorphism of Γ_i onto Γ_j.

We shall show that, if Γ has any nontrivial element of finite order, then all nontrivial elements of Γ have a common prime order. Let Γ contain a nontrivial element of finite order, hence one of prime order m. This element belongs to some Γ_i, and therefore each of the isomorphic Γ_j contains an element of order m. If g is any nontrivial element of Γ, it belongs to some Γ_i, and there exists an element h of order m in some Γ_j different from Γ_i. Now gh lies in some Γ_k different from both Γ_i and Γ_j. Since g lies in Γ_i, while gh lies in Γ_k, it follows that $(gh)^m = g^m h^m = g^m$ lies in both Γ_i and Γ_k, hence that $g^m = 1$.

COROLLARY 2.3. *For* $d = 1$, A(G) *possesses a representation over a group only if* n *is infinite or a power of a prime.*

We see this as follows. By virtue of Theorem 1, if A(G) possesses a representation over a group Γ, then it must be possible to equip the group Γ with the structure of an affine plane of order n. Hence Γ must have order n^2, and, by the lemma, n^2 must be infinite or a power of a prime.

Finally, Corollaries 1.3 and 2.3 together yield the following.

COROLLARY 2.4. *If there exists a finite projective plane whose order is not a power of a prime, then there exists a finite algebra that is representable, but not representable over a group.*

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