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Almost Gorenstein Rees Algebras of p_g-Ideals, Good Ideals, and Powers of the Maximal Ideals

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Abstract. Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring, and let I be an ideal of A. We prove that the Rees algebra $\mathcal{R}(I)$ is an almost Gorenstein ring in the following cases:

(1) $(A, \mathfrak{m})$ is a two-dimensional excellent Gorenstein normal domain over an algebraically closed field $K \cong A/\mathfrak{m}$, and I is a p_g-ideal;

(2) $(A, \mathfrak{m})$ is a two-dimensional almost Gorenstein local ring having minimal multiplicity, and $I = \mathfrak{m}^\ell$ for all $\ell \geq 1$;

(3) $(A, \mathfrak{m})$ is a regular local ring of dimension $d \geq 2$, and $I = \mathfrak{m}^{d-1}$. Conversely, if $\mathcal{R}(\mathfrak{m}^\ell)$ is an almost Gorenstein graded ring for some $\ell \geq 2$ and $d \geq 3$, then $\ell = d - 1$.

1. Introduction

In [6], the authors proved that for any $\mathfrak{m}$-primary integrally closed ideal I in a two-dimensional regular local ring $(A, \mathfrak{m})$, its Rees algebra $\mathcal{R}(I)$ is an almost Gorenstein graded ring. As a direct consequence, we have that the Rees algebra $\mathcal{R}(\mathfrak{m}^\ell)$ is an almost Gorenstein graded ring for every integer $\ell \geq 1$. The main purpose of this paper is to extend these results to other classes of rings and ideals.

The notion of almost Gorenstein rings in our sense was introduced by Barucci and Fröberg [1], where they dealt with one-dimensional analytically unramified local rings. Goto, Matsuoka, and Phuong [4] extended the notion to arbitrary (but still of dimension one) Cohen–Macaulay local rings. Goto, Takahashi, and Taniguchi [8] gave the definition of almost Gorenstein graded/local rings for higher-dimensional cases.

Let us recall the definition of almost Gorenstein rings.

Definition 1.1 (Goto et al. [8, Def. 3.3]). Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring that possesses the canonical module K_A. Then A is said to be an *almost Gorenstein local ring* if there exists an exact sequence

$$0 \to A \to \mathrm{K}_A \to C \to 0$$

of A-modules such that $\mu_A(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$, where $\mu_A(C)$ denotes the number of elements in a minimal system of generators of C, and $\mathrm{e}^0_{\mathfrak{m}}(C)$ is the multiplicity of C with respect to $\mathfrak{m}$. Note that such an A-module C is called an *Ulrich* A-module; see e.g. [8, Sec. 2].

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Let $R = \bigoplus_{n \geq 0} R_n$ be a Cohen–Macaulay graded ring such that $R_0 = A$ is a local ring. Suppose that R possesses the graded canonical module K_R. Then $\mathrm{a}(R) = -\min\{n \in \mathbb{Z} \mid [\mathrm{K}_R]_n \neq 0\}$ is called the a-*invariant* of R; see e.g. [9, Def. 3.14].

Definition 1.2 (Goto et al. [8, Def. 8.1]). Let R be as before. Let $\mathfrak{M}$ be the unique graded maximal ideal of R and set $a = \mathrm{a}(R)$. Then R is said to be an *almost Gorenstein graded ring* if there exists an exact sequence

$$0 \to R \to \mathrm{K}_R(-a) \to C \to 0$$

of graded R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{M}}(C)$, where $\mu_R(C)$ denotes the number of elements in a minimal system of generators of C, and $\mathrm{e}^0_{\mathfrak{M}}(C)$ is the multiplicity of C with respect to $\mathfrak{M}$. Here $\mathrm{K}_R(-a)$ denotes the graded R-module whose underlying R-module is the same as that of K_R whose grading is given by $[\mathrm{K}_R(-a)]_n = [\mathrm{K}_R]_{n-a}$ for all $n \in \mathbb{Z}$.

Note that the local ring $R_{\mathfrak{M}}$ is almost Gorenstein if R is an almost Gorenstein graded ring because $C_{\mathfrak{M}}$ is an Ulrich $R_{\mathfrak{M}}$-module and $\mathrm{K}_{R_{\mathfrak{M}}} \cong [\mathrm{K}_R]_{\mathfrak{M}}$. Unfortunately, the converse is not true in general (see e.g. [6, Thms. 2.7 and 2.8] and [8, Ex. 8.8]).

Any Gorenstein local ring is an almost Gorenstein local ring. Any rational singularity in dimension two is an almost Gorenstein local ring (see [8, Sec. 11]). All known examples of Cohen–Macaulay local rings of finite representation type are almost Gorenstein local rings (see [8, Sec. 12]). Moreover, a numerical semigroup ring $k[[H]]$ is an almost Gorenstein ring if and only if H is an almost symmetric semigroup [1]. Note that the notion of almost Gorenstein rings in our sense is different from that in [11].

Moreover, the following results are known as examples of higher-dimensional almost Gorenstein rings. For a parameter ideal Q in a regular local ring A of dimension $d \geq 3$:

(1) the Rees algebra $\mathcal{R}(Q) = \bigoplus_{n \geq 0} Q^n$ is an almost Gorenstein graded ring if and only if $Q = \mathfrak{m}$ (see [8, Thm. 8.3]);
(2) $\mathcal{R}(Q)_{\mathfrak{M}}$ is always an almost Gorenstein local ring, where $\mathfrak{M}$ denotes the unique graded maximal ideal of $\mathcal{R}(Q)$ [5].

The main results in this paper are the following theorems, which are extensions of the main result in [6]. Note that any $\mathfrak{m}$-primary integrally closed ideal I in a two-dimensional excellent regular local ring A over an algebraically closed field satisfies the assumption in the following theorem. Thus, this theorem essentially extends [6, Thm. 1.3].

Now let $(A, \mathfrak{m})$ be a two-dimensional excellent normal local ring over an algebraically closed field. For an $\mathfrak{m}$-primary ideal $I \subset A$, I is called a p_g*-ideal* (see [14]) if the Rees algebra $\mathcal{R}(I)$ is a Cohen–Macaulay normal domain; see the following section for the definition and basic properties of p_g-ideals.

Theorem 1.3 (See Theorem 2.4). *Let $(A, \mathfrak{m})$ be a two-dimensional excellent Gorenstein normal local ring over an algebraically closed field, and let $I \subset A$*

be a p_g-ideal. Then the Rees algebra $\mathcal{R}(I)$ of I is an almost Gorenstein graded ring.

Since any two-dimensional rational singularity is an almost Gorenstein local ring having minimal multiplicity, the following theorem provides a large class of local rings for which the Rees algebras of all powers of the maximal ideal are almost Gorenstein graded rings.

THEOREM 1.4 (See Theorem 3.5). *Let $(A, \mathfrak{m})$ be a two-dimensional almost Gorenstein local ring having minimal multiplicity. Then $\mathcal{R}(\mathfrak{m}^\ell)$ is an almost Gorenstein graded ring for every $\ell \geq 1$.*

COROLLARY 1.5. *Let $(A, \mathfrak{m})$ be a two-dimensional rational singularity. Then $\mathcal{R}(\mathfrak{m}^\ell)$ is an almost Gorenstein graded ring for every $\ell \geq 1$.*

The following theorem is a higher-dimensional analog of [6, Cor. 1.4]. We note that if $d = 5$ and $\ell = 2$, then $\mathcal{R}(\mathfrak{m}^2)_{\mathfrak{M}}$ is an almost Gorenstein local ring, but $\mathcal{R}(\mathfrak{m}^2)$ is not an almost Gorenstein graded ring.

THEOREM 1.6 (See Proposition 4.2 and Theorem 4.4). *Let $(A, \mathfrak{m})$ be a regular local ring of dimension $d \geq 2$ that possesses an infinite residue class field. Then*:

(1) *$\mathcal{R}(\mathfrak{m}^\ell)$ is an almost Gorenstein graded ring if and only if $\ell = 1$, $d = 2$ or $\ell = d - 1$;*
(2) *for $\ell \geq 2$ and $d \geq 3$, $\mathcal{R}(\mathfrak{m}^\ell)_{\mathfrak{M}}$ is an almost Gorenstein local ring if and only if $\ell \mid d - 1$, where $\mathfrak{M}$ denotes the graded maximal ideal of $\mathcal{R}(\mathfrak{m}^\ell)$.*

Note that under the assumption of Theorem 1.6, the associated graded ring $G(\mathfrak{m}^\ell)$ is Gorenstein if and only if $\ell \mid d - 1$; see [12, Thm. 2.4].

We now briefly explain how this paper is organized. The proof of Theorem 1.3 is given in Section 2. In Section 3, we prove Theorem 1.4. In Section 4, we prove Theorem 1.6.

In what follows, unless otherwise specified, $(A, \mathfrak{m})$ is a Cohen–Macaulay local ring. Let K_A denote the canonical module of A. For each finitely generated A-module M, let $\mu_R(M)$ (respectively $\mathrm{e}^0_{\mathfrak{m}}(M)$) denote the number of elements in a minimal system of generators for M (respectively, the multiplicity of M with respect to $\mathfrak{m}$).

2. Rees Algebras of p_g-Ideals (Proof of Theorem 1.3)

The purpose of this section is to prove Theorem 1.3. Throughout this section, let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension two and suppose that A is generically a Gorenstein ring and that it possesses a canonical ideal $\mathrm{K}_A \subset A$. Moreover, let $I \subset A$ be an $\mathfrak{m}$-primary ideal, and let Q be its minimal reduction of I. Suppose that I is stable, that is, $I^2 = QI$ and $I \neq Q$, and set $J = Q : I$. Then the Rees algebra

$$\mathcal{R} = \mathcal{R}(I) = A[It] \subseteq A[t]$$

of I is a Cohen–Macaulay ring [7] with $\mathrm{a}(\mathcal{R}) = -1$, where t denotes an indeterminate over A. We denote by $\mathfrak{M} = \mathfrak{m}\mathcal{R} + \mathcal{R}_+$ the graded maximal ideal of $\mathcal{R}$.

2.1. P_g-Ideals

Assume that A is an excellent normal local domain over an algebraically closed field and there exists a resolution of singularities $f\colon X \to \operatorname{Spec} A$ with $E = f^{-1}(\mathfrak{m}) = \bigcup_{i=1}^{r} E_i$. Then $p_g(A) = \ell_A(H^1(\mathcal{O}_X))$ is called the *geometric genus* of A, which is independent of the choice of the resolution of singularities. Recall that any $\mathfrak{m}$-primary integrally closed ideal I can be written as $I = H^0(\mathcal{O}_X(-Z))$ for some resolution of singularities $X \to \operatorname{Spec} A$ and some anti-nef cycle Z on X such that $I\mathcal{O}_X = \mathcal{O}_X(-Z)$; see e.g. [13, Sec. 18]. Then I is said to be *represented on X by Z.*

Now let us recall the notion of p_g-ideals. Fix a resolution of singularities $X \to \operatorname{Spec} A$. Let Z be an anti-nef cycle on X and assume that $\mathcal{O}_X(-Z)$ has no fixed component, that is, $H^0(\mathcal{O}_X(-Z)) \neq H^0(\mathcal{O}_X(-Z - E_i))$ for every $E_i \subset E$. Then we have

$$\ell_A(H^1(\mathcal{O}_X(-Z)) \leq p_g(A).$$

If the equality holds, then $\mathcal{O}_X(-Z)$ is generated by global sections (see [14, Thm. 3.1]), and Z is called a *p_g-cycle.*

Definition 2.1 (Okuma et al. [14, Def. 3.2]). An $\mathfrak{m}$-primary ideal $I \subset A$ is called a *p_g-ideal* if it is represented by a p_g-cycle Z on some resolution of singularities $X \to \operatorname{Spec} A$.

The following criterion for p_g-ideals is very useful.

Lemma 2.2 (Okuma et al. [15, Thm. 1.1]). *Let $I \subset A$ be an $\mathfrak{m}$-primary ideal that is not a parameter ideal. Then the following conditions are equivalent*:

(1) *I is a p_g-ideal*;
(2) *$I^2 = QI$ for some parameter ideal $Q \subset I$, and I^n is integrally closed for every $n \geq 1$*;
(3) *$\mathcal{R}(I)$ is a Cohen–Macaulay normal domain.*

Now let us recall the notion of rational singularities (of dimension two). The ring A is called a *rational singularity* if $p_g(A) = 0$. For instance, toric singularities and quotient singularities are typical examples of rational singularities. The notion of p_g-ideals can be regarded as analogous to the notion of integrally closed ideals in a rational singularity. In fact, Lipman [13, Thm. 12.1] showed that any $\mathfrak{m}$-primary integrally closed ideal of A is a p_g-ideal, provided that A is a rational singularity.

It is known that the Rees algebra $\mathcal{R}(I)$ of an integrally closed ideal I in a rational singularity possesses some good property (e.g. rationality) by Lipman [13]. Moreover, Lemma 2.2 implies that $\mathcal{R}(I)$ is a Cohen-Macaulay normal domain if I is a p_g-ideal of A. We pose the following conjecture.

CONJECTURE 2.3. If $I \subset A$ is a p_g-ideal, then $\mathcal{R}(I)$ an almost Gorenstein graded ring.

As a partial answer, we prove the following theorem, which implies Theorem 1.3.

THEOREM 2.4. *Assume that A is Gorenstein under the assumption in this section. Let $I \subset A$ be a p_g-ideal, and let Q be its minimal reduction. Put $J = Q \colon I$ and $\mathcal{R} = \mathcal{R}(I)$. Then there exist a short exact sequence and elements $f \in \mathfrak{m}$, $g \in I$, and $h \in J$ such that*

$$0 \to \mathcal{R} \xrightarrow{\varphi} \mathrm{K}_{\mathcal{R}}(1) \cong J\mathcal{R} \to C \to 0,$$

where $\varphi(1) = h$ and $\mathfrak{M}C = (f, gt)C$. In particular, $\mathcal{R}$ is an almost Gorenstein graded ring.

COROLLARY 2.5. *Assume that A is a rational double point, that is, it is a Gorenstein rational singularity that is not regular. Then $\mathcal{R}(I)$ is an almost Gorenstein normal graded ring for any $\mathfrak{m}$-primary integrally closed ideal $I \subset A$.*

Proof. Under this assumption, any $\mathfrak{m}$-primary integrally closed ideal is a p_g-ideal but not a parameter ideal. Thus we can apply Theorem 2.4. □

Since any regular local ring is a rational singularity, we have the following:

EXAMPLE 2.6. Let A be a regular local ring with $\dim A = 2$. Then $\mathcal{R}(I)$ is an almost Gorenstein graded ring for any integrally closed ideal $I \subset A$. In particular, $\mathcal{R}(\mathfrak{m}^{\ell})$ is an almost Gorenstein graded ring for every $\ell \geq 1$.

REMARK 2.7. Suppose that A is a Gorenstein local ring of dimension two and $I = (a, b)$ is a parameter ideal of A. Then $\mathcal{R}(I) \cong A[X, Y]/(aY - bX)$ is a Gorenstein ring.

Okuma et al. [14, Thm. 1.2] showed that any excellent normal local domain of dimension two admits a p_g-ideal. Therefore, we obtain from Theorem 1.3 the following corollary.

COROLLARY 2.8. *For any excellent normal Gorenstein local domain of dimension two over an algebraically closed field k, there exists an $\mathfrak{m}$-primary ideal I such that $\mathcal{R}(I)$ is an almost Gorenstein graded ring.*

EXAMPLE 2.9 (Okuma et al. [15]). Let $p \geq 1$ be an integer.

1. Let $A = k[[x, y, z]]/(x^2 + y^3 + z^{6p+1})$. Then $I_k = (x, y, z^k)$ is a p_g-ideal for every $k = 1, 2, \ldots, 3p$.
2. Let $A = k[[x, y, z]]/(x^2 + y^4 + z^{4p+1})$. Then $I_k = (x, y, z^k)$ is a p_g-ideal for every $k = 2, \ldots, 2p$. However, $I_1 = \mathfrak{m}$ is not.

When this is the case, $p_g(A) = p$.

2.2. *Proof of Theorem* 1.3

In what follows, we always assume that the assumption of Theorem 1.3 holds. The following lemma, which is a particular case of [2, Lemma 5.1], plays a key role in the proof. Note that if A is Gorenstein, then it is a particular case of [17, Thm. 2.7(a)].

Lemma 2.10. *Suppose that A possesses the canonical ideal $\mathrm{K} = \mathrm{K}_A$. Then $\mathrm{K}_{\mathcal{R}}(1) \cong (Q\mathrm{K} :_{\mathrm{K}} I)\mathcal{R}$ as graded $\mathcal{R}$-modules.*

Proof. Since $I^2 = QI$, if we put $\omega_i = Q^i\mathrm{K} :_{\mathrm{K}} I$, then $\omega_i = I^{i-1}\mathrm{K} = Q^{i-1}\mathrm{K}$ for every $i \geq 1$ and $\mathrm{K}_{\mathcal{R}}(1)$ is isomorphic to $\omega_1\mathcal{R} = (Q\mathrm{K} :_{\mathrm{K}} I)\mathcal{R}$. □

The following two lemmata play important roles in the proof of the main theorem.

Lemma 2.11 (Okuma et al. [15, Thm. 3.5]). *Assume that I is a p_g-ideal and J is an integrally closed $\mathfrak{m}$-primary ideal. Then there exist $a \in I$ and $b \in J$ such that $IJ = aJ + bI$.*

Lemma 2.12 (Okuma et al. [16]). *Assume that $I \subset A$ is a p_g-ideal. If Q is a minimal reduction of I, then $J = Q : I$ is also a p_g-ideal.*

We are now ready to prove Theorem 2.4.

Proof of Theorem 2.4. Assume that A is Gorenstein and I is a p_g-ideal. Then $J = Q : I$ is also a p_g-ideal by Lemma 2.12. Hence, Lemma 2.11 implies that there exist $f \in \mathfrak{m}$, $g \in I$, and $h \in J$ such that

$$IJ = gJ + Ih, \qquad \mathfrak{m}J = fJ + \mathfrak{m}h$$

because I, J are p_g-ideals and $\mathfrak{m}$ is integrally closed (see also [18]).

Claim. We claim that $\mathfrak{M} \cdot J\mathcal{R} \subset (f, gt)J\mathcal{R} + \mathcal{R}h$.

In fact,

$$[\mathfrak{M} \cdot J\mathcal{R}]_0 = \mathfrak{m}J = fJ + \mathfrak{m}h \subseteq fJ + Ah = [(f, gt)J\mathcal{R} + \mathcal{R}h]_0,$$
$$[\mathfrak{M} \cdot J\mathcal{R}]_1 = IJ + \mathfrak{m}JI = IJ = gJ + Ih \subseteq [(f, gt)J\mathcal{R} + \mathcal{R}h]_1,$$
$$[\mathfrak{M} \cdot J\mathcal{R}]_n = I^nJ = (gJ)I^{n-1} + I^{n-1}Ih \subseteq [(f, gt)J\mathcal{R} + \subseteq \mathcal{R}h]_n$$

for all $n \geq 2$.

Thus, we have proved the claim.

Since $\mathrm{K}_{\mathcal{R}}(1) \cong J\mathcal{R}$ by Lemma 2.10 and $\mathrm{a}(\mathcal{R}) = -1$, if we define an $\mathcal{R}$-linear map φ by $\varphi(1) = h$, then we have an exact sequence

$$\mathcal{R} \xrightarrow{\varphi} \mathrm{K}_{\mathcal{R}}(1) = J\mathcal{R} \to C \to 0,$$

so that $C/\mathfrak{M}C = C/(f, gt)C$. Hence, C is an Ulrich A-module by [8, Lemma 3.1]. As $\dim C_{\mathfrak{M}} \leq 2 < \dim \mathcal{R} = 3$, $\varphi_{\mathfrak{M}}$ is injective by [8, Lemma 3.1] again. This means φ is injective. Therefore, we conclude that $\mathcal{R}$ is an almost Gorenstein graded ring. □

3. Rees Algebras of Good Ideals

We first recall the notion of good ideals introduced in [3].

Definition 3.1. Let $I \subset A$ be an $\mathfrak{m}$-primary ideal. Then I is called a *good* ideal if $I^2 = QI$ and $Q : I = I$ for some minimal reduction Q of I.

Now assume that $\mathfrak{m}^2 = Q\mathfrak{m}$ for some minimal reduction Q of $\mathfrak{m}$, that is, A has minimal multiplicity. Then $\mathfrak{m}$ is an integrally closed good ideal. For instance, if A is a two-dimensional rational singularity, then $\mathfrak{m}$ is a good ideal.

The following proposition says that the definition of good ideals is independent of the choice of its minimal reduction.

Proposition 3.2. *Put $\mathcal{R} = \mathcal{R}(I)$ and $G = G(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$. Then the following conditions are equivalent*:

(1) *I is a good ideal*;
(2) *G is a Cohen–Macaulay ring with* $\mathrm{a}(G) = 1 - d$ *and* $\mathrm{Soc}(H^d_{\mathfrak{M}}(G)) \subset [H^d_{\mathfrak{M}}(G)]_{1-d}$.

Proof. (1) $\Longrightarrow$ (2) By definition there exists a minimal reduction Q of I such that $I^2 = QI$ and $Q : I = I$. In particular, G is a Cohen–Macaulay ring with $\mathrm{a}(G) = 1 - d$. Write $Q = (a_1, \dots, a_d)$ and $a_i^* := a_i + I^2 \in [G]_1$ for $i = 1, 2, \dots, d$. Since $a_1^*, a_2^*, \dots, a_d^*$ forms a regular sequence in G, we have

$$G/(a_1^*, \dots, a_d^*)G \cong G(I/Q) \cong A/I \oplus I/Q =: \overline{G}.$$

Then

$$H^d_{\mathfrak{M}}(G) \cong H^{d-1}_{\mathfrak{M}}(G/a_1^*G)(1) \cong \cdots \cong H^0_{\mathfrak{M}}(G/(a_1^*, \dots, a_d^*)G)(d) \cong \overline{G}(d).$$

Thus, it suffices to show that $\mathrm{Soc}(\overline{G}) \subset [\overline{G}]_1$. Now suppose that $\alpha = (x + I, y + Q) \in \mathrm{Soc}(\overline{I})$, where $x \in A$ and $y \in I$. By definition we have $(z + I)\alpha = 0$ for any $z \in I$, that is, $zx \in Q$ for any $z \in I$. Hence $x \in Q : I = I$ because I is good. Therefore, $\alpha = (0 + I, y + Q) \in [\overline{G}]_1$, as required.

(2) $\Longrightarrow$ (1) As G is a Cohen–Macaulay ring with $\mathrm{a}(G) = 1 - d$, we have $I^2 = QI$ for some minimal reduction Q of I. Write $Q = (a_1, \dots, a_d)$ and $a_i^* := a_i + I^2 \in [G]_1$ for $i = 1, 2, \dots, d$. Then $H^d_{\mathfrak{M}}(G) \cong \overline{G}(d)$, where $\overline{G} = A/I \oplus I/Q$.

Now suppose that I is not good, that is, $I \subsetneq Q : I$. Then we can choose $x \in I : \mathfrak{m} \setminus I$ so that $x \in Q : I$. If we put $\alpha = x + I \in [\overline{G}]_0$, then $\beta\alpha = 0$ in $\overline{G}$ for every $\beta \in \mathfrak{m}\overline{G} + [\overline{G}]_1$, the maximal ideal of $\overline{G}$, that is, $0 \neq \alpha \in [\mathrm{Soc}(\overline{G})]_0$. However, this contradicts the assumption. Therefore, $I = Q : I$, and I is a good ideal. $\square$

In what follows, we consider the two-dimensional case. As a corollary of Proposition 3.2, we can compute $\mathrm{K}_{\mathcal{R}(I)}$ for a good ideal I.

Corollary 3.3. *Assume that* $\dim A = 2$, *$I \subset A$ is a good ideal, and A possesses the canonical ideal* $\mathrm{K} = \mathrm{K}_A$. *Set $\mathcal{R} = \mathcal{R}(I)$ and $G = G(I)$. Then* $\mathrm{K}_{\mathcal{R}}(1) \cong I\mathrm{K}\mathcal{R}$.

Proof. Set $\omega_{-1} = \omega_0 = \mathrm{K}$, $\omega_1 = Q\mathrm{K} :_{\mathrm{K}} I$, and $\omega_i = I^{i-1}\omega_1$ for $i \geq 2$. Then $[\mathrm{K}_G]_i = \omega_{i-1}/\omega_i$ for every $i \geq 1$; see [2, Thm. 2.1]. By Proposition 3.2 and duality, K_G is generated by $[\mathrm{K}_G]_1$ as an G-module. Hence $\omega_1/\omega_2 = [\mathrm{K}_G]_2 \subset G \cdot \omega_0/\omega_1$, that is, $\omega_1 = I\omega_0 + \omega_2 = I\mathrm{K} + I\omega_1$. This implies that $\omega_1 = I\mathrm{K}$ by Nakayama's lemma. It follows from Lemma 2.10 that $\mathrm{K}_{\mathcal{R}}(1) \cong I\mathrm{K}\mathcal{R}$, as required. □

In this section, we consider the following question.

Question 3.4. Assume that I is a good ideal. When is $\mathcal{R}(I)$ an almost Gorenstein graded ring?

The following theorem is the main result in this section, which gives a partial answer to the question.

Theorem 3.5. *Let $(A, \mathfrak{m})$ be a two-dimensional almost Gorenstein local ring. Assume that $\mathfrak{m}$ is good. Then $\mathcal{R}(\mathfrak{m}^\ell)$ is an almost Gorenstein graded ring for every $\ell \geq 1$.*

Proof. Set $\mathcal{R} = \mathcal{R}(\mathfrak{m}^\ell)$. By [8, Rem. 3.2] the ring A contains the canonical ideal $\mathrm{K} = \mathrm{K}_A$. Fix an integer $\ell \geq 1$. Note that $\mathfrak{m}^\ell$ is a good ideal, so that $\mathrm{K}_{\mathcal{R}}(1) \cong \mathfrak{m}^\ell\mathrm{K}\mathcal{R}$ by Corollary 3.3. Then it suffices to prove the following claim.

Claim. There exist $f \in \mathfrak{m}$, $g \in \mathfrak{m}^\ell$, and $h \in \mathfrak{m}^\ell\mathrm{K}$ such that

$$\mathfrak{m}^{\ell+1}\mathrm{K} = f\mathfrak{m}^\ell\mathrm{K} + \mathfrak{m}h, \qquad \mathfrak{m}^{2\ell}\mathrm{K} = g\mathfrak{m}^\ell\mathrm{K} + \mathfrak{m}^\ell h.$$

Note that this gives a proof of the theorem. Indeed, by a similar argument as in the proof of Theorem 2.4 we have $\mathfrak{M} \cdot \mathrm{K}_{\mathcal{R}} \subset (f, gt)\mathrm{K}_{\mathcal{R}} + \mathcal{R}h$. This yields a graded short exact sequence

$$0 \to \mathcal{R} \xrightarrow{\psi} \mathrm{K}_{\mathcal{R}}(1) \cong \mathfrak{m}\mathrm{K}\mathcal{R} \to C \to 0,$$

where $\psi(1) = h$ and $\mathfrak{M}C = (f, gt)C$. Namely, $\mathcal{R}$ is an almost Gorenstein graded ring.

Let us prove the claim. First, suppose that A is Gorenstein. Then $\mathrm{K} = A$. By assumption we can take $y, z \in \mathfrak{m}$ such that $\mathfrak{m}^2 = (y, z)\mathfrak{m}$. If we set $f = y$, $g = y^\ell$, and $h = z^\ell$, then

$$\mathfrak{m}^{\ell+1} = f\mathfrak{m}^\ell + \mathfrak{m}h, \qquad \mathfrak{m}^{2\ell} = g\mathfrak{m}^\ell + \mathfrak{m}^\ell h,$$

as required.

Next, suppose that A is not Gorenstein. Then we have a short exact sequence

$$0 \to A \xrightarrow{\varphi} \mathrm{K} = \mathrm{K}_A \to C \to 0$$

such that C is an Ulrich A-module of $\dim C = 1$. If we put $x = \varphi(1)$, then $x \in \mathrm{K} \setminus \mathfrak{m}\mathrm{K}$ by [8, Cor. 3.10]. Choose $y, z \in \mathfrak{m}$ so that:

(i) (y, z) is a minimal reduction of $\mathfrak{m}$;
(ii) the image of (y) in $A/\operatorname{Ann}_A C$ is a minimal reduction of $\mathfrak{m}/\operatorname{Ann}_A C$.

From (i) we have $\mathfrak{m}^2 = (y,z)\mathfrak{m}$, and thus $\mathfrak{m}^\ell = (y,z)^{\ell-1}\mathfrak{m}$ and $\mathfrak{m}^{2\ell} = (y^\ell, z^\ell)\mathfrak{m}^\ell$. Then (ii) implies $\mathfrak{m}C = yC$. In particular, $\mathfrak{m}\mathrm{K} \subset y\mathrm{K} + xA$. Hence $\mathfrak{m}\mathrm{K} = y\mathrm{K} + x\mathfrak{m}$ because $x \notin \mathfrak{m}\mathrm{K}$. Multiplying this by $\mathfrak{m}^\ell$, we obtain

$$\begin{aligned}\mathfrak{m}^{\ell+1}\mathrm{K} &= y\mathfrak{m}^\ell\mathrm{K} + xz\mathfrak{m}^\ell \\ &= y\mathfrak{m}^\ell\mathrm{K} + xz(y,z)^{\ell-1}\mathfrak{m} \\ &= y\mathfrak{m}^\ell\mathrm{K} + xz^\ell\mathfrak{m}.\end{aligned}$$

Moreover, we have $\mathfrak{m}^{2\ell}\mathrm{K} = y^\ell\mathfrak{m}^\ell\mathrm{K} + z^\ell\mathfrak{m}^\ell\mathrm{K}$. On the other hand,

$$\begin{aligned}z^\ell\mathfrak{m}^\ell\mathrm{K} &= z^\ell\mathfrak{m}^{\ell-1}\mathfrak{m}\mathrm{K} \\ &= z^\ell\mathfrak{m}^{\ell-1}(y\mathrm{K} + x\mathfrak{m}) \\ &\equiv yz^\ell\mathfrak{m}^{\ell-1}\mathrm{K} \pmod{xz^\ell\mathfrak{m}^\ell} \\ &\equiv yz^\ell\mathfrak{m}^{\ell-2}(y\mathrm{K} + x\mathfrak{m}) \pmod{xz^\ell\mathfrak{m}^\ell} \\ &\equiv y^2z^\ell\mathfrak{m}^{\ell-2}K \pmod{xz^\ell\mathfrak{m}^\ell} \\ &\equiv \cdots \equiv y^\ell z^\ell\mathrm{K} \pmod{xz^\ell\mathfrak{m}^\ell}.\end{aligned}$$

Hence $\mathfrak{m}^{2\ell}\mathrm{K} = y^\ell z^\ell\mathrm{K} + xz^\ell\mathfrak{m}^\ell$. Setting $f = y$, $g = y^\ell \in \mathfrak{m}^\ell$, and $h = xz^\ell \in \mathfrak{m}^\ell\mathrm{K}$, we obtain the required equality. □

Let us explore the example to show how Theorem 3.5 works.

EXAMPLE 3.6. Let $r \geq 2$ be an integer. Let $A = k[[s^r, s^{r-1}t, \ldots, st^{r-1}, t^r]]$ be the rth Veronese subring of $k[[s,t]]$. Note that $(A, \mathfrak{m})$ is a rational singularity. Set

$$\mathrm{K} = (s^{r-1}t, s^{r-2}t^2, \ldots, st^{r-1}).$$

If we take

$$x = st^{r-1} \in \mathrm{K}, \qquad y = s^r, \qquad z = t^r,$$

then $\mathfrak{m}\mathrm{K} = y\mathfrak{m}\mathrm{K} + x\mathfrak{m}$ and $\mathfrak{m}^2 = (y,z)\mathfrak{m}$. Thus, $\mathcal{R}(\mathfrak{m})$ is an almost Gorenstein graded ring.

If A is a Cohen–Macaulay local ring of $\mathrm{e}^0_\mathfrak{m}(A) = 2$, then it has minimal multiplicity, and it is Gorenstein, that is, $\mathrm{K} = A$.

EXAMPLE 3.7. Suppose that $(A, \mathfrak{m})$ is a Cohen–Macaulay local ring of $\mathrm{e}^0_\mathfrak{m}(A) = 2$. Then $\mathcal{R}(\mathfrak{m})$ is an almost Gorenstein graded ring.

In the rest of this section, we consider Question 3.4 in the higher-dimensional case. To prove our result, we need the following lemma, which is very useful in proving the almost Gorensteinness of the Rees algebra.

LEMMA 3.8. *Let $(A, \mathfrak{m})$ be a regular local ring, and let $I \subsetneq A$ be an ideal of positive height. Set $\mathcal{R} = \mathcal{R}(I)$. If $\mathcal{R}_\mathfrak{M}$ is a Cohen–Macaulay local ring with $\mathrm{K}_\mathcal{R} = \sum_{i=1}^{c+1} \mathcal{R}t^i$ for some $c \geq 0$, then $\mathcal{R}_\mathfrak{M}$ is an almost Gorenstein local ring.*

Proof. Note that

$$\mathrm{K}_\mathcal{R} \cong At + At^2 + \cdots + At^c + \mathcal{R}t^{c+1}.$$

Set $C = \mathrm{K}_{\mathcal{R}}/\mathcal{R}t^{c+1}$. Then $C \cong A\overline{t} + A\overline{t^2} + \cdots + A\overline{t^c} \cong A^{\oplus c}$ as A-modules. If we set $\mathfrak{a} = \mathcal{R}_+^c$ and $\overline{\mathcal{R}} = \mathcal{R}/\mathfrak{a}$, then C is an $\overline{R}$-module because $\mathfrak{a}C = 0$. Moreover, $A \hookrightarrow \mathcal{R} \to \overline{\mathcal{R}}$ is a finite morphism, and $\mathfrak{m}\overline{\mathcal{R}}$ is a reduction of $\overline{\mathfrak{M}} = \mathfrak{M}/\mathfrak{a}$, so that we have

$$\mathrm{e}^0_{\mathfrak{M}}(C) = \mathrm{e}^0_{\overline{\mathfrak{M}}}(C) = \mathrm{e}^0_{\mathfrak{m}\overline{\mathcal{R}}}(C) = \mathrm{e}^0_{\mathfrak{m}}(C) = \mathrm{e}^0_{\mathfrak{m}}(A) \cdot \mathrm{rank}_A C = 1 \cdot c = c.$$

On the other hand, since $\mu_{\mathcal{R}}(C) = c$, $C_{\mathfrak{M}}$ is an Ulrich $\mathcal{R}_{\mathfrak{M}}$-module, and thus $\mathcal{R}_{\mathfrak{M}}$ is an almost Gorenstein local ring. □

The following theorem gives a complete answer to Question 3.4 in the higher-dimensional Gorenstein case.

THEOREM 3.9. *Suppose that A is a Gorenstein local ring of $d = \dim A \geq 3$ and that $I \subset A$ is a good ideal. Set $\mathcal{R} = \mathcal{R}(I)$. Then we have*:

(1) *The following conditions are equivalent*:
  (a) *$\mathcal{R}$ is an almost Gorenstein graded ring*;
  (b) *$\mathcal{R}$ is Gorenstein*;
  (c) *$d = 3$.*

(2) *If A is a regular local ring, then $\mathcal{R}_{\mathfrak{M}}$ is an almost Gorenstein local ring. Conversely, if $\mathcal{R}_{\mathfrak{M}}$ is an almost Gorenstein local ring but not Gorenstein, then A itself is a regular local ring.*

Proof. (1) $(c) \Longrightarrow (b)$ By [3, Prop. 2.2], $G(I)$ is a Gorenstein ring with $\mathrm{a}(G(I)) = 1 - d = -2$. Hence $\mathcal{R}(I)$ is Gorenstein by the Goto–Shimoda theorem (see [7]).

$(b) \Longrightarrow (a)$ This is trivial.

$(a) \Longrightarrow (c)$ Now suppose that $\mathcal{R}$ is an almost Gorenstein graded ring but not Gorenstein. Then there exists an exact sequence of graded $\mathcal{R}$-modules

$$0 \to \mathcal{R} \xrightarrow{\varphi} \mathrm{K}_{\mathcal{R}}(1) \to C \to 0$$

such that $\mu_{\mathcal{R}}(C) = \mathrm{e}^0_{\mathfrak{M}}(C)$. As $\varphi(1) \notin \mathfrak{M}\mathrm{K}_{\mathcal{R}}(1)$, we may assume that $\varphi(1) = t$. Then

$$C = A/I \oplus A/I^2 \oplus \cdots \oplus A/I^{d-2} \oplus I/I^{d-1} \oplus \cdots \supset \mathcal{R}/I^{d-2}\mathcal{R}(-d+2),$$

and $\dim C = \dim \mathcal{R}/I^{d-2}\mathcal{R}$ $(= d)$. This implies that $\mathrm{e}^0_{\mathfrak{M}}(C) \geq \mathrm{e}^0_{\mathfrak{M}}(\mathcal{R}/I^{d-2}\mathcal{R})$. For each prime $P \in \mathrm{Assh}(\mathcal{R}/I\mathcal{R})$, we have $\dim \mathcal{R}_P = \mathrm{ht} I\mathcal{R} = 1$ and $I^{d-2}\mathcal{R}_P \subsetneq \cdots \subsetneq I\mathcal{R}_P \subsetneq \mathcal{R}_P$. Thus the associative formula implies that

$$\mathrm{e}^0_{\mathfrak{M}}(C) \geq \mathrm{e}^0_{\mathfrak{M}}(\mathcal{R}/I^{d-2}\mathcal{R}) \geq \ell_{\mathcal{R}_P}(\mathcal{R}_P/I^{d-2}\mathcal{R}_P) \geq d-2.$$

On the other hand,

$$\mathrm{e}^0_{\mathfrak{M}}(C) = \mu_{\mathcal{R}}(C) = \mu_{\mathcal{R}}(\mathrm{K}_{\mathcal{R}}) - 1 = (d-2) - 1 = d-3.$$

This is a contradiction.

(2) First, suppose that A is a regular local ring. Since I is a good ideal, we have that $\mathcal{R}$ is a Cohen–Macaulay ring with $\mathrm{K}_{\mathcal{R}} \cong \mathcal{R}t + \mathcal{R}t^2 + \cdots + \mathcal{R}t^{d-2}$. Hence, $\mathcal{R}_{\mathfrak{M}}$ is an almost Gorenstein local ring by Lemma 3.8.

Conversely, assume that $\mathcal{R}_{\mathfrak{M}}$ is an almost Gorenstein local ring but not Gorenstein. We may assume that $d \geq 4$. Take an exact sequence

$$0 \to \mathcal{R}_{\mathfrak{M}} \to \mathrm{K}_{\mathcal{R}_{\mathfrak{M}}} \to C_{\mathfrak{M}} \to 0$$

such that $C_{\mathfrak{M}}$ is an Ulrich $\mathcal{R}_{\mathfrak{M}}$-module. Then we obtain

$$\mu_{\mathcal{R}}(\mathfrak{M}\mathrm{K}_{\mathcal{R}}) \leq \mu_{\mathcal{R}}(\mathfrak{M}) + \mu_{\mathcal{R}}(\mathfrak{M}C) \leq \mu_{\mathcal{R}}(\mathfrak{M}) + d \cdot (\mu_{\mathcal{R}}(\mathrm{K}_{\mathcal{R}}) - 1).$$

Since $\mu_{\mathcal{R}}(\mathrm{K}_{\mathcal{R}}) = d - 2$, $\mu_{\mathcal{R}}(\mathfrak{M}) = \mu_A(\mathfrak{m}) + \mu_A(I)$, and

$$\mathfrak{M} \cdot \mathrm{K}_{\mathcal{R}} = (\mathfrak{m}t, \mathfrak{m}t^2, \ldots, \mathfrak{m}t^{d-2}, It^{d-1}),$$

we obtain

$$(d-2)\mu_A(\mathfrak{m}) + \mu_A(I) \leq \mu_A(\mathfrak{m}) + \mu_A(I) + d(d-3).$$

This implies $\mu_A(\mathfrak{m}) = d$, that is, A is a regular local ring, as required. □

4. Higher-Dimensional Case (Proof of Theorem 1.6)

In this section, we prove Theorem 1.6. In what follows, let $(A, \mathfrak{m})$ be a regular local ring of dimension $d \geq 2$ with infinite residue class field, and let $\ell \geq 1$ be an integer.

Remark 4.1. First, suppose that $\ell = 1$. Then $\mathcal{R}(\mathfrak{m}^{\ell}) = \mathcal{R}(\mathfrak{m})$ is an almost Gorenstein graded ring because the maximal ideals $\mathfrak{m}$ of a regular local ring is a parameter ideal; see [5, Thm. 1.3].

Next, suppose that $d = 2$. Then [6, Cor. 1.4] implies that $\mathcal{R}(\mathfrak{m}^{\ell})$ is an almost Gorenstein graded ring for every $\ell \geq 1$.

Finally, suppose that $\ell = d - 1$. Then $\mathcal{R}(\mathfrak{m}^{d-1})$ is a Gorenstein ring, and thus it is an almost Gorenstein graded ring; see e.g. [5, Prop. 2.3].

Thus, to prove Theorem 1.6, we restrict our attention to the case where $\ell \geq 2$ and $d \geq 3$.

Proposition 4.2. *Let $\ell \geq 2$ and $d \geq 3$ be integers. Assume that $\mathcal{R}(\mathfrak{m}^{\ell})_{\mathfrak{M}}$ is an almost Gorenstein local ring. Then ℓ is a divisor of $d - 1$.*

Proof. The graded canonical module of the Rees algebra $\mathcal{R}(\mathfrak{m})$ is given by

$$\mathrm{K}_{\mathcal{R}(\mathfrak{m})} \cong At + At^2 + \cdots + At^{d-2} + \sum_{n \geq d-1} \mathfrak{m}^{n-d+1}t^n;$$

see e.g. [2, Lemma 5.1]. This formula and [10, Prop. 2.5] imply

$$\mathrm{K}_{\mathcal{R}(\mathfrak{m}^{\ell})} \cong \sum_{n=1}^{b} At^n + \sum_{n \geq b+1} \mathfrak{m}^{n\ell-d+1}t^n, \quad \text{where } b = \left\lfloor \frac{d-2}{\ell} \right\rfloor = \left\lceil \frac{d-1}{\ell} \right\rceil - 1.$$

Now, set $J = \mathfrak{m}^{(b+1)\ell-d+1}$, $\mathcal{R} = \mathcal{R}(\mathfrak{m}^\ell)$, and $\mathfrak{M} = \mathfrak{m}\mathcal{R} + \mathcal{R}_+$. Then $\mathrm{K} := \mathrm{K}_{\mathcal{R}(\mathfrak{m}^\ell)}$ is generated by $t, t^2, \dots, t^b$, and Jt^{b+1} as an $\mathcal{R}$-module. Hence

$$\mu_{\mathcal{R}}(\mathrm{K}) = b + \mu_A(J). \tag{4.1}$$

Similarly, since $\mathfrak{M}\mathrm{K}$ is generated by $\mathfrak{m}t, \mathfrak{m}t^2, \dots, \mathfrak{m}t^b, \mathfrak{m}Jt^{b+1}$, and $\mathfrak{m}^\ell Jt^{b+2}$, we have

$$\mu_{\mathcal{R}}(\mathfrak{M}\mathrm{K}) = b \cdot \mu_A(\mathfrak{m}) + \mu_A(\mathfrak{m}J) + \mu_A(\mathfrak{m}^\ell J). \tag{4.2}$$

Now assume that $\mathcal{R}_{\mathfrak{M}}$ is an almost Gorenstein local ring. Then we must prove the following claim.

Claim. $\mu_A(\mathfrak{m}J) + \mu_A(\mathfrak{m}^\ell J) \le \mu_A(\mathfrak{m}^\ell) + d \cdot \mu_A(J)$.

We consider the exact sequence

$$0 \to \mathcal{R}_{\mathfrak{M}} \to \mathrm{K}_{\mathcal{R}_{\mathfrak{M}}} \to C \to 0$$

of $\mathcal{R}_{\mathfrak{M}}$-modules with $\mu_{\mathcal{R}_{\mathfrak{M}}}(C) = \mathrm{e}^0_{\mathfrak{M}\mathcal{R}_{\mathfrak{M}}}(C)$. Then [8, Cor. 3.10] implies that

$$\mu_{\mathcal{R}_{\mathfrak{M}}}(C) = \mu_{\mathcal{R}_{\mathfrak{M}}}(\mathrm{K}_{\mathcal{R}_{\mathfrak{M}}}) - 1 = \mu_{\mathcal{R}}(\mathrm{K}_{\mathcal{R}}) - 1$$

and

$$0 \to \mathfrak{M}\mathcal{R}_{\mathfrak{M}} \to \mathfrak{M}\mathrm{K}_{\mathcal{R}_{\mathfrak{M}}} \to \mathfrak{M}C \to 0$$

is exact. Moreover, as C is an Ulrich $\mathcal{R}_{\mathfrak{M}}$-module and A is regular, we have

$$\begin{aligned}
\mu_{\mathcal{R}}(\mathfrak{M}\mathrm{K}) = \mu_{\mathcal{R}_{\mathfrak{M}}}(\mathfrak{M}\mathrm{K}_{\mathcal{R}_{\mathfrak{M}}}) &\le \mu_{\mathcal{R}_{\mathfrak{M}}}(\mathfrak{M}\mathcal{R}_{\mathfrak{M}}) + \mu_{\mathcal{R}_{\mathfrak{M}}}(\mathfrak{M}C)\\
&\le \mu_{\mathcal{R}}(\mathfrak{M}) + d \cdot \mu_{\mathcal{R}_{\mathfrak{M}}}(C)\\
&= \mu_A(\mathfrak{m}) + \mu_A(\mathfrak{m}^\ell) + d \cdot (\mu_{\mathcal{R}}(\mathrm{K}) - 1)\\
&= \mu_A(\mathfrak{m}^\ell) + d \cdot \mu_{\mathcal{R}}(\mathrm{K}).
\end{aligned}$$

By substituting (4.1) and (4.2) for this, we obtain the desired claim.

Note that $\mu_A(\mathfrak{m}^k) = \binom{k+d-1}{d-1}$ and $J = \mathfrak{m}^{(b+1)\ell-d+1}$. Using the claim, we have

$$\binom{(b+1)\ell+1}{d-1} + \binom{(b+2)\ell}{d-1} \le \binom{\ell+d-1}{d-1} + d \cdot \binom{(b+1)\ell}{d-1}. \tag{4.3}$$

The opposite inequality follows from the following lemma, and thus ℓ is a divisor of $d-1$. $\square$

LEMMA 4.3. *Let $\ell \ge 2$ and $d \ge 3$ be integers and set $b = \lfloor \frac{d-2}{\ell} \rfloor$. Then*

$$\binom{(b+1)\ell+1}{d-1} + \binom{(b+2)\ell}{d-1} \ge \binom{\ell+d-1}{d-1} + d \cdot \binom{(b+1)\ell}{d-1}. \tag{4.4}$$

In addition, equality holds if and only if ℓ is a divisor of $d-1$.

Proof. Set $i = d-2-b\ell$. Then $0 \le i \le \ell-1$. Note that

$$\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1} \quad \text{for } n, m \ge 2.$$

By substituting this equality we have

$$\begin{aligned}
\binom{(b+2)\ell}{d-1} &= \binom{(b+1)\ell+\ell-1}{d-1} + \binom{(b+1)\ell+\ell-1}{d-2} \\
&= \binom{(b+1)\ell+\ell-2}{d-1} + \binom{(b+1)\ell+\ell-2}{d-2} + \binom{(b+1)\ell+\ell-1}{d-2} \\
&= \cdots = \binom{(b+1)\ell+i+1}{d-1} + \sum_{j=i+1}^{\ell-1} \binom{(b+1)\ell+j}{d-2} \\
&= \binom{\ell+d-1}{d-1} + \sum_{j=i+1}^{\ell-1} \binom{(b+1)\ell+j}{d-2}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
d \cdot \binom{(b+1)\ell}{d-1} &= \binom{(b+1)\ell}{d-1} + (d-1)\binom{(b+1)\ell}{d-1} \\
&= \binom{(b+1)\ell}{d-1} + ((b+1)\ell-d+2)\binom{(b+1)\ell}{d-2} \\
&= \binom{(b+1)\ell}{d-1} + (\ell-i)\binom{(b+1)\ell}{d-2} \\
&= \binom{(b+1)\ell+1}{d-1} + (\ell-i-1)\binom{(b+1)\ell}{d-2}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\binom{(b+1)\ell+1}{d-1} + \binom{(b+2)\ell}{d-1} - \left\{\binom{\ell+d-1}{d-1} + d \cdot \binom{(b+1)\ell}{d-1}\right\} \\
&= \sum_{j=i+1}^{\ell-1} \left\{\binom{(b+1)\ell+j}{d-2} - \binom{(b+1)\ell}{d-2}\right\} \geq 0
\end{aligned}$$

because $i \leq \ell - 1$. Therefore, the equality of (4.4) holds if and only if $i = \ell - 1$. $\square$

To complete the proof of Theorem 1.6, we give the following theorem.

THEOREM 4.4. *Suppose that $\ell \geq 2$, $d \geq 3$, and $\ell \mid d-1$. Set $\mathcal{R} = \mathcal{R}(\mathfrak{m}^\ell)$. Then*:

(1) *$\mathcal{R}_\mathfrak{M}$ is an almost Gorenstein local ring*;

(2) *if $\ell \neq d-1$, then $\mathcal{R}$ is* not *an almost Gorenstein graded ring*.

Proof. Set $b = \frac{d-1}{\ell} - 1$. Then $b \geq 0$ and

$$\mathrm{K}_\mathcal{R} \cong \mathcal{R}t + \mathcal{R}t^2 + \cdots + \mathcal{R}t^b + \mathcal{R}t^{b+1}.$$

(1) This follows from Lemma 3.8.

(2) Now suppose that $\mathcal{R}$ is an almost Gorenstein graded ring. Then there exists a short exact sequence

$$0 \to \mathcal{R} \xrightarrow{\varphi} \mathrm{K}_\mathcal{R}(1) \to C \to 0$$

Table 1 When is $\mathcal{R}(\mathfrak{m}^\ell)$ almost Gorenstein?

$d \setminus \ell$	1	2	3	4	5	6	7	8	9
2	**Gor**	**AG**	**AG**	**AG**	**AG**	**AG**	**AG**	**AG**	**AG**
3	**AG**	**Gor**	X	X	X	X	X	X	X
4	**AG**	X	**Gor**	X	X	X	X	X	X
5	**AG**	**AGL**	X	**Gor**	X	X	X	X	X
6	**AG**	X	X	X	**Gor**	X	X	X	X
7	**AG**	**AGL**	**AGL**	X	X	**Gor**	X	X	X
8	**AG**	X	X	X	X	X	**Gor**	X	X
9	**AG**	**AGL**	X	**AGL**	X	X	X	**Gor**	X
10	**AG**	X	**AGL**	X	X	X	X	X	**Gor**

of graded $\mathcal{R}$-modules, so that C is an Ulrich $\mathcal{R}$-module. Since $\varphi(1)$ is part of a minimal set of generators of $[\mathrm{K}_\mathcal{R}]_1$ by [8, Cor. 3.10], we may assume that $\varphi(1) = t$ without loss of generality. Then

$$C = \mathrm{K}_\mathcal{R}(1)/\mathcal{R}t \cong \mathcal{R}\overline{t^2} + \cdots + \mathcal{R}\overline{t^{b+1}}$$

yields that $\mu_\mathcal{R}(C) \le b < \frac{d-1}{\ell}$.

If we put $I = \mathfrak{m}^\ell$, then

$$C = \sum_{n=2}^{\infty} C_n \cong A/I \oplus A/I^2 \oplus \cdots \oplus A/I^b \oplus I/I^{b+1} \oplus \cdots .$$

Thus

$$C \supset \mathcal{R}\overline{t^{b+1}} \cong \mathcal{R}/I^b\mathcal{R}(-(b+1)),$$

and hence

$$\mathrm{e}^0_\mathfrak{M}(C) \ge \mathrm{e}^0_\mathfrak{M}(\mathcal{R}/I^b\mathcal{R}) \ge \mathrm{e}^0_\mathfrak{M}(\mathcal{R}/I\mathcal{R}) = \mathrm{e}^0_\mathfrak{M}(G(I)) = \mathrm{e}^0_I(A) = \ell^d \cdot \mathrm{e}^0_\mathfrak{m}(A) = \ell^d.$$

As $\ell \ge 2$ and $b \ge 1$, we have $\ell^d > \frac{d}{\ell} > \frac{d-1}{\ell} = b + 1$. Therefore $\mathrm{e}^0_\mathfrak{M}(C) > b + 1 \ge \mu_\mathcal{R}(C)$, which contradicts the assumption. □

In Table 1 we present part of the list of (d, ℓ) for which, over a d-dimensional regular local ring $(A, \mathfrak{m})$, the Rees algebra $\mathcal{R}(\mathfrak{m}^\ell)$ is a Gorenstein ring (**Gor**), an almost Gorenstein graded ring (**AG**), or $\mathcal{R}(\mathfrak{m}^\ell)_\mathfrak{M}$ is an almost Gorenstein local ring (**AGL**).

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