POWER MAPS IN RINGS

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It requires very little technique, and no knowledge of any deep results, to discuss groups in which, for a fixed integer $n > 1$, $(xy)^n = x^n y^n$ for all x and y in the group. One cannot expect the same situation to hold for rings, because certain special cases involving such an identity provide interesting theorems. For instance, the theorem of Jacobson which asserts that a ring in which $x^n = x$ for all x is commutative should be a corollary of any results obtained about a ring in which $(xy)^n = x^n y^n$.

In this paper we first study rings in which $(xy)^n = x^n y^n$; later we consider rings in which $(x + y)^n = x^n + y^n$. In the last section we assume that both relations hold. Our theorems then say that while the rings need not be commutative, they are nearly commutative in the sense that all commutators turn out to be nilpotent. The existence of nil rings in which $x^n = 0$ for all x probably rules out the possibility of any stronger result.

1. RINGS IN WHICH $(xy)^n = x^n y^n$

In this section we assume that R is a ring in which $(xy)^n = x^n y^n$ for all $x, y \in R$ and a fixed integer $n > 1$. We prove

THEOREM 1. *Let* R *be a ring in which* $(xy)^n = x^n y^n$ *for all* $x, y \in R$ *and a fixed integer* $n > 1$; *then every commutator* $ab - ba$ *in* R *is nilpotent. Moreover, the nilpotent elements of* R *form an ideal.*

Proof. Using the Jacobson structure theory and settling the theorem first for division rings, we shall ascend to general rings.

So suppose that R is a division ring. Since $(xy)^n = x^n y^n$, cancelling an x on the left and a y on the right in this identity we see that $(yx)^{n-1} = x^{n-1} y^{n-1}$ for all $x, y \in R$. Hence

$$y^n x^n = (yx)^n = (yx)(yx)^{n-1} = (yx)(x^{n-1} y^{n-1}) = yx^n y^{n-1}.$$

This leads us to $y^{n-1} x^n = x^n y^{n-1}$ for all $x, y \in R$. Let D be the subdivision ring of R generated by all the x^n. D is invariant under all inner automorphisms of R, so that by the Brauer-Cartan-Hua theorem [2] either $D = R$ or D is contained in Z, the center of R. If $D \subset Z$, then, since $x^n \in D \subset Z$ for all $x \in R$, it follows from a theorem of Kaplansky [5] that R is a commutative field. On the other hand, if $D = R$, then since $y^{n-1} x^n = x^n y^{n-1}$ for all $x, y \in R$, y^{n-1} commutes with the generators of D, hence with every element of D, and therefore with every element of R. Thus $y^{n-1} \in Z$ for all $y \in R$; again by Kaplansky's theorem we can conclude that R is commutative. Thus a division ring satisfying $(xy)^n = x^n y^n$ must be a field.

Suppose now that R is a primitive ring; if it should happen that R is not a division ring, then the 2-by-2 matrices over some division ring would be a homomorphic

Received August 22, 1960.

This research was supported by a contract at Cornell University from the Office of Ordnance Research.

image of a subring of R and would thus inherit the property that $(xy)^n = x^n y^n$. This is manifestly false for $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Thus, if R is a primitive ring, it must be a division ring, whence by the above discussion it is a field.

Since any ring R semi-simple in the sense of Jacobson is a subdirect sum of primitive rings R_α each of which is a homomorphic image of R, each R_α is a field, so that R is commutative.

Now let R be any ring in which $(xy)^n = x^n y^n$. Pick $a, b \in R$, and let S be the subring of R generated by a and b; let J(S) be the radical of S. Since S is finitely generated and satisfies the polynomial identity $(xy)^n - x^n y^n = 0$, J(S) is a nil ring by a result of Amitsur [1]. However, S/J(S), being semi-simple, must be commutative. Thus all commutators in S, and in particular $ab - ba$, are in J(S), which is a nil ring. The conclusion that $ab - ba$ is nilpotent then follows.

We still have to prove that the nilpotent elements of R form an ideal. If $a^k = 0$ and $x \in R$, then $(ax)^{n^k} = a^{n^k} x^{n^k} = 0$, so that aR is a nil right ideal. Thus at any rate a belongs to J(R), the radical of R. We must show that if a and b in R are nilpotent then $a + b$ is also nilpotent. Let T be the subring of R generated by a and b, and let J(T) be the radical of T. As before, J(T) is a nil ring. We claim that both a and b are in J(T) for, as before, the right ideals aT, bT of T are nil. Thus $a + b$ belongs to J(T) and so is nilpotent. We have established that the nilpotent elements of R form an ideal of R. This completes the proof of the theorem.

An immediate consequence of the theorem, and itself of independent interest, is the

COROLLARY. *If* R *is as in* Theorem 1 *and has no nontrivial nil ideals, then* R *is commutative.*

2. RINGS IN WHICH $(x + y)^n = x^n + y^n$

We now turn our attention to the second analog, namely, to rings R in which $(x + y)^n = x^n + y^n$ for all $x, y \in R$ and some fixed integer $n > 1$.

THEOREM 2. *Let* R *be a ring in which for some fixed integer* $n > 1$, $(x + y)^n = x^n + y^n$ *for all* $x, y \in R$. *Then every commutator in* R *is nilpotent, and the nilpotent elements of* R *form an ideal.*

Proof. The procedure will be as in the proof of Theorem 1, although each stage of the procedure requires an argument different from the one used before.

Suppose then that R is a division ring in which $(x + y)^n = x^n + y^n$. Since R satisfies a polynomial identity, it is finite-dimensional over its center Z, by a theorem due to Kaplansky [4]. If, in addition, Z is finite, then R is finite, so that by Wedderburn's theorem on finite division rings it is commutative. If Z is infinite for $\lambda \in Z$, then $(x + \lambda y)^n = x^n + \lambda^n y^n$; expanding this, we get

$$\lambda p_1(x, y) + \lambda^2 p_2(x, y) + \cdots + \lambda^{n-1} p_n(x, y) = 0,$$

where

$$p_1(x, y) = x^{n-1}y + x^{n-2}yx + \cdots + xyx^{n-2} + yx^{n-1}.$$

Since Z is infinite, we can find $\lambda_1, \cdots, \lambda_{n-1}$ in Z such that

$$\begin{vmatrix} \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ \vdots & \vdots & & \vdots \\ \lambda_{n-1} & \lambda_{n-1}^2 & \cdots & \lambda_{n-1}^{n-1} \end{vmatrix} \neq 0.$$

Therefore, $p_1(x, y) = 0$. Computing $xp_1(x, y) - p_1(x, y)x = 0$, we see that $x^n y = yx^n$ for all $x, y \in R$. Thus $x^n \in Z$ for all $x \in R$, so that by the result of Kaplansky's used in the proof of Theorem 1, we can conclude that R is a field. The argument used involving $xp_1(x, y) - p_1(x, y)x$ is similar to that used by Forsythe and McCoy [3] in proving that a ring in which $x^p = x$ and $px = 0$, p a prime, is commutative.

Suppose that R is primitive; as before, if R were not a division ring, the 2-by-2 matrices over some division ring would be a homomorphic image of a subring of R and so would inherit the identity $(x + y)^n = x^n + y^n$; this is false for $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Thus R is a division ring and therefore a field. From this we can see that if R is semi-simple, then it is commutative.

Therefore all commutators of an arbitrary ring satisfying $(x + y)^n = x^n + y^n$ belong to the radical of this ring. As before, taking the subring generated by two elements leads us to the fact that commutators are nilpotent. To see that the nilpotent elements form an ideal, we first note that the sum of two nilpotent elements is nilpotent; for if $x^k = y^\ell = 0$, then

$$(x + y)^{n^{k+\ell}} = x^{n^{k+\ell}} + y^{n^{k+\ell}} = 0.$$

Suppose that $a^k = 0$, $r \in R$. Let S be the subring of R generated by a and r, and let J(S) be the radical of S. As before, J(S) is a nil ring; also, S/J(S), being semi-simple, is commutative and therefore has no nilpotent elements. It follows that $a \in J(S)$ and so both ra and ar are in J(S) also, in consequence of which both ar and ra are nilpotent. This proves the theorem.

3. RINGS IN WHICH $x \to x^n$ IS A HOMOMORPHISM

We now suppose that R is a ring in which both $(xy)^n = x^n y^n$ and $(x + y)^n = x^n + y^n$. In order to sharpen the previous results and to avoid nil rings in which $a^n = 0$, for all a, we further assume that the mapping is onto. That is, given $x \in R$, $x = y^n$ for some $y \in R$. This is somewhat restrictive.

LEMMA. *If* $a \in R$ *is nilpotent, then it is in the center of* R.

Proof. We use induction over the degree of nilpotence of a.

Suppose that $a^r = 0$. Since $a^2, a^3, \cdots$ are of lower nilpotence degrees, they are in the center of R, by our induction hypothesis. If $x \in R$, let

$$y = (1 + a)x(1 - a + a^2 - a^3 + \cdots) = (1 + a)x(1 + a)^{-1}$$

$$= (x + ax) - (x + ax)a^2 + \cdots.$$

Since $a^2, a^3, \cdots$ are in the center,

$$y = (1 + a)x(1 - a) + (1 + a)(a^2 - a^3 + \ldots)x = (1 + a)x(1 - a) + a^2 x.$$

A simple verification, from the unsimplified form of y, shows that

$$y^n = (1 + a)x^n(1 - a + a^2 + \cdots) = (1 + a)x^n(1 - a) + a^2x^n.$$

On the other hand,

$$y^n = ((1 + a)x(1 - a) + a^2x)^n = (x + ax - xa - axa + a^2x)^n$$

$$= x^n + (ax)^n - (xa)^n - (axa)^n + (a^2x)^n$$

$$= x^n + a^nx^n - x^na^n - a^nx^na^n + a^{2n}x^n.$$

Since $n > 1$, $a^n \in Z$ by induction, so that the right-hand side of this reduces to x^n. Comparing the two expressions for y^n, we see that $ax^n - x^na - ax^na + a^2x^n = 0$. Thus $(1 + a)(ax^n - x^na) = 0$, and since a is nilpotent, $1 + a$ is invertible, whence $ax^n - x^na = 0$. Since every $u \in R$ is some x^n, we conclude that $a \in Z$.

THEOREM 3. *If* R *is a ring in which the mapping* $x \to x^n$ *for a fixed integer* $n > 1$ *is a homomorphism onto, then* R *is commutative.*

Proof. By Theorem 1, all $ab - ba$ are nilpotent, so that by the lemma $ab - ba$ is in the center of R. Now $(ab - ba)a$ is also nilpotent, so it too belongs to the center of R. Thus $(ab - ba)ab = b(ab - ba)a = (ab - ba)ba$, and this leads to $(ab - ba)^2 = 0$. Since $n \geq 2$, $(ab - ba)^n = 0$. However, the mapping $x \to x^n$ is a ring homomorphism by assumption, whence

$$0 = (ab - ba)^n = a^nb^n - b^na^n$$

for all $a, b \in R$. If $x, y \in R$ then, since the mapping $u \to u^n$ is onto, $x = a^n$ and $y = b^n$ for some $a, b \in R$. Thus $xy - yx = a^nb^n - b^na^n = 0$. This proves the theorem.

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