REPRESENTATIONS OF A CLASS OF INFINITE GROUPS

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1. INTRODUCTION

The results of this paper are contained in two theorems.

THEOREM 1. *A free group with a countable number of generators can by faithfully represented by a group of 2-by-2 unimodular matrices with rational integral entries (that is, by a subgroup of the modular group).*

THEOREM 1. *Let* G *be a group with a countable number of generators* t_j $(j = 1, 2, \cdots)$ *and a finite number of generating relations*

$$t_1^{q_1} = t_2^{q_2} = \cdots = t_n^{q_n} = 1 \qquad (q_j = \text{integer} \geq 2)\,. \tag{1}$$

Let $\mathscr{F}_n$ *be the algebraic number field obtained by adjoining the quantities*

$$\lambda_j = 2\cos \pi/q_j \qquad (j = 1, 2, \cdots, n) \tag{2}$$

to the rational field. Then G *can be faithfully represented by a group of* 2-*by*-2 *unimodular matrices whose entries are integers in* $\mathscr{F}_n$.

The proof will show that in each case infinitely many inequivalent representations are possible.

It will also be clear from the construction that Theorem 2 is valid for a group G whose generators fall into two classes A and B; those in A have power relations of type (1), those in B are free. Either A or B or both may be finite or denumerably infinite; either may be empty.

An application of these results to the theory of discontinuous groups of linear transformations of the plane is made in Section 5.

Such representations are useful in the construction of subgroup topologies for groups of the above types. (See [3].) The construction will be carried out in a future publication.

2. THE ISOMETRIC CIRCLE

Our main tool will be the isometric circle of L. R. Ford [2, p. 23 ff]. Let z be a complex variable. Given a linear transformation of the plane $z' = (az + b)/(cz + d)$, with $ad - bc = 1$ and $c \neq 0$, we define the circle

$$I(T):\ |cz + d| = 1$$

and call it the isometric circle of T. If T has an isometric circle (that is, if $c \neq 0$), then so has T^{-1}. The isometric circle I(T) together with its interior will be called the isometric disk of T, and we shall denote it by K(T).

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It is readily verified that T maps $I(T)$ onto $I(T^{-1})$, and that it maps the interior (exterior) of $I(T)$ onto the exterior (interior) of $I(T^{-1})$; [2, p. 25].

3. FREE GROUPS

Let G be a free group with generators $t_1, t_2, \cdots$. Let T_j be the transformation with matrix

$$T_j = \begin{pmatrix} -r_j & -1 + r_j^2 \\ 1 & -r_j \end{pmatrix} \quad (j = 1, 2, \cdots), \tag{3}$$

where r_j is a rational integer and

$$r_{j+1} - r_j \geq 3, \qquad r_1 \geq 2.$$

Write T_{-j} for T_j^{-1}. The isometric circle of T_j is $|z - r\ | = 1$; that of T_j^{-1} is $|z + r_j| = 1$. The isometric disks $K(T_j)$ $(j = \pm 1, \pm 2, \cdots)$ are pairwise disjoint, because of the restrictions on the r_j imposed above.

Let G^* be the group generated by T_j $(j = 1, 2, \cdots)$. We shall show that G^* is a free group.

Let $S_k S_{k-1} \cdots S_1$ be a word in G^*. Each S_i is a T_j. It may happen that $S_i = S_{i+1}$, but we never have $S_{i+1} = S_i^{-1}$.

Let P be a point in the plane lying outside of every isometric disk $K(T_j)$ $(j = 1, 2, \cdots)$. Then $S_1(P)$ lies inside $K(S_1^{-1})$. Since $S_1(P)$ lies outside $K(S_2)$—this is true even if $S_2 = S_1$—we see that $S_2 S_1(P)$ lies inside $K(S_2^{-1})$. Continuing in this way, we conclude that $Q = S_k S_{k-1} \cdots S_1(P)$ lies inside $K(S_k^{-1})$. Since P was outside $K(S_k)$, it follows that $Q \neq P$. Hence $S_k S_{k-1} \cdots S_1 \neq 1$. No nontrivial word of G^* is equal to the identity, and G^* is a free group.

Since T_j has rational integral entries and determinant 1, the same is true of each element of G^*.

The mapping $t_j \leftrightarrow T_j$ $(j = \pm 1, \pm 2, \cdots)$ now establishes an isomorphism between G and G^* and completes the proof of Theorem 1.

Since the integers r_j can be selected in infinitely many ways, there exist infinitely many representations of the above type. Obviously the sets $\{r_j\}$ can be chosen so that these are mutually inequivalent.

4. GROUPS WITH RELATIONS

Let G satisfy the hypotheses of Theorem 2. Define the matrices

$$\begin{aligned} T_j &= \begin{pmatrix} p_j & p_j(\lambda_j - p_j) - 1 \\ 1 & \lambda_j - p_j \end{pmatrix} \quad (j = 1, 2, \cdots, n), \\ T_j &= \begin{pmatrix} -r_j & -1 + r_j^2 \\ 1 & -r_j \end{pmatrix} \quad (j = n + 1, n + 2, \cdots), \end{aligned} \tag{4}$$

where λ_j is defined by (2), p_j and r_j are rational integers, and

$$p_{j+1} - p_j > 2 + \lambda_{j+1},$$

$$r_{j+1} - r_j \geq 3, \qquad r_{n+1} \geq 3 + p_n, \qquad -r_{n+1} \leq p_1 - \lambda_1 - 3.$$

The effect of these inequalities is to make the disks $K(T_j)$ ($|j| > n$) disjoint from each other and from the disks $K(T_j)$ ($|j| < n$). The latter form intersecting pairs $\{K(T_j), K(T_{-j})\}$ ($j = 1, 2, \cdots, n$), but each pair is disjoint from any other pair. (In case $q_j = 2$, $I(T_j)$ and $I(T_{-j})$ coincide.)

If we write T_j ($j \leq n$) in normal form, we get

$$\frac{z' - \xi_j}{z' - \xi_j'} = e^{-2\pi i/q_j} \frac{z - \xi_j}{z - \xi_j'} \qquad (\xi_j, \xi_j' = p_j - e^{\mp \pi i/q_j}).$$

Hence

$$T_j^{q_j} = 1 \qquad (j = 1, 2, \cdots, n). \tag{5}$$

At this point we bring to bear the theory of discontinuous groups. Let G^* be the group generated by all $\{T_j\}$ ($j = 1, 2, \cdots$), and let H_j^* be the group generated by T_j. H_j^* is a discontinuous group, and

$$F_j = \text{complement of } K(T_j) \cup K(T_{-j}) \tag{6}$$

is a fundamental region for H_j^*. (See [2, Thm. 7, p. 45, and p. 53].

Now G^* is the free product of all H_j^*. Since F_j contains the exterior of F_i for each $i \neq j$, we can assert, by Klein's principle of *Ineinanderschiebung*, that G^* is a discontinuous group and that the complement of $\bigcup_{j=-\infty}^{\infty} K(T_j)$ is a fundamental region R for G^*. (See [2, pp. 56-59], where this principle is called "the method of combination;" in particular, see the Theorem on p. 58.)

What are the relations in G^*? According to a classical result of Poincare, the relations determined by small circuits surrounding the vertices of R constitute a set of generating relations in G^*. (See [1, pp. 143-145].) The only vertices in R are the points where $K(T_j)$ and $K(T_{-j})$ intersect ($j = 1, 2, \cdots, n$). These vertices give rise to the relations (5), which are therefore a set of generating relations for G^*. (If $K(T_j)$ and $K(T_{-j})$ coincide, we must consider the highest and lowest points of $I(T_j)$ as vertices, and thus we obtain the relation $T_j^2 = 1$.)

It follows that G and G^* are isomorphic under the mapping $t_j \leftrightarrow T_j$ ($j = 1, 2, \cdots$).

Let $\mathscr{F}_n$ be the field obtained by adjoining $\lambda_1, \lambda_2, \cdots, \lambda_n$ to the rationals. Since λ_n is an integer in $\mathscr{F}_n$, each T_j has entries that are integers in $\mathscr{F}_n$. Hence each matrix of G^* has entries that are integers in $\mathscr{F}_n$. This completes the proof of Theorem 2.

5. DISCONTINUOUS GROUPS

If Γ is a Fuchsian group with signature $(p, m; q_1, q_2, \cdots, q_n)$ $(p + m > 0, m \geq n)$, then the relations in Γ are as follows:

$$E_1^{q_1} = E_2^{q_2} = \cdots = E_n^{q_n} = 1, \tag{7}$$

$$P_1 P_2 \cdots P_{m-n} E_1 E_2 \cdots E_n A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_p B_p A_p^{-1} B_p^{-1} = 1. \tag{8}$$

(See [1, p. 198].) The P_i are parabolic elements of Γ.

Suppose there exists at least one P_i. Then we can solve (8) for P_1. The group Γ is isomorphic to a group Γ' whose generators are those of Γ with P_1 deleted, and whose generating relations are given by (7). Γ' is a group satisfying the hypotheses of Theorem 2. Therefore we have

THEOREM 3. *Every Fuchsian group of signature* $(p, m; q_1, q_2, \cdots, q_n)$ *containing parabolic elements has a faithful representation by a group of* 2-*by*-2 *unimodular matrices with entries in* $\mathscr{F}_n$.

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