Tree-Lattice Zeta Functions and Class Numbers

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ABSTRACT. We extend the theory of Ihara zeta functions to noncompact arithmetic quotients of Bruhat–Tits trees. This new zeta function turns out to be a rational function despite the infinite-dimensional setting. In general, it has zeros and poles in contrast to the compact case. The determinant formulas of Bass and Ihara hold if we define the determinant as the limit of all finite principal minors. From this analysis we derive a prime geodesic theorem, which, applied to special arithmetic groups, yields new asymptotic assertions on class numbers of orders in global fields.

Introduction

The Ihara zeta function, introduced by Yasutaka Ihara in [19; 20] is a zeta function counting prime elements in discrete subgroups of rank one p-adic groups. It can be interpreted as a geometric zeta function for the corresponding finite graph, which is a quotient of the Bruhat–Tits building attached to the p-adic group [26]. Over time it has been generalized in stages by Sunada, Hashimoto, and Bass [29; 30; 18; 14; 15; 16; 17; 1; 21]. Comparisons with number-theoretical zeta functions can be found in the papers of Stark and Terras [27; 28; 31]. This zeta function is defined as the product

$$Z(u) = \prod_p (1 - u^{l(p)})^{-1},$$

where p runs through the set of prime cycles in a finite graph X. The product, being infinite in general, converges to a rational function, in fact, the inverse of a polynomial, and satisfies the famous *Ihara determinant formula*

$$Z(u)^{-1} = \det(1 - uA + u^2 Q)(1 - u^2)^{-\chi},$$

where A is the adjacency operator of the graph, $Q + 1$ is the valency operator, and χ is the Euler number of the graph. One of the most remarkable features of the Ihara formula is that in the case of $X = \Gamma \backslash Y$, where Y is the Bruhat–Tits building of a p-adic group G, and Γ is a cocompact arithmetic subgroup of G, then the right-hand side of the Ihara formula equals the nontrivial part of the Hasse–Weil zeta function of the Shimura curve attached to Γ, thus establishing the only known link between geometric and arithmetic zeta- or L-functions.

In recent years, several authors have asked for a generalization of these zeta functions to infinite graphs. The paper [25] considers the arithmetic situation,
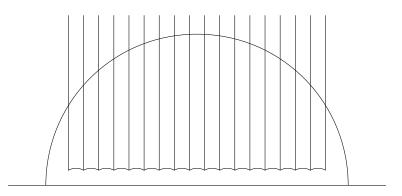
where the graph is the union of a compact part and finitely many cusps. The zeta function is defined by plainly ignoring the cusps, so that, indeed, it is a zeta function of a finite graph. In [5] and [4], the zeta function of a finite graph is generalized to an L^2-zeta function where a finite trace on a group von Neumann algebra is used to define a determinant. In [11], an infinite graph is approximated by finite ones, and the zeta function is defined as a suitable limit. In [12; 13], a relative version of the zeta function is considered on an infinite graph that is acted upon by a group with finite quotient. In [3], the idea of the Ihara zeta function is extended to infinite graphs by counting not all cycles, but only those passing through a given point. In [22], the zeta function is extended to an infinite graph acted upon by a groupoid and equipped with an invariant measure.

In the present paper, we take a different approach by considering cycles that come from geodesics in the universal covering. This idea goes back to Bass [1], who incorporated torsion in this way. Surprisingly, this yields a convergent Euler product, which extends to a rational function. This works for graphs of "Lie type", that is, quotients of Bruhat–Tits buildings by lattices in rank one p-adic groups. These graphs are "cuspidal" in the sense that they consist of a compact part and a finite number of cusps. To derive determinant expressions of the zeta function, we introduce the notion of operators of "determinant class", which means that the net of all finite principal minors converges, the limit being called the determinant of the operator. It turns out that with this notion, the classical determinant formulas of Bass and Ihara actually hold without change. Also, the analysis of the resulting rational function is precise enough to deduce a version of the prime geodesic theorem in this context. In the case of the Bruhat–Tits tree of GL_2, the prime geodesic theorem can be applied to obtain the following asymptotic result on class numbers:
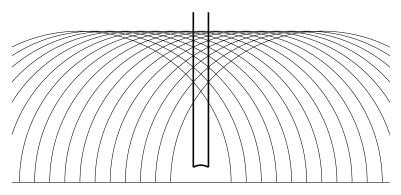
$$\sum_{\Lambda:R(\Lambda)=m} h(\Lambda) = \Delta \mathbf{1}_{\Delta \mathbb{Z}}(m) q^m + O((q-\varepsilon)^m).$$

Here the sum ranges over all quadratic orders over the coordinate ring of an affine curve over a finite field of q elements, $h(\Lambda)$ is the class number of the order Λ, and $R(\Lambda)$ is the regulator. This result is the function-field analogue of [24]; see also [8].

Let us now introduce the geometric idea behind this approach. The classical predecessor of Ihara's zeta function is the zeta function of Selberg, which counts closed geodesics in compact Riemann surfaces. The latter generalizes to noncompact surfaces as long as they have finite hyperbolic volume. In this case, such a surface is a union of a compact set and finitely many cusp sectors, and the typical behavior of a closed geodesic is that it winds around a cusp, going out for a while and then winds back to the compact core. We further have drawn this in the case of the quotient of the upper half-plane by $SL_2(\mathbb{Z})$. The first picture shows how a geodesic in the universal covering runs through translates of the standard fundamental domain.

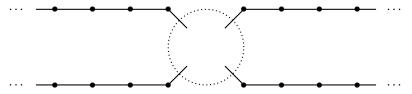Instead, one can also leave the fundamental domain fixed and replace the geodesic by the union of its translates, as in the second picture.

Finally, to understand the behavior of the geodesic in the quotient, we only look at what happens in the fundamental domain, where we now nicely see how, after identifying the left and right boundaries of the domain, the geodesic winds up and down again.

The analogue of the upper half-plane in the p-adic setting is the Bruhat–Tits tree Y of a rank one group, together with a lattice Γ acting on the tree. In this setting, a cusp sector in the quotient $X = \Gamma \backslash Y$ is an infinite ray emanating from a compact core. The following picture shows an example of a graph with four cusps; the compact core is not drawn.

Now winding up and down along a cusp means that we consider cycles that move out on the ray, then return once, and go back to the compact core. This corresponds nicely to what happens to a "geodesic" in the tree when projected down to the quotient graph. Note that noncompact quotients with cusps can only occur when the group Γ has torsion, so Γ is not the fundamental group of the quotient X, and that is why we cannot consider the quotient alone but have to take the action of Γ on Y into the picture. This is the idea behind Hyman Bass's approach to the zeta function [1].

In the case of the Selberg zeta function, in certain arithmetic cases, it is possible to derive class number growth assertions [24; 6; 8; 9]. This can, in principle, also be done for Ihara zeta functions [7]. In this paper, we apply this technique also in the case of an infinite graph, thus obtaining the above-mentioned growth assertions for class numbers of orders in global fields as follows: Let $\mathcal{C}$ be a smooth projective curve with field of constants k of q elements, let ∞ be a closed point of $\mathcal{C}$, and let A be the coordinate ring of the affine curve $\mathcal{C} \smallsetminus \{\infty\}$. Then there exist $\Delta \in \mathbb{N}$ and $\varepsilon > 0$ such that, as $m \to \infty$, we have

$$\sum_{\Lambda : R(\Lambda) = m} h(\Lambda) = \Delta \mathbf{1}_{\Delta \mathbb{Z}}(m) q^m + O((q - \varepsilon)^m),$$

where the sum runs over all quadratic A-orders Λ, and $h(\Lambda)$ is the class number of Λ.

1. Cuspidal Tree Lattices

A *tree lattice* is a group Γ together with an action on a locally finite tree Y such that all stabilizer groups Γ_e of edges e are finite and such that

$$\sum_{e \ \mathrm{mod}\ \Gamma} \frac{1}{|\Gamma_e|} < \infty.$$

As an additional condition, we always assume that the tree Y is *uniform*, that is, the quotient graph $G \backslash Y$ is finite, where $G = \mathrm{Aut}(Y)$ is the automorphism group of the tree Y. The compact-open topology makes G a totally disconnected locally compact group. The action of Γ on Y defines a group homomorphism $\alpha : \Gamma \to G$, and Γ is a tree lattice if and only if α has finite kernel and the image $\alpha(\Gamma)$ is a

lattice in the group G, that is, $\alpha(\Gamma)$ is a discrete subgroup such that on G/Γ there exists a G-invariant Radon measure of finite positive volume.

By replacing Γ with a subgroup of index two if necessary, we can assume that Γ acts orientation preservingly on Y. We will always assume this, as it simplifies the presentation. In this way, oriented edges of the quotient graph $X = \Gamma \backslash Y$ will be Γ-orbits of oriented edges on Y.

A tree lattice of *Lie type* is a lattice in a semisimple p-adic group H of rank one, acting on the Bruhat–Tits building Y of H [2].

Let Y be a uniform tree. A *path* in Y is a sequence $p = (e_1, e_2, \ldots, e_n)$ of oriented edges such that the end point $t(e_j)$ of e_j is the starting point $o(e_{j+1})$ of e_{j+1} for each $j \in \{1, \ldots, n-1\}$. We say that the path is *reduced* if $e_{j+1} \neq e_j^{-1}$ for every $1 \leq j \leq n-1$, where e^{-1} denotes the reverse of the oriented edge e. A *ray* in Y is an infinite reduced path $r = (r_1, r_2, \ldots)$. Two rays r, s are *equivalent* if they join at some point, that is, if there exist $N \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that $r_{j+k} = s_j$ for all $j \geq N$. An equivalence class of rays is called an *end* of Y. The set ∂Y of all ends is called the *boundary* or *visibility boundary* of Y. For a given vertex x_0 and any point $c \in \partial Y$, there exists a unique ray in c that starts at the point x_0. So the visibility boundary is what you see when you look around from any point in the tree.

Let $c \in \partial Y$ be a boundary point. Two vertices x, y of Y lie in the same *horocycle* with respect to c if there are rays $r, s \in c$ such that $r_0 = x$, $s_0 = y$ and $n \in \mathbb{N}_0$ such that $r_n = s_n$. So a horocycle is the set of all vertices that have the "same distance" to the boundary point c. Let P_c be the stabilizer of the boundary point c in the automorphism group $G = \mathrm{Aut}(Y)$. Let N_c be the subgroup of P_c of all elements that stabilize a horocycle (and hence all horocycles) with respect to c.

The point c is called a *cusp* of the tree lattice Γ if $\Gamma \cap P_c = \Gamma \cap N_c$ and there exists a horocycle H on which $\Gamma \cap N_c$ acts transitively. It then acts transitively on every horocycle which is nearer to c than H. In this case, any ray in c, or rather its image in $X = \Gamma \backslash Y$, is called a *cusp section* of Γ. Let c be a ray in Y giving a cusp section in Γ. Recall that the *valency* of a vertex x is the number of edges ending at x. The valency of the vertices in the ray c can generally behave irregularly. We say that c, and so the cusp, is *periodic* if the sequence of valencies is eventually periodic, that is, if the vertices of c are $(x_1, x_2, \ldots)$, then c is periodic if there exist $N \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $\mathrm{val}(x_n) = \mathrm{val}(x_{n+k})$ for all $n \geq N$, where $\mathrm{val}(x)$ is the valency of the vertex x. The smallest possible such k is called the *period* of the cusp. If Y is the Bruhat–Tits tree of a semisimple p-adic group, then every cusp is periodic of period one or two [23].

We say that the tree lattice Γ is *cuspidal* if $X = \Gamma \backslash Y$ is the union of a finite graph and finitely many periodic cusp sections. Any tree lattice of Lie type is cuspidal.

2. The Bass–Ihara Zeta Function

Let Γ be a cuspidal tree lattice of the uniform tree Y, and let $X = \Gamma \backslash Y$ be the quotient graph. A path $p = (e_1, \ldots, e_n)$ in X is called *closed* if $t(e_n) = o(e_1)$.

Then the shifted path $\tau p = (e_2, \ldots, e_n, e_1)$ is closed again, and the shift induces an equivalence relation on the set of closed paths, whereby two closed paths are equivalent if one is obtained from the other by finitely many shifts. A *cycle* is an equivalence class of closed paths. If c is a cycle, then any power c^n, $n \in \mathbb{N}$, obtained by running the same path for n times is again a cycle, and a given cycle c_0 is called a *prime cycle* if it is not a power of a shorter one. For every given cycle, there exists a unique prime cycle c_0 such that $c = c_0^m$ for some $m \in \mathbb{N}$. The number $m = m(c)$ is called the *multiplicity* of c. A path $p = (e_1, \ldots, e_n)$ is called *reduced* if $e_{j+1} \neq e_j^{-1}$ for every $1 \leq j \leq n - 1$. A cycle is called reduced if it consists of reduced paths only.

For an edge e of Y, we set

$$w(e) = |\Gamma_{o(e)}e|,$$

which is the cardinality of the $\Gamma_{o(e)}$-orbit of e. For an edge e of $X = \Gamma \backslash Y$, we write $w(e) = w(\tilde{e})$, where $\tilde{e}$ is any preimage of e in Y. For two consecutive edges e and e', that is, if $t(e) = o(e')$, we write

$$w(e, e') = \begin{cases} w(e'), & e' \neq e^{-1}, \\ w(e') - 1, & e' = e^{-1}. \end{cases}$$

For a closed path $p = (e_1, \ldots, e_n)$, let

$$w(p) = \prod_{j \mod n} w(e_j, e_{j+1}).$$

DEFINITION 2.1. The Bass–Ihara zeta function for the tree lattice Γ is defined to be

$$Z(u) = \prod_c (1 - w(c)u^{l(c)})^{-1},$$

where the product runs over all prime cycles, and $l(c)$ is the length of the cycle c. In [1], it is shown that this in general infinite product converges for $|u|$ small to a rational function if X is compact. We will now show the same in the cuspidal case.

Note the special case of all trivial stabilizer groups. In this case, we get

$$Z(u) = \prod_c (1 - u^{l(c)})^{-1},$$

where now the product runs over *reduced* prime cycles only.

THEOREM 2.2. *Suppose that the tree lattice Γ is cuspidal. Then, for small $|u|$, the product $Z(u)$ converges to a rational function.*

The proof requires a new form of determinant for operators on infinite-dimensional spaces, which is given in the next section.

3. Determinant Class Operators

For a set I, consider the formal complex vector spaces

$$P = P(I) = \prod_{i \in I} \mathbb{C}i, \qquad S = S(I) = \bigoplus_{i \in I} \mathbb{C}i.$$

We denote elements of these spaces as formal sums $\sum_{i \in I} a_i i$, where in the second case, the sums are finite in the sense that $a_i = 0$ outside a finite set. For a linear operator $A : S \to P$ and any finite subset $F \subset I$, we define the operator A_F as the composition

$$S(F) \hookrightarrow S(I) \xrightarrow{A} P(I) \twoheadrightarrow P(F),$$

where the first arrow is the natural inclusion, and the last is the natural projection.

The system of finite subsets F of I is a directed set when equipped with the partial order by inclusion. Therefore the map $F \mapsto \det(A_F)$ is a net with values in $\mathbb{C}$.

DEFINITION 3.1. We say that A is of *determinant class* if the limit of the net of all principal minors,

$$\det(A) \overset{\text{def}}{=} \lim_F \det(A_F),$$

exists and is not zero.

Define a pairing $\langle \cdot, \cdot \rangle : P \times S \to \mathbb{C}$ by

$$\left\langle \sum_{i \in I} a_i i, \sum_{i \in I} b_i i \right\rangle = \sum_{i \in I} a_i b_i.$$

A permutation $\sigma : I \to I$ is called *finite* if $\sigma(i) = i$ outside a finite set F. In this case, we write $\text{sgn}(\sigma)$ for the sign of the permutation $\sigma|_F$. It does not depend on the choice of F.

LEMMA 3.2 (Computation of the determinant). *Let $A : S \to P$ be a linear operator such that*

- *there is a finite set $F_0 \subset I$ with $\langle Ai, i \rangle \neq 0$ outside F_0 and $\sum_{i \in I \smallsetminus F_0} |\log\langle Ai, i \rangle| < \infty$, and*
- *$\sum_{\sigma \text{ finite}} |\prod_{i \in I} \langle Ai, \sigma i \rangle| < \infty$.*

Then A is of determinant class, and we have

$$\det(A) = \sum_{\sigma \text{ finite}} \text{sgn}(\sigma) \prod_{i \in I} \langle Ai, \sigma i \rangle.$$

An operator satisfying the condition of this lemma is said to be of strong determinant class.

Proof. Suppose that A satisfies the conditions of the lemma. Then

$$\sum_\sigma \text{sgn}(\sigma) \prod_{i \in I} \langle Ai, \sigma i \rangle = \lim_F \underbrace{\left(\sum_{\sigma \in \text{Per}(F)} \text{sgn}(\sigma) \prod_{i \in F} \langle Ai, \sigma i \rangle \right)}_{= \det(A_F)} \underbrace{\left(\prod_{i \in I \smallsetminus F} \langle Ai, i \rangle \right)}_{\to 1}.$$

The second factor tends to one, so we get the claim. □

EXAMPLES 3.3. • If I is finite, then every operator is of determinant class, and this formula gives the usual determinant.

• If the set I is an orthonormal basis of a Hilbert space H on which $T : H \to H$ is a linear operator, then T gives rise to an operator in the algebraic sense, again written T and if T is of trace class, then $1 - T$ is of determinant class, and the determinant coincides with the *Fredholm determinant*

$$\det(1 - T) = \sum_{k=0}^{\infty} (-1)^k \, \text{tr} \bigwedge^k T.$$

• The operators we consider here cannot generally be composed. There are, however, two classes of operators that allow composition. Firstly, an operator A as before is called a *finite column operator* if it maps $S(I)$ to $S(I) \subset P(I)$. The name comes from the fact that the corresponding matrix indeed has finite columns, that is, for every $i \in I$, the set $\{j \in I : \langle Ai, j \rangle \neq 0\}$ is finite.

  If, on the other hand, for each $j \in I$, the set $\{i \in I : \langle Ai, j \rangle \neq 0\}$ is finite, then we call A a *finite row operator*. A finite row operator possesses a canonical extension to a linear operator $P \to P$.

• Let $I = \mathbb{N}$. Then any operator is given by an infinite matrix. If this matrix is upper triangular with ones on the diagonal, then the operator is of determinant class. The same holds for lower triangular matrices. The product of a lower triangular matrix and an upper triangular matrix, however, not always is of determinant class. This means that the determinant class is not closed under multiplication.

We say that an operator $T : S \to P$ is *traceable* if $\sum_{i \in I} |\langle Ti, i \rangle| < \infty$. If T is traceable, then we define its *trace* by

$$\text{Tr}(T) = \sum_{i \in I} \langle Ti, i \rangle.$$

LEMMA 3.4. *Suppose that operators A and B such that A has finite rows or B has finite columns, so that the product AB exists. Assume further that A, B, and AB are of determinant class. Then*

$$\det(AB) = \det(A) \det(B).$$

Proof. Viewed as infinite matrices, we have

$$\langle ABj, i \rangle = (AB)_{i,j} = \sum_k A_{i,k} B_{k,j}.$$

Under either condition, this sum is finite. In the rest of the proof, we assume that A is a finite row operator, the other case being similar. For a finite set $E \subset I$, we write A^E for the operator given by $A^E_{i,k} = \delta_{i,k}$ unless i, k are both in E, in which latter case we have $A^E_{i,j} = A_{i,j}$. Then A^E has finite rows and columns, and $\det(A^E)$ tends to $\det(A)$ as $E \to I$. Now if $F \supset E$, then we get $(A^E B)_F = A^E_F B_F$, and so

$$\det(A^E B) = \lim_F \det(A^E_F B_F) = \det(A^E) \det(B).$$

The right-hand side tends to $\det(A) \det(B)$ as $E \to I$. The fact that A is of finite rows implies that, for every finite set $F \subset I$, there exists a finite set E with $F \subset E \subset I$ such that $(A^E B)_F = (AB)_F$. This implies that $\det(A^E B) \to \det(AB)$ as $E \to I$, and the lemma is proven. $\qquad\square$

3.1. Connectedness

Let $A : S(I) \to P(I)$ be an operator. For $j \in I$, write $Aj = \sum_{i \in I} a_i i$ and let $\tilde{A}(\{j\})$ denote the subset of all $i \in I$ with $a_i \neq 0$. For any subset $M \subset I$, set

$$\tilde{A}(M) = \bigcup_{j \in M} \tilde{A}(\{j\}).$$

A subset $M \subset I$ is called *A-stable* if $\tilde{A}(M) \subset M$. The same applies in the relative situation: If $F \subset I$ is any subset, not necessarily finite, then we say that F is *A-irreducible* if F and $\emptyset$ are the only $\tilde{A}_F$-stable subsets of F. Further, F is called *A-connected* if whenever $F = M \cup N$ with *A*-stable disjoint sets M, N, then $M = \emptyset$ or $N = \emptyset$.

Further, we say that A is *connected* if I is *A*-connected.

LEMMA 3.5. *If I is A-irreducible, then the A-connected finite subsets $F \subset I$ are cofinal in the directed set of all finite subsets of I. So in particular, any net $F \mapsto n(F)$ has a subnet obtained by restricting to A-connected sets F, called the A-connected subnet.*

Proof. We have to show that any given finite subset $F \subset I$ is contained in an *A*-connected finite subset $C \subset I$. So let $j_0 \in F$. The set $\bigcup_{n \in \mathbb{N}_0} \tilde{A}(\{j_0\})$ is *A*-stable in I and therefore coincides with I. So for each $j \in C$, there is a chain $j_0, j_1, \dots, j_k = j$ such that $j_{\nu+1} \in \tilde{A}(\{j_\nu\})$ for every $\nu = 0, \dots, k-1$. The set $K(j) = \{j_0, j_1, \dots, j_k = j\}$ is finite and *A*-connected. Let C be the union of all $F(j)$, where j ranges over F. Then C is *A*-connected and finite and contains F. $\qquad\square$

DEFINITION 3.6. The operator A is of *connected determinant class* if I is *A*-irreducible and $\lim_F \det(A_F)$ exists, where the limit is taken over all *A*-connected

finite subsets $F \subset I$. If this is the case, we still write

$$\det(A) = \lim_F \det(A_F),$$

where the limit is taken over connected sets F only.

EXAMPLE 3.7. Let I be the vertex set of a graph X, and let A be the adjacency operator, that is,

$$Ax = \sum_{x'} x',$$

where the sum runs over all neighbors x' of x. We have that I is A-irreducible if and only if I is A-connected if and only if X is connected. For every subset $F \subset I$, we write X_F for the full subgraph of X with vertex set F. We then get

$$X_F \text{ is connected} \quad \Leftrightarrow \quad F \text{ is } A\text{-connected}.$$

PROPOSITION 3.8. *If I is A-irreducible and A is of determinant class, then A is of connected determinant class. There are operators that are of connected determinant class but not of determinant class.*

Proof. To prove the first statement, we need to show that A-connected finite sets are cofinal in the set of all finite subsets of I. So we need to show that, for each finite subset $F \subset I$, there exists an A-connected finite subset $C \subset I$ with $F \subset C$. For this, let X be the graph with vertex set I, where $i, j \in I$ are connected if $|\langle Ai, j\rangle| + |\langle Aj, i\rangle|$ is nonzero. For $F \subset I$, let X_F be the full subgraph with vertex set F. Then F is A-connected if and only if the graph X_F is connected. So the assertion boils down to the fact that in a connected graph each finite set of vertices is contained in a connected finite subgraph.

 An example of an operator that is of connected determinant class but not of determinant class is given in Section 5. □

4. The Zeta Function as a Determinant

We now apply the theory of determinant class operators to give a proof of Theorem 2.2. So we assume that Γ is a cuspidal tree lattice acting on the tree Y. We write $X = \Gamma \backslash Y$ for the quotient graph. Let $I = \mathrm{OE}(X)$ be the set of all oriented edges of X and define the operator $T : S(I) \to S(I)$ by

$$Te = \sum_{e'} w(e, e')e',$$

where the sum runs over all edges e' with $o(e') = t(e)$. Note that T is exactly the pushdown of the operator $\tilde{T}$ on $S(J)$, where $J = \mathrm{OE}(Y)$, given by

$$\tilde{T}e = \sum_{e'} e',$$

where here the sum runs over all edges $e' \neq e^{-1}$ with $o(e') = t(e)$. We make this a bit more precise. As T is an operator of finite rows and columns, we can as well

consider it as $T : P(I) \to P(I)$. Now $P(I) = P(\mathrm{OE}(X))$ can be identified with

$$P(\mathrm{OE}(Y))^{\Gamma},$$

that is, the Γ-invariants in $P(J)$, where $J = \mathrm{OE}(Y)$. Hence any operator on $P(J)$ that commutes with the Γ-action defines an operator on $P(I)$. In this way the operator T corresponds to the operator $\tilde{T} : P(J) \to P(J)$ given before.

LEMMA 4.1. *For any given $n \in \mathbb{N}$, the operator T^n is traceable, and the trace is*

$$\mathrm{Tr}(T^n) = \sum_{c:l(c)=n} l(c_0) w(c),$$

where the sum runs over all cycles c of length n, and c_0 is the underlying prime cycle of a given cycle c.

Proof. Recall that X is a union of a finite graph X_{fin} and a finite number of cusp sections. The best way of thinking of T is that it sends potentials from an edge to the following edges and therefore an edge $e \in I$ only gives a nonzero contribution to the trace, that is, $\langle T^n e, e \rangle \neq 0$, if e lies on some cycle of length n. Now if e lies on a cusp section and its distance to X_{fin} is greater than n, then it cannot lie on such a cycle, as potentials on a cusp section that move inward, that is, toward X_{fin}, cannot reverse on the cusp section but have to move all the way to X_{fin} before returning. Therefore the sum $\sum_e \langle T^n e, e \rangle$ is actually finite, and so T^n is traceable. The claimed formula is clear. $\qquad\square$

With this lemma, we compute, formally at first,

$$Z(u)^{-1} = \prod_{c_0} (1 - w(c_0) u^{l(c_0)}) = \exp\left(\sum_{c_0} \log(1 - w(c_0) u^{l(c_0)})\right)$$

$$= \exp\left(-\sum_{c_0} \sum_{n=1}^{\infty} \frac{w(c_0)^n u^{nl(c_0)}}{n}\right) = \exp\left(-\sum_{c} \frac{w(c) u^{l(c)}}{l(c)} l(c_0)\right)$$

$$= \exp\left(-\sum_{n=1}^{\infty} \frac{u^n}{n} \sum_{c:l(c)=n} w(c) l(c_0)\right) = \exp\left(-\sum_{n=1}^{\infty} \frac{u^n}{n} \mathrm{Tr}(T^n)\right).$$

We say that a sequence A_n of operators *converges weakly* to an operator $A : S \to P$ if for all $i, j \in I$, the sequence of complex numbers $\langle A_n i, j \rangle$ converges to $\langle A i, j \rangle$.

LEMMA 4.2. *There is $\alpha > 0$ such that, for $u \in \mathbb{C}$ with $|u| < \alpha$, the series $-\sum_{n=1}^{\infty}(u^n/n)T^n$ converges weakly to an operator we call $\log(1 - uT)$. This operator is traceable, and we have*

$$Z(u)^{-1} = \exp(\mathrm{Tr}(\log(1 - uT)))$$

for every $u \in \mathbb{C}$ with $|u| < \alpha$.

Proof. As Y is uniform, there is an upper bound M to the valency of vertices. It follows that $\sum_{e'} w(e, e') \leq M$ for every edge e, where the sum runs over all edges e' with $o(e') = t(e)$. It follows that, for any two $i, j \in I$, we have

$$|\langle T^n i, j \rangle| \leq M^n C_n(i, j),$$

where $C_n(i, j)$ is the number of paths of length n connecting i and j. Let r denote the number of cusp sections in X, and let s be the number of oriented edges in X_{fin}. For counting the number of paths connecting any two given edges, it suffices to replace each cusp section with a vertex and one oriented edge going out and one in. Going out a long stretch on a cusp section then is replaced by iterating the loop. In that way, we see that

$$|\langle T^n i, j \rangle| \leq M^n (s + 2r)^n,$$

from which the convergence assertion follows. The trace assertion is in fact an assertion of changing the order of summation because the trace is itself a sum over I. For $u > 0$, all summands are positive, so there is no problem with this interchange of order; for general u, we use absolute convergence, that is, a Fubini argument, to reach the same conclusion. □

THEOREM 4.3. *For $|u|$ small enough, the operator $1 - uT$ is of strong determinant class, and we have*

$$Z(u)^{-1} = \det(1 - uT).$$

This is a rational function of u.

Proof. We consider one cusp at a time and for simplicity assume that the period is one. Then all vertices along the cups can be assumed to have the same valency, say $q + 1$. The modifications for the general case are easy. Let $(e_0, e_1, e_2, \dots)$ be a ray representing the cusp. Writing $f_j = e^{-1}$, we get

$$Te_j = e_{j+1} + (q - 1)f_j, \quad j \geq 0,$$
$$Tf_j = qf_{j-1}, \quad j \geq 1.$$

We find that the operator $1 - uT$ is represented by the matrix

		e_0	f_0	e_1	f_1	e_2	f_2
A		α					
e_0	β	1					
f_0		a	1	0	$a+b$		
e_1		b		1			
f_1				a	1		$a+b$
e_2				b		1	
f_2						a	1

where $a = -u(q - 1)$ and $b = -u$. Further, the operator A represents what is going on outside the current cusp, α is a finite column vector, and β is a finite row vector. Note that if we choose to go up by one unit in the cusp, meaning that e_j will be replaced by e_{j+1} and f_j by f_{j+1}, then the vectors α and β are disjoint in

the sense that there cannot be a permutation σ that gives a nonzero contribution to the determinant such that $\sigma(e_0) = f_0$. It follows that a finite permutation σ that gives a nonzero contribution to the determinant must satisfy $\sigma(e_0) = e_0$ or $\sigma(e_0) = e_1$. In the first case, it follows that σ is the identity on the whole cusp section. So, if σ is not the identity on the cusp section, then the factor $\langle T e_0, \sigma e_0 \rangle$ gives a factor $b = u(q - 1)$. We find that there are not very many choices for such a σ and that they come with growing powers of u, which for $|u|$ small, force in convergence of the determinant series. Therefore $1 - uT$ is of strong determinant class. We compute

$$
\det(1 - uT) = \lim_F \det(1 - uT_F)
$$

$$
= \lim_F \exp\left(-\sum_n \frac{u^n}{n} \operatorname{tr} T_F^n \right)
$$

$$
= \exp\left(-\lim_F \sum_n \frac{u^n}{n} \operatorname{tr} T_F^n \right),
$$

where we have used the continuity of the exponential function. Next, the limit can be interchanged with the sum for small $|u|$ by using dominated convergence by means of a crude estimate of $\operatorname{tr} T_F^n$ similar to the proof of Lemma 4.2. Finally, we have $\lim_F \operatorname{tr} T_F^n = \operatorname{Tr} T^n$ by the same estimate, so that we end up with the claim. The other cusps are dealt with in the same fashion.

It remains to show that $\det(1 - uT)$ is a rational function in u. For this, we again look at one cusp only, and again we assume it to be of period one. As in the proof of Theorem 4.3, we see that $\det(1 - uT)$ is the determinant of the matrix

			e_0	f_0	e_1	f_1	e_2	f_2
	A			α				
e_0	β		1					
f_0			a	1	0	$a+b$		
e_1			b		1			
f_1					a	1		$a+b$
e_2					b		1	
f_2							a	1

By starting the cusp section one step later, that is, moving out on the cusp, we can assume that the vectors α and β each have at most one nonzero entry. Note that by Lemma 3.2 we are able to compute the determinant of $1 - uT$ using row and column reduction as finite matrices. By first using column reduction on the column f_0 and then applying row reduction on the row e_0, there exist a submatrix A' of A and a number $c \in \mathbb{C}$ such that the determinant above is equal to $\det(A)$ plus $cu^2 \det(A')$ times the determinant of the infinite matrix

$$
\begin{vmatrix}
a & 0 & a+b & & & & \\
b & 1 & & & & & \\
 & a & 1 & a+b & & & \\
 & b & & 1 & & & \\
 & & a & 1 & a+b & & \\
 & & b & & 1 & & \\
 & & & & & \ddots &
\end{vmatrix}
$$

Let us denote by D this determinant. Using Laplace expansion along the first row, we see that D equals a plus $(a+b)$ times the determinant of

$$
\begin{vmatrix}
b & 1 & & & & & \\
a & & a+b & & & & \\
b & 1 & & & & & \\
 & a & 1 & & a+b & & \\
 & b & & 1 & & & \\
 & & a & 1 & & & \\
 & & b & & & \ddots &
\end{vmatrix}
$$

But this is b times D, so we get $D = a + (a+b)bD$, or

$$
D = \frac{a}{1-(a+b)b} = \frac{(1-q)u}{1-qu^2}.
$$

The proof of Theorem 4.3 and therefore of Theorem 2.2 is finished. In the case of higher period than one, the reduction pattern used before to compute D will repeat itself only later, but will still lead to a recursion formula giving rationality. Other cusps are treated in the same way, so the claim follows in full generality. $\square$

5. The Ihara Formula

In [1], it is shown that in the case of a uniform tree lattice Γ acting on a tree Y, the zeta function satisfies

$$
Z(u)^{-1} = \frac{\det(1 - uA + u^2 Q)}{(1 - u^2)^{\chi}},
$$

where $A : S(VY) \to S(VY)$ is the adjacency operator of Y, that is,

$$
Ay = \sum_{y'} y',
$$

where the sum runs over all vertices y' adjacent to $y \in VY$. Further, Q is the valency operator minus one, that is,

$$
Q(y) = (\mathrm{val}(y) - 1)y,
$$

where val(y) is the valency of the vertex y. Finally, χ is the Euler number of the finite graph X. Here we use the identification of $S(VX)$ with the set of Γ-invariants in $S(VY)$.

Proof of Bass's theorem. For the convenience of the reader, we give a short account of Bass's proof. Let C_0 be the complex vector space of all maps from VY to $\mathbb{C}$ and C_1 the complex vector space of all maps from $OE(Y)$ to $\mathbb{C}$. We write the elements of C_0 as formal sums $\sum_{p\in VY} c_p p$ with $c_p \in \mathbb{C}$ and likewise for C_1. Let $\partial_0, \partial_1 : C_1 \to C_0$ be the two boundary operators mapping an edge e to its origin and terminus, respectively. Let $J : OE(Y) \to OE(Y)$ be the flip or orientation change operator. Let $\sigma : C_0 \to C_1$ be defined as

$$\sigma(p) = \sum_{e:\partial_0 e=p} e.$$

All these operators commute with the Γ-action, so they preserve the finite-dimensional subspace $C_0^\Gamma \oplus C_1^\Gamma$ of Γ-invariants. For a given $u \in \mathbb{C}$, we consider the following operators on this finite-dimensional space:

$$L = \begin{pmatrix} 1-u^2 & u\partial_0 - \partial_1 \\ 0 & 1 \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & -u\partial_0 + \partial_1 \\ u\sigma & 1-u^2 \end{pmatrix}.$$

A simple computation shows that

$$LM = \begin{pmatrix} 1 - uA + u^2 Q & 0 \\ u\sigma & 1-u^2 \end{pmatrix}$$

and

$$ML = \begin{pmatrix} 1-u^2 & 0 \\ (1-u^2)\sigma & (1-uT)(1-uJ) \end{pmatrix}.$$

Note that $\dim C_0^\Gamma = |VX|$ and $\dim C_1^\Gamma = |OE(X)| = 2|EX|$. As $\det(ML) = \det(LM)$, we get

$$\det(1 - uA + u^2 Q)(1-u^2)^{2|EX|} = (1-u^2)^{|VX|} \det(1-uT)\det(1-uJ).$$

Now $\det(1 - uJ) = (1-u^2)^{|EX|}$, so we get the claim. $\qquad\square$

THEOREM 5.1. *Let Γ be a cuspidal tree lattice. For $|u|$ small enough, the operator $1 - uA + u^2 Q$ is of connected determinant class, and we have*

$$Z(u)^{-1} = \frac{\det(1 - uA + u^2 Q)}{(1-u^2)^{\chi(X_{\mathrm{fin}})}},$$

where X_{fin} is the finite part of X, that is, it is X minus the cusp sections. If X has at least one cusp, then $1 - uA + u^2 Q$ is not of determinant class, providing the example promised in Proposition 3.8.

Proof. For any finite subset F of VX, let X_F be the full finite subgraph with vertex set F. Assume that X_F is connected and contains X_{fin}. Let Y_F be the preimage of X_F in Y. Then Y_F equals Y minus a disjoint union of horoballs,

so Y_F is a tree, acted upon by Γ with compact quotient X_F. So Bass's theorem applies to Y_F, giving

$$Z_F(u)^{-1} = \det(1 - uT_F)$$
$$= \frac{\det(1 - uA_F + u^2 Q_F)}{(1 - u^2)^{\chi(X_F)}}.$$

If X_F is connected and contains X_{fin}, then $\chi(X_F) = \chi(X_{\text{fin}})$ as cusps do not contribute to the Euler number. Therefore we conclude that, as $\lim_F \det(1 - uT_F)$ exists, the connected limit over $\det(1 - uA_F + u^2 Q_F)$ also exists, proving all but the last assertion of the theorem. It remains to show that $1 - uA_F + u^2 Q_F$ is not of determinant class. For this, let F be large enough that X_F contains X_{fin}. Then each connected component of X_F that does not contain X_{fin} contributes a factor $1 - u^2$ to the rational function $(1 - u^2)^{\chi(X_F)}$. So we see that

$$\frac{\det(1 - uA_F + u^2 Q_F)}{(1 - u^2)^{|\pi_0(X_F)|}}$$

converges as $F \to I$, where $\pi_0(X_F)$ is the set of connected components of X_F. As the denominator alone does not converge, the enumerator does not either. $\square$

6. L-Functions

The Bass–Ihara zeta function can be twisted with a finite-dimensional unitary representation $\omega : \Gamma \to GL(V)$ of the tree lattice Γ. For better distinction, in this section, we denote the oriented edges of Y by $e, e', e_1, e_2, \ldots$ and the oriented edges of X by $f, f', f_1, f_2, \ldots$ For each $e \in OE(Y)$, we denote by V_e a copy of the space V, so between any V_e and any $V_{e'}$, there is a natural identification $V_e \cong V_{e'}$. For each $f \in OE(X)$, we let

$$V_f = \left(\prod_{e \in f} V_e \right)^{\Gamma}$$

denote the space of Γ-invariants in the product of which Γ acts by $(\gamma v)_e = \omega(\gamma)v_{\gamma^{-1}e}$. Recall that f is an edge of $\Gamma\backslash Y$, so f is a Γ-orbit of edges in Y. The space V_f is finite-dimensional and isomorphic with $V_e^{\Gamma_e}$ for any $e \in f$. If f, f' are consecutive edges in X, so that $t(f) = o(f')$, then we define a map $W(f, f') : V_f \to V_{f'}$ by

$$W(f, f')v_f = \sum_{e \in f} \sum_{\substack{e':e \to e' \\ e' \neq e^{-1}}} v_{e'}.$$

For a closed path $p = (f_1, \ldots, f_n)$ in X, we define $W(p) : V_{f_1} \to V_{f_1}$ by

$$W(p) = W(f_n, f_1) \circ \cdots \circ W(f_2, f_3) \circ W(f_1, f_2).$$

Then $W(p)$ does depend on the path p, whereas $\det(1 - u^{l(p)}W(p))$ only depends on the cycle of p. Therefore the product

$$L(\omega, u) = \prod_c \det(1 - u^{l(c)}W(c))^{-1}$$

is well defined as a product over all prime cycles in X. On the space $\bigoplus_{f \in OE(X)} V_f$, we consider the operator

$$T_\omega(v_f) = \sum_{f'} W(f, f')v_f,$$

where the sum runs over all $f' \in OE(X)$ with $o(f') = t(f)$.

Similarly, for a vertex y if Y, we let V_y denote a copy of V, and for x of X, we set

$$V_x = \left(\prod_{y \in x} V_y\right)^\Gamma.$$

On the space $\bigoplus_{x \in VX} V_x$, we consider the adjacency operator

$$A_\omega(v_x) = \sum_{y \in x} \sum_{y'} v_{y'},$$

where the first sum runs over all y in the Γ-orbit x, and the second is extended over all neighbors y' of y in Y. Finally, $v_{y'}$ is the image of v_x under the canonical identification $V_x \cong V \cong V_{y'}$. Further, let

$$Q(v_x) = (\text{val}(y) - 1)v_x,$$

where y is any element of x, and $\text{val}(y)$ is the valency of the vertex y in the tree Y.

THEOREM 6.1. *Let Γ be a cuspidal lattice. For $|u|$ small enough, the operator $1 - uT_\omega$ is of determinant class, and we have*

$$L(\omega, u)^{-1} = \det(1 - uT_\omega).$$

This is a rational function of u. For $|u|$ small enough, the operator $1 - uA + u^2Q$ is of connected determinant class, and we have the Ihara formula

$$L(\omega, u)^{-1} = \frac{\det(1 - uA_\omega + u^2Q)}{(1 - u^2)^{d\chi(X_{\text{fin}})}},$$

where X_{fin} is the finite part of X, that is, it is X minus the cusp sections, and $d = \dim(V_\omega)$.

Proof. Analogously to Lemma 4.1, we see that

$$\text{Tr}(T^n) = \sum_{c:l(c)=n} l(c_0)\,\text{tr}(W(c)),$$

where the sum runs over all cycles c of length n, and $l(c_0)$ is the prime cycle underlying c. The first identity follows as in Theorem 4.3. The argument for rationality is analogous to the proof of Theorem 4.3, and the Ihara formula follows as in Theorem 5.1. □

7. An Arithmetic Example

For background material on this section, see [26]. Let $q = p^k$ be a prime power, and let $\mathbb{F} = \mathbb{F}_q$ be the finite field of q elements. On the function field $\mathbb{F}_q(t)$, we put the discrete valuation corresponding to the "point at infinity":

$$v\left(\frac{a}{b}\right) = \deg(b) - \deg(a). \qquad \text{(polynomial degree)}$$

Let $K = \widehat{\mathbb{F}(t)}$ denote the local field obtained by completing $\mathbb{F}(t)$, and let $\mathcal{O} \subset K$ be the corresponding complete discrete valuation ring,

$$\mathcal{O} = \{x \in K : v(x) \geq 0\}.$$

Then $\pi = 1/t$ is a uniformizer in $\mathcal{O}$. We consider the locally compact group $G = \mathrm{GL}_2(K)$. Its Bruhat–Tits tree Y can be described as follows. The vertices are homothety classes of $\mathcal{O}$-lattices in K^2. Two such lattice classes $[L]$ and $[L']$ are connected by an edge if and only if the representatives may be chosen so that

$$\pi L \subset L' \subset L.$$

The graph described in this way is a tree Y that has constant valency $q + 1$. The natural action of G on $\mathcal{O}$-lattices induces an action of G on the tree Y. The group $\Gamma = \mathrm{GL}_2(\mathbb{F}[t])$ is a discrete subgroup of G. In [26], the quotient $\Gamma \backslash Y$ is described as follows. For each $n \in \mathbb{N}_0$, let

$$L_n = \mathcal{O}e_1 \oplus \pi^n \mathcal{O}e_2,$$

where e_1, e_2 is the standard basis of K^2. Write x_n for the vertex given by the class $[L_n]$. Then $L_0, L_1, \dots$ is a complete set of representatives for $\Gamma \backslash Y$, the only edges being (L_n, L_{n+1}) for $n \geq 0$. Put $\Gamma_0 = \mathrm{GL}_2(\mathbb{F})$ and, for $n \geq 1$,

$$\Gamma_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, d \in \mathbb{F}^\times, b \in \mathbb{F}[t], \deg(b) \leq n \right\}.$$

Then for each $n \geq 0$, the group Γ_n is the stabilizer group of x_n. The group Γ_0 acts transitively on the set of edges with origin x_0, and for $n \geq 1$, the edge (L_n, L_{n+1}) is fixed by Γ_n. Finally, the group Γ_n acts transitively on the set of edges with origin x_n distinct from (x_n, x_{n+1}). For this, see [26], Proposition I.1.3.

So the quotient $X = \Gamma \backslash Y$ is a single ray. As in the proof of Theorem 4.3, we let $a = -(q - 1)u$ and $b = -u$, and we see that $\det(1 - uT)$ equals the determinant

of

$$
\begin{vmatrix}
1 & a+b \\
a & 1 & 0 & a+b \\
b & & 1 \\
& & a & 1 & & a+b \\
& & b & & 1 \\
& & & & a & 1 & & a+b \\
& & & & b & & 1 \\
& & & & & & & \ddots
\end{vmatrix}
$$

Again as in the theorem, we see that

$$
Z(u)^{-1} = \det(1 - uT) = \frac{1 - q^2 u^2}{1 - q u^2}.
$$

8. The Prime Geodesic Theorem

Now suppose $X = \Gamma \backslash Y$, where Γ is a cuspidal tree lattice. If we write

$$
u \frac{Z'}{Z}(u) = \sum_{m=1}^{\infty} N_m u^m,
$$

then from the Euler product we get

$$
N_m = \sum_{c:l(c)=m} w(c) l(c_0),
$$

where the sum runs over all cycles c, and c_0 denotes the primitive cycle underlying c. Note that this implies that N_m is a nonnegative integer for every $m \in \mathbb{N}$. On the other hand, using Theorem 2.2, we get that $Z(u)$ is a rational function such that

$$
Z(u) = \frac{\prod_{j=1}^{r}(1 - a_j u)}{\prod_{k=1}^{t}(1 - b_k u)}.
$$

Then

$$
N_m = \sum_{j=1}^{r} a_j^m - \sum_{k=1}^{t} b_k^m.
$$

When Y is $(q + 1)$-regular tree and X has s cusps, as shown in the proof of Theorem 4.3, we have $b_j = \pm\sqrt{q}$ and

$$
N_m = \sum_{j=1}^{r} a_j^m - 2s q^{m/2} \mathbf{1}_{2\mathbb{Z}}(m),
$$

where $\mathbf{1}_{2\mathbb{Z}}$ denotes the indicator function of $2\mathbb{Z}$, so the negative summand at the end only occurs for even m.

If Γ is cuspidal and $N \in \mathbb{N}$, fix a numeration $e_{j,1}, e_{j,2}, \ldots$ of the outward edges of each cusp, where we have s cusps and $j = 1, \ldots, s$. For a given $N \in \mathbb{N}$, denote by X_N the finite subgraph obtained from X by cutting off all edges after e_N.

For the graph $X = \Gamma \backslash Y$, we define the formal space

$$C_1(X) = \bigoplus_{e \in \mathrm{OE}(X)} \mathbb{C}e,$$

where the direct sum runs over the set $\mathrm{OE}(X)$ of all oriented edges of X. We define the operator $T : C_1(X) \to C_1(X)$ by

$$Te = \sum_{e' : o(e') = t(e)} w(e, e')e'.$$

THEOREM 8.1. *Suppose that Y is $(q+1)$-regular and Γ is cuspidal. We also write T for the matrix of T with respect to the basis $(e)_e$ of $C_1(X)$ consisting of all oriented edges.*

(a) *If X is a finite graph, then the matrix T has positive entries only and is connected in the sense of Section 3.1.*
(b) *We have $\max_{j=1}^r |a_j| = q$.*
(c) *(Prime geodesic theorem) There exists $\Delta \in \mathbb{N}$ such that, as $m \to \infty$, we have*

$$N_m = \Delta \mathbf{1}_{\Delta \mathbb{Z}}(m) q^m + O((q - \varepsilon)^m)$$

for some $\varepsilon > 0$.

Proof. Claim (a) is clear as the graph X is connected. For the other assertions, let us first assume that Γ is uniform, so X is finite. The matrix T has positive entries only. On $C_1(X)$, we define the 1-norm by

$$\|v\|_1 = \sum_e |v(e)|,$$

and we denote by $\| \cdot \|_{\mathrm{op}}$ the corresponding operator norm. We claim that $\|T\|_{\mathrm{op}} \le q$. For this, note that $\|T(e)\| = q$ for every edge e, so that for an arbitrary element $v = \sum_e v_e e$ of $C_1(X)$, we have

$$\|Tv\| \le \sum_e |v_e| \|Te\| = q \sum_e |v_e| = q \|v\|.$$

Next, we show that T has q as an eigenvalue. Since $\|Te\| = q$ for each edge, it follows that every column of the matrix T has sum q, or, in other words, we have $T^t v = qv$, where $v = \sum_e e$. Since the transpose T^t has the same eigenvalues as T, the number q is an eigenvalue. This proves assertion (a), and (b) then follows from the Perron–Frobenius theorem [10, 13.2.2] (note that a connected matrix is called a *nondecomposable* matrix in [10]).

Now we prove (b) and (c) in the case of an infinite graph X. For simplicity, we still assume that X has only one cusp $(e_0, e_1, \ldots)$ and $f_i = e_i^{-1}$. On the other hand, the argument can be easily applied to multicusps. Analogously, we define the formal space $C_1(X_N)$ and on it the operator A_N given by

$$A_N(e_N) = q f_N, \qquad A_N(f_N) = (q-1) f_{N-1} + e_N,$$

and $A_N(e) = T(e)$ otherwise.

LEMMA 8.2. (a) *For all N and* $m \in \mathbb{N}$, *we have*

$$\mathrm{tr}\, A_N^m = \mathrm{tr}\, T^m + \mathbf{1}_{2\mathbb{Z}}(m)q^{m/2}.$$

(b) A_N *has q for an eigenvalue, and every eigenvalue* λ *of A satisfies* $|\lambda| \leq q$.

Proof. We have $\mathrm{tr}\, T^m = \sum_c w(c)l(c_0)$, where the sum runs over all closed cycles of length m in X. Likewise, we have $\mathrm{tr}\, A_N^m = \sum_c w'(c)l(c_0)$, where the sum runs over all closed cycles of length m in X_N. There is an injective map from the set of all cycles in X to the set of all cycles in X_N. Whenever a cycle in X contains e_{N+1}, the stretch outside X_N is replaced by the a number of loops (f_N, e_N) to match up the length. Every cycle in X_N of length m lies in the image of this map, with the exception of any cycle of the form $(f_N, e_N)^{m/2}$ when m is even. Counting the contributions of these cycles gives claim (a).

(b) We claim that $\|A_N\|_{\mathrm{op}} \leq q$, where $\|\cdot\|_{\mathrm{op}}$ is the operator norm with respect to $\|\cdot\|_1$. For this, note that $\|A_N e\|_1 = q$ for ever edge e. Therefore, for given $v = \sum_e v_e e \in C_1(X_N)$, we obtain

$$\|A_N v\|_1 \leq \sum_e v_e \|A_N e\|_1 = q\|v\|_1,$$

which implies $\|A_N\|_{\mathrm{op}} \leq q$, and so $|\lambda| \leq q$ for every eigenvalue of A_N. Let A_N also denote the matrix of A_N with respect to the basis of edges. Then we note that all entries of this matrix are ≥ 0 and the sum over each column is exactly q. The eigenvalues of A_N coincide with the eigenvalues of the transpose matrix A_N^t, and now the sum over each row equals q, or, equivalently,

$$A_N^t \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = q \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

So A_N^t has q for an eigenvalue, and so does A_N. The lemma is proven. $\square$

As the graph X_N is connected, the matrix A_N is connected, so the Perron–Frobenius theorem implies that

$$\mathrm{tr}(A_N^m) = \mathbf{1}_{\Delta\mathbb{Z}}(m)q^m + O((q - \varepsilon)^m)$$

for some $\varepsilon > 0$, and so the theorem follows. $\square$

DEFINITION 8.3. Let Y be a tree. We call an element $\gamma \in \mathrm{Aut}(Y)$ a *hyperbolic* element if its *length*

$$l(\gamma) = \min\{d(\gamma y, y) : y \in |Y|\}$$

is greater than 0, where y runs through the points of a geometric realization of Y.

If γ is hyperbolic, then there exists an infinite line $\dots, v_{-1}, v_0, v_1, \dots$ in Y, called the *axis* of γ and written $\mathrm{ax}(\gamma)$, such that γ is a translation along this line, that is, $\gamma v_j = v_{j+l}$ with $l = l(\gamma)$; see [26].

Recall that an element $\sigma \in \Gamma$ is called *primitive* if the equation $\sigma = \tau^n$ with $n \in \mathbb{N}$ and $\tau \in \Gamma$ implies $n = 1$. Any hyperbolic element $\gamma \in \Gamma$ is a positive power

$\gamma = \gamma_0^m$, $m \in \mathbb{N}$, of a primitive element γ_0, which is uniquely determined up to torsion.

PROPOSITION 8.4. *Let Γ be a cuspidal tree lattice on the uniform tree Y, and let N_m be defined as before. Then, for $m \in \mathbb{N}$, we have*

$$N_m = \sum_{\substack{[\gamma] \subset \Gamma_{\mathrm{hyp}} \\ l(\gamma) = m}} \frac{l(\gamma_0)}{|\mathrm{Cent}(\gamma, \Gamma_{\mathrm{ax}(\gamma)})|},$$

where the sum runs over all conjugacy classes $[\gamma]$ in Γ of hyperbolic elements of length m, $\Gamma_{\mathrm{ax}(\gamma)}$ denotes the pointwise stabilizer in Γ of the axis of γ, and $\mathrm{Cent}(\gamma, \Gamma_{\mathrm{ax}(\gamma)})$ is the centralizer of γ in $\Gamma_{\mathrm{ax}(\gamma)}$.

Proof. We have $N_m = \sum_{c:l(c)=m} w(c)l(c_0)$. Pick a cycle $c = (\overline{e}_0, \ldots, \overline{e}_n = \overline{e}_0)$ of length m. We say that the hyperbolic element $\gamma \in \Gamma$ *closes* c if the axis of γ can be written as $\ldots, e_{-1}, e_0, e_1, \ldots$ such that e_0 maps to $\overline{e}_0$, e_n maps to $\overline{e}_n$, and $\gamma e_0 = e_n$.

We now show that, for a given c, there are $w(c)$ many conjugacy classes of $\gamma \in \Gamma$ closing c.

We first construct one. For this, pick an oriented edge $\overline{e}_0$ in c pointing in the direction of the cycle. Choose a preimage e_0 of $\overline{e}_0$ in Y. Next, let $\overline{e}_1$ be the next edge on c following $\overline{e}_0$ and choose a lift e_1 of $\overline{e}_1$ that follows e_0 and is different from the inverse of e_0. Note that the number of possible choices for e_1 is $w(e_0, e_1)$. Repeat this step until you finished the full cycle c. Above, in Y, you then have a path $(e_0, \ldots, e_n)$, where e_n is another preimage of $\overline{e}_0$. Note that, given e_0, there are $w(c)$ many different choices for e_n. Then there exists $\gamma_0 \in \Gamma$ with $\gamma_0 e_0 = e_n$. As e_0 and $\gamma_0 e_0 = e_n$ are in a line and have the same orientation, γ_0 is hyperbolic, and e_0 and $\gamma_0 e_0$ lie on its axis. So we have found γ_0 that closes c.

For any given $\gamma \in \Gamma$ closing c, fix a preimage $f_0 \in \mathrm{ax}(\gamma)$ of $\overline{e}_0$. There exists $\tau \in \Gamma$ with $\tau f_0 = e_0$. Conjugating by τ, we see that the set of all Γ-conjugacy classes of elements γ closing c is in bijection with the set of Γ_{e_0}-conjugacy classes of $\Gamma_{\mathrm{hyp},c,e_0}$, where the latter is the set of all $\gamma \in \Gamma$ closing c and having e_0 in their axis. We now show that, for $\gamma \in \Gamma_{\mathrm{hyp},c,e_0}$, the stabilizer of γ in Γ_{e_0} equals $\mathrm{Cent}(\gamma, \Gamma_{\mathrm{ax}(\gamma)})$, since if $\tau \in \Gamma_{e_0}$ and $\tau \gamma \tau^{-1} = \gamma$ with $\gamma \in \Gamma_{\mathrm{hyp},c,e_0}$, then it follows that $\tau(\mathrm{ax}(\gamma)) = \mathrm{ax}(\gamma)$, and as τ fixes e_0, it fixes $\mathrm{ax}(\gamma)$ pointwise, and hence $\tau \in \Gamma_{\mathrm{ax}(\gamma)}$. As τ centralizes γ, we conclude that $\tau \in \mathrm{Cent}(\gamma, \Gamma_{\mathrm{ax}(\gamma)})$.

The proposition will finally follow if we show that

$$|\Gamma_{\mathrm{hyp},c,e_0}| = w(c)|\Gamma_{e_0}|.$$

As mentioned before, the number of possible choices for e_n is $w(c)$, and if two elements $\gamma, \gamma' \in \Gamma_{\mathrm{hyp},c,e_0}$ have the same e_n, that is, $\gamma e_0 = \gamma' e_0$, then it follows that $\gamma^{-1}\gamma' \in \Gamma_{e_0}$, which indeed implies $|\Gamma_{\mathrm{hyp},c,e_0}| = w(c)|\Gamma_{e_0}|$. $\qquad\square$

9. Class Numbers

Let $\mathcal{C}$ be a smooth projective curve over the finite field k, and let $K = k(\mathcal{C})$ denote the function field. We assume that k in fact is the constant field of K, which means that k is algebraically closed in K. The closed points of $\mathcal{C}$ correspond to the valuations on K. Fix a closed point ∞ of $\mathcal{C}$ and let $\mathcal{C}^{\text{aff}} = \mathcal{C} \smallsetminus \{\infty\}$. We denote the coordinate ring of this affine curve by A. Then A is a Dedekind domain whose prime ideals correspond to the points of $\mathcal{C}^{\text{aff}}$. Let K_∞ be the completion of K at ∞. Then $\Gamma = \mathrm{GL}_2(A)$ is a lattice in the locally compact group $G = \mathrm{GL}_2(K_\infty)$. We consider the action of Γ on the Bruhat–Tits tree Y of G. Then Γ acts as a cuspidal tree lattice of period one. The number of cusps equals the number $\mathrm{GL}_2(A) \backslash \mathbb{P}^2(K)$, and so this number is the class number $h(A)$ of the Dedekind ring A.

LEMMA 9.1. *For every hyperbolic $\gamma \in \Gamma$, we have*

$$|\mathrm{Cent}(\gamma, \Gamma_{\mathrm{ax}(\gamma)})| = (q-1)^2,$$

where q is the cardinality of k.

Proof. Let $\gamma \in \Gamma$ be hyperbolic. Then Γ is a split semisimple element of G, so its centralizer in G is a maximal torus T, and this maximal torus happens to be split, so $T \cong K_\infty^\times \times K_\infty^\times$. Also, the axis $\mathrm{ax}(\gamma)$ equals the apartment attached to the torus T, whence the group of $t \in T$ fixing $\mathrm{ax}(\gamma)$ pointwise is isomorphic to $k^\times \times k^\times$. $\qquad\square$

We write $\mathrm{M}_2(K)$ for the algebra of K-valued 2×2 matrices. Let $\gamma \in \Gamma$ be a hyperbolic element. Then its centralizer in $\mathrm{GL}_2(K)$ is a nonsplit torus, so its centralizer $L_\gamma = \mathrm{M}_2(K)_\gamma$ in $\mathrm{M}_2(K)$ is a field, a quadratic extension of K. The set $\Lambda_\gamma = L_\gamma \cap \mathrm{M}_2(A)$ is an A-order in L_γ. Its unit group is

$$\Lambda_\gamma^\times = \langle \gamma_0 \rangle \times F_\gamma,$$

where F_γ is a finite group. The number $R(\Lambda_\gamma) = l(\gamma_0)$ is called the *regulator* of order Λ_γ. We get a map $\psi : \gamma_0 \mapsto \Lambda_{\gamma_0}$ from the set of primitive hyperbolic conjugacy classes in Γ to the set of isomorphy classes of A-orders in quadratic extensions of K.

PROPOSITION 9.2. *The map ψ is surjective, and each given A-order Λ in a quadratic extension L/K has $h(\Lambda)(q-1)^2$ preimages.*

Proof. Let Λ be an A-order in a quadratic extension L of K. Let v_1, v_2 be an A-basis of the free A-module Λ. Then v_1, v_2 also is a K-basis of L, and we get an embedding $\sigma : L \hookrightarrow \mathrm{M}_2(K)$ by sending $y \in L$ to the matrix of the map $x \mapsto xy$ in this basis. Then $\Lambda = \sigma^{-1}(\mathrm{M}_2(A))$. Let $\lambda_0 \in \Lambda$ be a generator of the $\Lambda^\times / \Lambda_{\mathrm{tors}}^\times$, and let $\gamma_0 = \sigma(\lambda_0)$. Then $\sigma(L) = L_{\gamma_0}$ and $\sigma(\Lambda) = \Lambda_{\gamma_0}$, so surjectivity is established.

For a given embedding $\sigma : L \hookrightarrow \mathrm{M}_2(K)$, we write Λ_σ for the A-order $\sigma^{-1}(\mathrm{M}_2(A))$. For a given A-order $\Lambda \subset L$, we write $\Sigma(\Lambda)$ for the set of all

$\sigma : L \hookrightarrow M_2(K)$ with $\lambda_\sigma = \Lambda$. The group $\Gamma = GL_2(A)$ acts on $\Sigma(\Lambda)$ by conjugation.

LEMMA 9.3. *The quotient $\Sigma(\Lambda)/\Gamma$ is finite and has cardinality $h(\Lambda)$.*

Proof. The proof is similar to the proof of Lemma 2.4 in [7] or Lemma 2.3 in [6].
□

As γ_0 in a given Λ is only unique up to multiplication by an element of $\mathrm{Cent}(\gamma_0, \Gamma_{\mathrm{ax}(\gamma_0)})$, we get the proposition from Lemma 9.1 and Lemma 9.3. □

We write
$$N_m = N_m^P + N_m^R,$$
where N_m^P is the sum over all primitive conjugacy classes $[\gamma]$ with $l(\gamma) = m$.

LEMMA 9.4. *As $m \to \infty$, we have*
$$N_m^P = N_m + O(q^{m/2}).$$

Proof. The expression $N_m = \sum_{j=1}^r a_j^m - 2sq^{m/2}\mathbf{1}_{2\mathbb{Z}}(m)$, together with $|a_j| \le q$, yields $N_m \le rq^m$. Using this, we estimate
$$N_m^R = \sum_{\substack{d|m \\ d<m}} \sum_{\substack{[\gamma_0] \\ l(\gamma_0)=d}} \frac{l(\gamma_0)}{(q-1)^2} \le \sum_{\substack{d|m \\ d<m}} N_d \le \sum_{\substack{d|m \\ d<m}} rq^d \le \frac{mr}{2}q^{m/2}.$$

The claim follows. □

We finally can put things together now to get the following class number estimate.

THEOREM 9.5 (Class number asymptotics). *Let $\mathcal{C}$ be a smooth projective curve with field of constants k of q elements, fix a closed point ∞ of $\mathcal{C}$, and let A be the coordinate ring of the affine curve $\mathcal{C} \smallsetminus \{\infty\}$. Then there exist $\Delta \in \mathbb{N}$ and $\varepsilon > 0$ such that*
$$\sum_{\Lambda : R(\Lambda) = m} h(\Lambda) = \Delta \mathbf{1}_{\Delta\mathbb{Z}}(m)q^m + O((q-\varepsilon)^m),$$
where the sum runs over all quadratic A-orders Λ, and $h(\Lambda)$ is the class number of Λ.

Proof. The theorem follows from the prime geodesic theorem, Proposition 8.4, and the considerations of this section. □

COROLLARY 9.6. *In the special case of the polynomial ring $A = k[x]$, we get*
$$\sum_{\Lambda : R(\Lambda) = m} h(\Lambda) = 2q^m \mathbf{1}_{2\mathbb{Z}}(m) + O(q^{m/2}),$$
where the sum runs over all quadratic A-orders.

Proof. This follows as the theorem, together with the explicit computation of the zeta function in this case. □

10. Nagao Rays

DEFINITION 10.1. Assume that the quotient graph $X = \Gamma \backslash Y$ is a ray $(x_0, x_1, x_2, \ldots)$, where x_i are vertices of X. Let $e_i = (x_i, x_{i+1})$ be a directed edge, and denote its inverse by f_i. The ray X is called a *Nagao ray* if for $n \in \mathbb{N}$, we have
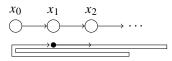
$$w(e_0) = q_0 + 1,$$
$$w(e_n) = 1,$$
$$w(f_0) = q_1,$$
$$w(f_n) = q_{n+1}.$$

Here we assume that all $q_i \in \mathbb{N}$. Note that the valency of the preimage of x_i in X is equal to $q_i + 1$.

DEFINITION 10.2. Let P_n be the collection of all closed paths p of length $2n$ and $w(p) \neq 0$. Note that the closed paths in P_n do not have backtracking of the form (f_i, e_i) for $i \geq 1$, but they may contain backtracking of the form (f_0, e_0), called left backtracking, and the backtracking of the form (e_i, f_i) for some $i \geq 0$, called right backtracking.

The picture below shows a typical closed path in P_n with two instances of right backtracking and two of left backtracking.

Now, given a closed path $p = (p_0, \ldots, p_n = p_0)$ in P_n, where the p_j are oriented edges, let i be the smallest index satisfying $p_i = f_0$, and let j be the second smallest index (or the smallest index if there is only one such index) satisfying $(p_j, p_{j+1}) = (e_k, f_k)$ for some k. We call the number k the right backtrack index. Then we have two injective maps $\rho_1, \rho_2 : P_n \mapsto P_{n+1}$ given by

$$\rho_1(p) = (p_0, \ldots, p_i, e_0, f_0, p_{i+1}, \ldots, p_n)$$

and

$$\rho_2(p) = (p_0, \ldots, p_j, e_{k+1}, f_{k+1}, p_{j+1}, \ldots, p_n).$$

Note that there are unique closed paths $\mathfrak{p}_n$ and $\mathfrak{p}'_n$ in P_n starting from e_n and f_n, respectively.

LEMMA 10.3. *For $n \geq 0$, P_{n+1} is the disjoint union of $\rho_1(P_n)$, $\rho_2(P_n)$, and $\{\mathfrak{p}_{n+1}, \mathfrak{p}'_{n+1}\}$. Especially,*

$$|P_1| = 2 \quad \text{and} \quad |P_{n+1}| = 2|P_n| + 2 \quad \text{for all } n > 0.$$

Proof. This is easy to see. $\square$

THEOREM 10.4. *Let X be a Nagao ray of Lie type. Then it is periodic of period one or two.*

(a) *If X is of period one, then $q_i = q$ for some q and all i. In this case, the Bass–Ihara zeta function is*

$$Z(u) = \frac{(1 - qu^2)}{(1 - q^2u^2)}.$$

Moreover, the prime geodesic theorem becomes

$$N_m = 2q^m \mathbf{1}_{2\mathbb{Z}}(m) + O(q^{m/2}).$$

(b) *If X is of period two, then $q_{2i} = q_0$ and $q_{2i+1} = q_1$ for all $i \geq 0$. In this case, the Bass–Ihara zeta function is*

$$Z(u) = \frac{(1 + q_0u^2)(1 - q_0q_1u^2)}{(1 - q_0q_1u^4)}.$$

Moreover, the prime geodesic theorem is

$$N_m = 2(q_0q_1)^{m/2} \mathbf{1}_{2\mathbb{Z}}(m) + O((q_0q_1)^{m/4} + q_0^{m/2}).$$

Proof. (a) See Section 7.

(b) Now assume that X is of period two, so that $q_i = q_0$ when i is even and $q_i = q_1$ when i is odd. In this case, we split P_n into two parts: A_n and B_n, where A_n and B_n contain closed paths with odd and even right backtrack indexes. Note that when $n = 2k$ is even, we have $\mathfrak{p}_n, \mathfrak{p}'_n \in B_n$ and

$$w(\mathfrak{p}_n) = w(\mathfrak{p}'_n) = q_0^k q_1^k (q_0 - 1).$$

When $n = 2k + 1$ is odd, we have $\mathfrak{p}_n, \mathfrak{p}'_n \in A_n$ and

$$w(\mathfrak{p}_n) = w(\mathfrak{p}'_n) = q_0^{k+1} q_1^k (q_1 - 1).$$

On the other hand, for any $p \in P_n$, we have $\rho_1(p) \in A_{n+1}$, $\rho_2(p) \in B_{n+1}$ if $p \in A_n$, and $\rho_2(p) \in A_{n+1}$ if $p \in B_n$. Denote $\sum_{p \in A_n} w(p)$ by $N_{1,2n}$ and $\sum_{p \in B_n} w(p)$ by $N_{2,2n}$ for short. Then we have $N_{2n} = N_{1,2n} + N_{2,2n}$ for all n.

Moreover, when $n = 2k$,

$$N_{1,2n+2} = \sum_{p' \in P_n} w(\rho_1(p')) + \sum_{p' \in B_n} w(\rho_2(p')) + 2w(\mathfrak{p}_n)$$

$$= q_0(q_1 - 1)N_{2n} + \frac{(q_1 - 1)q_0}{(q_0 - 1)} N_{2,2n} + 2q_0^{k+1}q_1^k(q_1 - 1)$$

and

$$N_{2,2n+2} = \sum_{p' \in A_n} w(\rho_2(p')) = \frac{(q_0 - 1)q_1}{(q_1 - 1)} N_{1,2n};$$

when $n = 2k - 1$,

$$N_{1,2n+2} = \sum_{p' \in P_n} w(\rho_1(p')) + \sum_{p' \in B_n} w(\rho_2(p'))$$

$$= q_0(q_1 - 1)N_{2n} + \frac{(q_1 - 1)q_0}{(q_0 - 1)} N_{2,2n}$$

and

$$N_{2,2n+2} = \sum_{p' \in A_n} w(\rho_2(p')) + 2w(\mathfrak{p}_n)$$

$$= \frac{(q_0 - 1)q_1}{(q_1 - 1)} N_{1,2n} + 2q_0^k q_1^k (q_0 - 1).$$

Combining these formulas, we can find a recursive formula for N_n as follows.

For $n = 2k$, we have

$$N_{2n+2} = N_{1,2n+2} + N_{2,2n+2}$$

$$= q_0(q_1 - 1)N_{2n} + \frac{(q_1 - 1)q_0}{(q_0 - 1)} N_{2,2n} + 2q_0^{k+1} q_1^k (q_1 - 1)$$

$$+ \frac{(q_0 - 1)q_1}{(q_1 - 1)} N_{1,2n}$$

$$= q_0(q_1 - 1)N_{2n} + q_0 q_1 N_{1,2n-2} + 4q_0^{k+1} q_1^k (q_1 - 1)$$

$$+ q_0 q_1 N_{1,2n-2} + q_0 q_1 (q_0 - 1) N_{2n-2}$$

$$= q_0(q_1 - 1)N_{2n} + q_0^2 q_1 N_{2n-2} + 4q_0^{k+1} q_1^k (q_1 - 1).$$

Similarly, for $n = 2k - 1$, we have

$$N_{2n+2} = q_0(q_1 - 1)N_{2n} + q_0^2 q_1 N_{2n-2} + 4q_0^k q_1^k (q_0 - 1).$$

Together with the initial conditions $N_0 = 0$ and $N_2 = 2q_0(q_1 - 1)$, we can show by induction that

$$N_{4k-2} = 2q_0^{2k-1} q_1^{2k-1} - 2q_0^{2k-1} \quad \text{and} \quad N_{4k} = 2q_0^{2k} q_1^{2k} + 2q_0^{2k} - 4q_0^k q_1^k.$$

We conclude that the Bass–Ihara zeta function in this case equals

$$Z(u) = \exp\left(\sum_{n=1}^{\infty} \frac{N_n}{n} u^n \right) = \frac{(1 + q_0 u^2)(1 - q_0 q_1 u^2)}{(1 - q_0 q_1 u^4)}. \qquad \square$$

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