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A Note on Brill–Noether Existence for Graphs of Low Genus

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Abstract. In an influential 2008 paper, Baker proposed a number of conjectures relating the Brill–Noether theory of algebraic curves with a divisor theory on finite graphs. In this note, we examine Baker's Brill–Noether existence conjecture for special divisors. For $g \leq 5$ and $\rho(g,r,d)$ nonnegative, every graph of genus g is shown to admit a divisor of rank r and degree at most d. As further evidence, the conjecture is shown to hold in rank 1 for a number families of highly connected combinatorial types of graphs. In the relevant genera, our arguments give the first combinatorial proof of the Brill–Noether existence theorem for metric graphs, giving a partial answer to a related question of Baker.

1. Introduction

1.1. Statement of Main Results

The last decade has seen a number of results exploring the interplay between the divisor theory of algebraic curves and an analogous theory on graphs, developed by Baker and Norine [4]. These theories are interlinked by Baker's specialization lemma [2, Lemma 2], which states that the rank of a divisor on an algebraic curve over a valued field can only increase upon specialization to a skeleton. This has led to numerous applications in algebraic geometry and number theory; see the survey [3]. For example, combinatorial Brill–Noether theory has been successfully employed to produce tropical proofs of the Brill–Noether and Gieseker–Petri theorems in algebraic geometry and has provided insights on the maximal rank conjecture [10; 15; 16]. Divisors on graphs are also of purely combinatorial interest, for instance, through connections as diverse as G-parking functions [22] and cryptosystems [23].

In his paper on the specialization lemma [2], Baker conjectured a number of combinatorial results concerning the divisor theory of graphs based on theorems in algebraic geometry. Many of these conjectures have now been proved [10; 14] and have been the basis for substantial additional progress. In this paper, we study one of the remaining open questions, the combinatorial counterpart to the existence part of the Brill–Noether theorem.[1]

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[1] Experts have recorded a gap in the proof of this conjecture that appears in [7, Thm. 6.3]. We direct the reader to the discussion in [3, Rem. 4.8 and Footnote 5].

Recall that for nonnegative integers g, r, and d, the Brill–Noether number is defined to be $\rho(g, r, d) = g - (r + 1)(g - d + r)$.

CONJECTURE 1 (Brill–Noether existence conjecture for graphs). *If $\rho(g, r, d)$ is nonnegative, then every graph of genus g admits a divisor D with* $\mathrm{rk}(D) = r$ *and* $\deg(D) \leq d$.

A number of researchers have demonstrated an intricate Brill–Noether theory entirely within the realm of (finite or metric) graphs; see, for instance, [5; 8; 19; 20]. Conjecture 1 is a central question in this area. In this paper, we confirm Baker's conjecture in genera up to 5.

MAIN THEOREM. *The Brill–Noether existence conjecture holds for all finite graphs of genus at most* 5.

The specialization lemma immediately implies the Brill–Noether existence conjecture for all *metric* graphs, where chips may need to be placed in the interiors of edges. Baker asks the following question.

QUESTION 1. Can the Brill–Noether existence theorem for metric graphs be proved using purely combinatorial methods?

The proof of the main theorem, with superficial changes, furnishes such a proof for all metric graphs of genus at most 5.

In a complementary direction, we could ask for an algebro-geometric proof of Brill–Noether existence for finite graphs. This question is closely related to the existence of divisors on curves over discretely valued fields that are expressible as sums of rational points and bounds on degrees of ramified base changes in semistable reduction. We are not aware of any substantial progress in this direction.

As further evidence for the conjecture, we exhibit a highly connected homeomorphism classes of graphs in increasing genus, for which the existence conjecture holds in rank 1 for all representatives of that class. These results are stated precisely in Section 6.

1.2. Context from Algebraic Geometry

The fact that when $\rho \geq 0$, every algebraic curve admits a divisor of rank r and degree at most d was proved by Kempf, Kleiman, and Laksov [17; 18]. It is considered to be the easier part of the Brill–Noether theorem. The harder direction, showing the nonexistence of special divisors when ρ is negative, was proved by Griffiths and Harris [13]. Kempf, Kleiman, and Laksov's proof of the existence of special divisors follows from Schubert calculus techniques and the Thom–Porteous determinantal formula. However, such techniques are not available in the discrete setting. On the other hand, the harder direction, the existence of Brill–Noether general graphs in every genus was proved purely combinatorially by Cools, Draisma, Payne, and Robeva [10] and implies the harder direction of the Brill–Noether theorem.

1.3. Related Results

Baker [2] shows that any finite graph G can be uniformly rescaled to a graph G' for which Conjecture 1 holds. More precisely, there exists an integer m_G such the inflated graph G' obtained by putting $m_G - 1$ bivalent vertices on each edge of G satisfies Conjecture 1. Conjecture 1 then asserts that we can always pick $m_G = 1$ for every finite graph G. No effective bounds on the value of m_G are known. Conjecture 1 remains open for $r = 1$ and $g \geq 6$, where it is equivalent to the following:

CONJECTURE 2 (Gonality conjecture). *The gonality of any graph of genus g is at most $\lfloor (g+3)/2 \rfloor$.*

Recall that the gonality of an algebraic curve is the smallest degree of a rank one divisor. After the results of this paper, the next outstanding case of Conjecture 1 is $g = 6, r = 1, d = 4$.

For $g \geq 6$, the strongest result concerning the gonality conjecture is a recent result of Cools and Draisma [9]. They show that for any topologically trivalent genus g graph $G = (V, E)$ there exists a nonempty open cone $C_G \subseteq \mathbb{R}^{|E|}_{>0}$ whose image in $\mathcal{M}_g^{\mathrm{trop}}$ consists entirely of metric graphs with gonality exactly $d := \lfloor (g+3)/2 \rfloor$. Furthermore, any graph corresponding to a lattice point of C_G satisfies the existence conjecture. Their approach relies on studying harmonic morphisms to trees and the techniques of [1]. We are not aware of any systematic results in higher genus for which the cone C_G is known to be the entire orthant.

1.4. Outline of the Paper

In Section 2, we briefly recall the Baker–Norine theory of divisors on finite graphs, reduced divisors, and Dhar's burning algorithm. In Section 3, we reduce the existence conjecture to the rank 1 case. In Sections 4 and 5, we prove the Main Theorem for graphs of genus 4 and 5, respectively. In both sections, we produce divisors of prescribed degree and rank for topologically trivalent and then degenerate the construction for general graphs. In Section 6, we exhibit families of graphs of increasing genus for which the existence conjecture in rank 1 holds.

2. Divisor Theory on Finite Graphs

The main reference for this section is the original paper of Baker and Norine [4]. A graph G will mean a finite connected graph possibly with loops and multiple edges. The vertex and edge sets of G will be denoted $V(G)$ and $E(G)$, respectively. The *genus* of G, denoted $g(G)$, is defined to be

$$g(G) := |E(G)| - |V(G)| + 1.$$

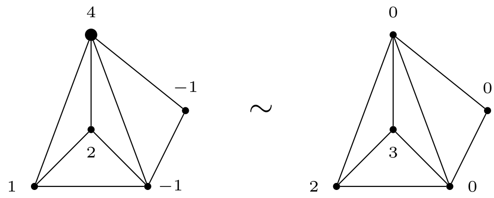


Figure 1 The larger vertex fires once to move from the left configuration to the right configuration

A *divisor* D on a graph G is a formal $\mathbb{Z}$-linear combination on its vertices

$$D = \sum_{v \in V(G)} D(v) \cdot v.$$

The *degree* of a divisor, denoted $\deg(D)$, equals $\sum_{v \in V(G)} D(v)$, and a divisor is said to be *effective* if $D(v) \geq 0$ for all $v \in V(G)$. The set of all divisors on a graph G is denoted by $\mathrm{Div}(G)$ and has a natural grading $\mathrm{Div}(G) = \bigoplus_{d \in \mathbb{Z}} \mathrm{Div}^d(G)$ induced by the degree. The same holds for $\mathrm{Div}_+(G)$, the set of all effective divisors.

It is often useful to think of the integers $D(v)$ as the number of *chips* or *antichips* placed on $v \in V(G)$. Given D and a vertex v, we may obtain a new divisor by means of a *chip firing move* as follows. The vertex v sends one chip to its neighbors, along each of the outgoing edges connecting them. Thus, $D(v)$ decreases by the valence of v, and for each w a neighbor of v, $D(w)$ increases by the number of edges between v and w. Chip-firing generates an equivalence relation on the set $\mathrm{Div}(G)$ of divisors on G known as *linear equivalence.* See Figure 1 for an illustration. The class $[D]$ is said to be *effective* if it contains an effective representative. For an alternative definition of this equivalence in terms of piecewise linear functions, see [7].

It is a well-known fact that every two same degree divisors on a tree are equivalent. For that reason, when studying divisors on graphs, no information is lost by contracting all grafted trees and assuming that all vertices have valency at least two.

The central invariant in the divisor theory of graphs is the rank of a divisor. If $[D]$ is effective, then the *rank* of D is defined as

$$\mathrm{rk}(D) := \max\{k \in \mathbb{Z}_{\geq 0} | [D - E] \text{ is effective } \forall E \in \mathrm{Div}_+^k(G)\}.$$

If $[D]$ is not effective, then we set $\mathrm{rk}(D) = -1$. Motivated by the classical result in the theory of algebraic curves, Baker and Norine [4] exhibited a Riemann–Roch theorem for graphs.

Theorem 2.1 (Riemann–Roch for graphs). *Let D be a divisor on G. Then*

$$\mathrm{rk}(D) - \mathrm{rk}(K_G - D) = \deg(D) - g + 1,$$

where $K_G = \sum_{v \in V(G)} (\mathrm{val}(v) - 2)(v)$.

2.1. Reduced Divisors and Dhar's Burning Algorithm

Given a divisor D on G and a vertex v_0, we say that D is v_0-*reduced* if

(1) $D(v) \geq 0$ for all $v \neq v_0$, and
(2) every nonempty set $A \subseteq V(G)\backslash\{v_0\}$ contains a vertex v such that $\text{outdeg}_A(v) > D(v)$.

Here $\text{outdeg}_A(v)$ denotes the outdegree of v with respect to A, that is, the number of edges connecting v to a vertex not in A. Every divisor is equivalent to a unique v_0-reduced divisor. Moreover, a divisor class is effective if and only its reduced form is effective. As a result, reduced divisors are central to calculating ranks of divisors. There is an efficient computational procedure to yield a reduced divisor known as *Dhar's burning algorithm.*

Suppose that D is such that $D(v) \geq 0$ for all $v \neq v_0$. At each vertex $v \neq v_0$, place $D(v)$ "firefighters." Each firefighter is capable of controlling precisely one fire. Start a fire at v_0. The fire spreads through the graph, so that an edge burns if one of its endpoints burns. A vertex burns if the number of burning edges incident to it exceeds the number of firefighters placed on it. If the entire graph burns, then D is v_0-reduced. If not, we chip fire all the unburnt vertices and repeat the procedure on the newly obtained divisor. The algorithm terminates at the v_0-reduced representative. For a detailed description, see [6, Section 5.1] and [11].

3. Reduction to Rank 1

In this short section, we show that for genera up to 5 proving the Brill–Noether conjecture reduces to establishing the validity of the gonality conjecture. We take advantage of the relatively high rank of the canonical divisor for graphs of small genus.

Let G be a genus g graph, and let $D \in \text{Div}(G)$. Since $\text{rk}(D) \geq -1$, the Riemann–Roch theorem implies that $\text{rk}(D) \geq \deg(D) - g$. This inequality and the following result are sufficient to prove Conjecture 1 for $g \leq 3$.

LEMMA 3.1 ([2, Lemma 2.7]). *Let G be a graph, and let $D \in \text{Div}(G)$. If $\text{rk}(D) \geq 0$, then $\text{rk}(D - v) = \text{rk}(D) - 1$ for some $v \in V(G)$.*

The same argument may be applied to reduce the Brill–Noether existence conjecture to rank 1 in the genera of interest.

PROPOSITION 3.1. *Fix $g \geq 0$ and suppose $r \geq \lfloor g/2 \rfloor$. If $d \geq 0$ is such that $\rho(g, r, d) \geq 0$, then every graph G of genus g has a divisor D with $\deg(D) \leq d$ and $\text{rk}(D) = r$.*

COROLLARY 3.1. *Let G be a graph of genus 4 or 5. Then Brill–Noether existence conjecture holds for G if and only if the gonality conjecture does, that is, if every G of genus 4 (resp. 5) admits a degree 3 divisor (resp. 4) of rank at least 1.*

4. Brill–Noether Existence for Graphs of Genus 4

Notation. *In the rest of this paper, we will use a large number of figures. To support the exposition, divisors on graphs will be depicted by placing chips on vertices that are larger in size.*

4.1. Auxiliary Results

Let G be a connected graph. A vertex $v \in V(G)$ is said to be *topological* if it has valency at least three. A path between two topological vertices consisting solely of bivalent edges will be considered as a *topological edge*. It may be visualized as the edges of the finite graph obtained erasing all the bivalent edges. Graphs G and G' are *homeomorphic* if G' is an inflation (resp. deflation) of G obtained by placing (resp. removing) bivalent vertices on the edges of G. We reserve the Greek letter ε to denote topological edges. The *length* of a topological edge ε is equal to one more than the number of bivalent vertices on ε. It is equal to the length of the path ε in a geometric realization of G where all edge lengths are 1.

We will need the following elementary lemma.

LEMMA 4.1. *Every genus g graph has at most $2g-2$ topological vertices.*

An edge e is called a *bridge* if its removal increases the number of connected components. A graph has a (g_1, g_2)*-bridge decomposition* if it has a bridge separating two components of genera g_1 and g_2, respectively.

LEMMA 4.2 (Bridge lemma). *Let g_1 and g_2 be positive integers, at least one among which is even. If the gonality conjecture holds for all graphs of genus g_1 and g_2, then it is also true for all graphs G with (g_1, g_2)-bridge decomposition.*

Proof. Let g_1 and g_2 be as before and consider a genus g graph G with (g_1, g_2)-bridge decomposition. Let e be a bridge connecting two connected subgraphs G_1 and G_2 of genera g_1 and g_2. Let $u \in V(G_1)$ and $v \in V(G_2)$ be its endpoints. Since the gonality conjecture holds for G_1 and G_2, there exists $D_i \in \mathrm{Div}(G_i)$ of $\deg(D_i) \leq \lfloor (g_i+3)/2 \rfloor$ and $\mathrm{rk}(D_i) \geq 1$ for $i = 1, 2$. By the definition of the rank there exist effective $D_1' \sim D_1 - (u)$ and $D_2' \sim D_2 - (v)$. Therefore, $D_1' + (u) \sim D_1$ and $D_2' + (v) \sim D_2$. Set $D := D_1' + D_2' + (u)$. By firing all vertices of G_1 we see that $D \sim D_1' + D_2' + (v)$. Pick $w \in V(G)$ and without loss of generality assume that $w \in G_1$. Then $D - (w)$ is equivalent to an effective divisor, because $D_1 - (w)$ is, and we can fire $G_2 \cup \{e\}$ in place of u. Since w was arbitrary, $\mathrm{rk}(D) \geq 1$. Since g_1 and g_2 are not simultaneously odd, this concludes the proof. □

As an immediate consequence, the Brill–Noether existence conjecture holds for all graphs with $(2, 2)$-bridge decomposition.

LEMMA 4.3 (Loop lemma). *The gonality conjecture holds for any genus g graph with at least one topological loop if $g = 5$, or with at least two topological loops if $g = 4$.*

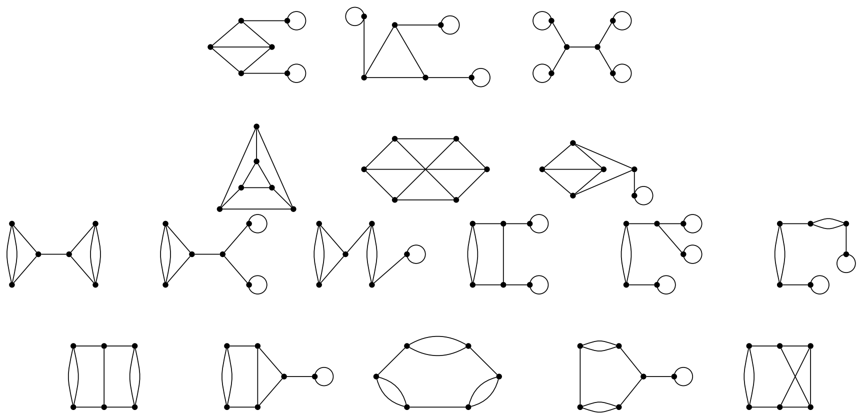


Figure 2 Topological types of trivalent genus 4 graphs, possibly with loops

Proof. Let G be a graph genus 4 with at least two loops. Suppose that two of its loops, denoted ε_1 and ε_2, are located at vertices v and w. If $v = w$, then the claim follows by the Bridge lemma, so suppose $v \neq w$. Let G' be the graph obtained by contracting both loops. It is of genus 2. If G' has only two vertices, then it must necessarily be a banana graph with two loops attached, which has a divisor of degree 3 and rank at least 1. Otherwise, G' has another vertex u, and then the divisor $D := 2 \cdot (v) + 2 \cdot (w) - (u)$ on G' has rank at least 1 by Riemann–Roch. It is not hard to see that $\mathrm{rk}(D) \geq 1$ when viewed as a divisor on G as well. A similar argument works for the genus 5 statement. □

4.2. Topologically Trivalent Graphs

A graph G is said to be *topologically trivalent* if all of its vertices have valency 2 or 3. Starting from a trivalent graph G, we elongate the edges by inserting bivalent vertices on its edges. In this manner, we produce graphs homeomorphic to G. The set of topologically trivalent graphs is in natural bijection with the integral points in the interiors of maximal dimensional cells in the moduli space $\mathcal{M}_g^{\mathrm{trop}}$ of tropical curves of genus g.

A topologically trivalent graph has genus 4 if and only if it has precisely six topological vertices. Using this characterization, we generate all trivalent graphs of genus 4, shown in Figure 2. These were verified with the help of the database [21].

We can apply the bridge and loop lemmas to all graphs homeomorphic to the ones on the first and third rows of Figure 2, obtaining the chip configurations in Figure 3. Bridges and loops are denoted by dashed lines, and the vertices decorated with nonzero integers are made large.

The next result reduces the search for divisors D with $\deg(D) = 3$ and $\mathrm{rk}(D) \geq 1$ to one of finding decompositions of graphs into appropriately chosen connected subgraphs, determined by running Dhar's burning algorithm once. Note that an

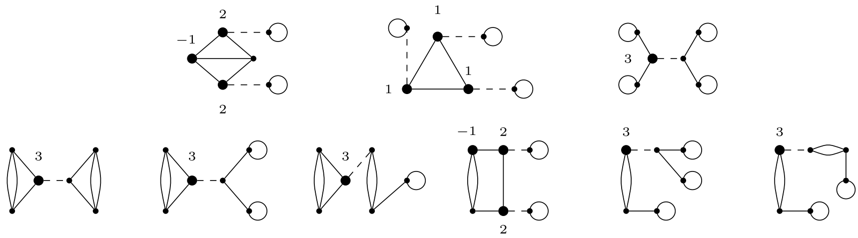


Figure 3 Genus 4 graphs with bridges or more than one loop

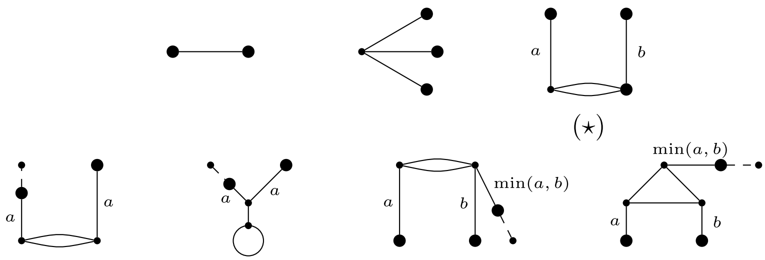
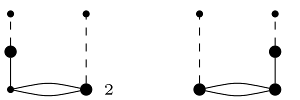


Figure 4 Possible divisors D_v on G_v. Note that G_v consists of the solid edges

effective divisor D on G is of rank at least 1 if, for any $v \in V(G)$, the algorithm applied to $D - (v)$ terminates in an effective divisor.

LEMMA 4.4. *Let G be a graph of genus* 4, *and let* $D \in \mathrm{Div}_+(G)$ *be of degree* 3. *For any* $v \in V(G) \setminus \mathrm{supp}(D)$, *run Dhar's burning algorithm for the divisor* $D - (v)$, *starting the fire at* v. *Let* G_v *be the closure of the connected graph consisting of all burnt vertices and edges on the first run of the algorithm. Let* D_v *be the restriction from* D *to* G_v. *If, for every* $v \in V(G) \setminus \mathrm{supp}(D)$, *the corresponding* G_v *and* D_v *are among those in Figure* 4, *then* $\mathrm{rk}(D) \geq 1$.

Proof. To show that $\mathrm{rk}(D) \geq 1$, choose a vertex $v \in V(G) \setminus \mathrm{supp}(D)$. By assumption, G_v and D_v are among the above configurations. Note that $\mathrm{rk}(D_v) \geq 1$ on G_v by running Dhar's burning algorithm. This can be easily checked on all cases with the possible exception of $(\star)$. For this particular case, we consider two cases: either $a \geq b$, or $a < b$. We can chip fire the configuration as shown further.



Note that at each run of the Dhar's burning for the divisor $D - (v)$, the set of unburnt vertices is contained in G_v. Thus, to find an effective representative

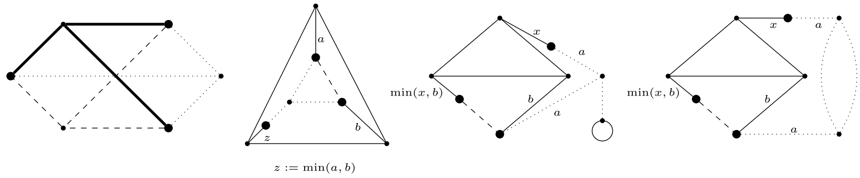


Figure 5 In the graphs, chips are placed on the large vertices. These divisors have degree 3 and rank at least 1

of $D-(v)$, we simply run Dhar's burning algorithm, chip firing the unburned vertices. Since $D_v-(v)\sim E_v$ for an effective $E_v\in \mathrm{Div}(G_v)$, it follows that $D-(v)\sim E$ for an effective $E\in \mathrm{Div}(G)$. The choice of v was arbitrary, so $\mathrm{rk}(D)\geq 1$. □

REMARK 4.1. Note that in the third (resp. in the fourth) configuration on the top row of Figure 4, we can chip fire away from the cycle (resp. the loop) until one of the two chips lands on a topological vertex. Hereafter, we assume that whenever D has a D_v among these two configurations, at least one of the two chips sits on a topological vertex. For the configurations on the second row, we assume that the chip at distance $\max(a,b)$ sits on a topological vertex.

For each of the remaining families of topologically trivalent graphs of genus 4, we separately construct a divisor D with $\deg(D)=3$ and $\mathrm{rk}(D)\geq 1$. The families are numbered in the order in which they appear in Figure 2. The first three are simple graphs with loops, and the remaining five are multigraphs. Further, in all figures, the letters a,b,c,d,x,y denote the length of the topological edge situated next to them. Many of the shown divisors have rank at least 1 as a consequence of Lemma 4.4. For the cases where Lemma 4.4 applies, we draw all edges participating in the same configuration G_v with corresponding D_v as having the same edge pattern (dotted, dashed, etc.). For each divisor, the vertices with chips are made larger.

4.2.1. Straightforward Cases. Some homeomorphic families of graphs admit a degree 3 divisor of rank at least 1, for any choice of edge length. For these families, such divisors are shown in Figure 5. They have rank at least 1 by Lemma 4.4.

4.2.2. First Family. Let b be the length of the top right topological edge, and let c be that of the bottom right topological edge. For this family, we consider three separate cases, depicted in Figure 6. The leftmost depicts the rank 1 divisor when $b\geq c$. The rank calculation follows immediately from Lemma 4.4. The next two cases depict the situation where $b<c$. Set $y=c-b$. The second one occurs when $y\geq \min(x,d)$, and the third occurs otherwise. That these divisors have rank 1 follows by running Dhar's algorithm.

4.2.3. Second Family. Place the first two chips as depicted in Figure 7. The third chip is placed at a distance $\min(x,c+b)$ from the grey vertex along the dashed

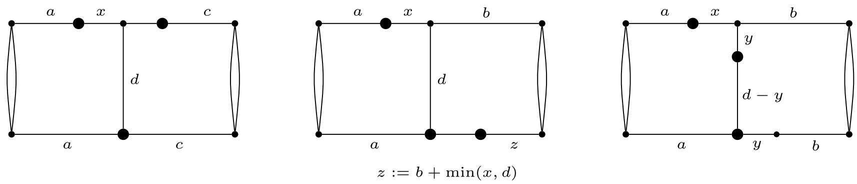


Figure 6 Degree 3 and rank at least 1 divisors on graphs of this combinatorial type, depending on the edge lengths as indicated. Lengths of edges are denoted by small letters adjacent to the edge

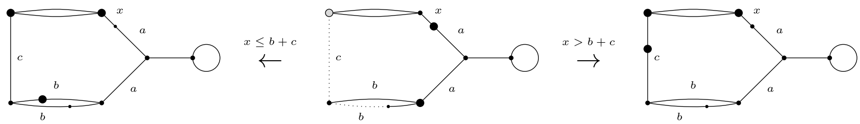


Figure 7 Degree 3 and rank at least 1 configurations on the second family. The leftmost and rightmost graphs depict the special divisor depending on which among x and $b + c$ is larger

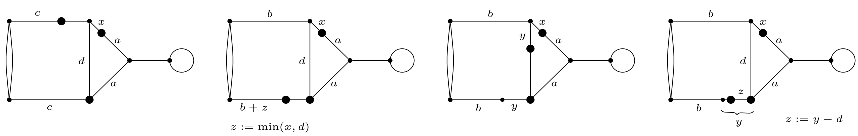


Figure 8 Cases from left to right are as follows: (1) $b \geq c$. (2) $c \geq b$ and $y \geq x$. (3) $c \geq b$, $y < x$, and $d \geq y$. (4) $c \geq b$, $y < x$, and $d < y$.

path. As in the previous case, we see that all divisors are of rank at least 1. The left and right divisors are the v-reduced representatives in the class of the special divisor, where v is the dashed vertex.

4.2.4. Third Family. For this family, we consider four cases. Let $y = c - b$, where b and c are the lengths of the top and bottom topological edges, respectively. Each divisor has rank at least 1, which follows by Dhar's burning algorithm. It suffices to run the algorithm for one vertex on each topological edge. See Figure 8.

4.2.5. Fourth Family. Let a, b, c be the lengths of the simple edges. Assume that $a \leq c \leq b$ and place the chips as shown in Figure 9. The third chips is placed at length $\min(x, c + d)$ from the gray vertex along the dotted path. To show that the divisor has rank at least 1, we run Dhar's burning algorithm for one vertex on each topological edge.

4.3. General Graphs of Genus 4 Via Edge Contractions

Let G be a genus 4 graph, and let ε be any of its topological edges, which is not a loop. Denote by G^{ε} the graph obtained by *contracting along* ε (see Figure 10), and let $\varphi^{\varepsilon} : V(G) \to V(G^{\varepsilon})$ be the *contraction map* fixing all vertices outside ε

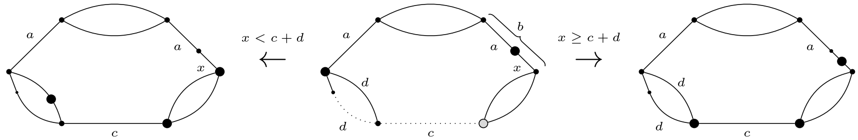


Figure 9 Degree 3 and rank at least 1 configurations on the fourth family. The leftmost and rightmost graphs depict the special divisor depending on which among x and $c+d$ is larger

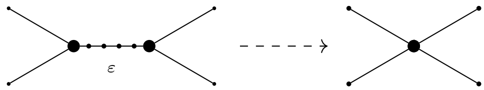


Figure 10 Edge contraction along the topological edge ε

and collapsing all vertices of ε to one. Then the *pushforward* $\varphi_*^\varepsilon : \mathbb{Z}[V(G)] \to \mathbb{Z}[V(G^\varepsilon)]$ is the map obtained by linearly extending φ^ε. For a divisor D of G, viewed as an element of $\mathbb{Z}[V(G)]$, we set $D^\varepsilon = \varphi_*^\varepsilon(D)$.

Every genus 4 graph can be obtained by (a series of) edge contractions from a suitably chosen topologically trivalent one. Furthermore, we can assume that each of these contractions is performed along a topological edge of *shortest* length.

We record the following useful observation.

LEMMA 4.5. *If G is a graph with a (g_1, g_2)-bridge decomposition and ε is a topological edge, then the graph G^ε also has (g_1, g_2)-bridge decomposition.*

This result shows that if we start with a trivalent genus 4 graph G with $(2,2)$-bridge decomposition, then any genus 4 graph G' obtained from G by repeated edge contractions satisfies the gonality conjecture. Note that edge contractions do not affect the total number of loops; hence, Lemma 4.3 applies as well.

PROPOSITION 4.1. *Let G be a genus 4 graph and let $D \in \mathrm{Div}_+(G)$ be of degree 3. Suppose all G_v and D_v, $v \in V(G)$, as defined in Lemma 4.4, are among the configurations in Figure 4. Then, for any set of topological edges $\varepsilon_1, \ldots, \varepsilon_k$ of G, the graph $G^{\varepsilon_1,\ldots,\varepsilon_k}$ admits a degree 3 divisor of rank at least 1.*

Proof. Consider an edge contraction along a topological edge ε. Given $v_0 \in (V(G) \cap \varepsilon) \setminus \mathrm{supp}(D)$, record D_{v_0}. If D_{v_0} is any of the configurations on the first row of Figure 4, D^ε, the pushforward from D to G^ε, has $\mathrm{rk}(D^\varepsilon) \geq 1$. Indeed, if ε is fully contained in G_{v_0}, then contraction along ε leaves the closure of $G \setminus G_{v_0}$ unchanged, and D_{v_0} remains of rank at least 1 under contraction along any edge. Recall the assumptions on D from the remark after Lemma 4.4.

Otherwise, suppose that v_0 is such that D_{v_0} is any of the configurations on the second row and that ε is the topological edge of length a. Consider D'_{v_0}, the divisor on G_v^ε obtained by substituting $a = 0$ for all edges of length a from the

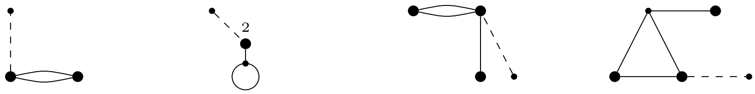


Figure 11 Edge contractions of G_{v_0}. Note that G_{v_0} consists only of the solid edges

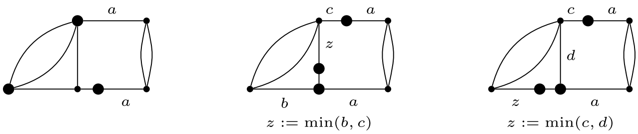


Figure 12 Edge contraction of the first family

second row of Figure 4 as shown in Figure 11. Note that $\mathrm{rk}(D'_{v_0}) \geq 1$ when viewed as a divisor on G^{ε}, since it satisfies Lemma 4.4.

We can analogously deal with all subsequent edge contractions. Indeed, by inspection we verify that each D_v from Figure 4 remains of rank at least 1 under any number of edge contractions. Suppose G admits an effective degree 3 divisor D such that all D_v are from the first row of Figure 4. Arguing as before, we see that so does the graph G^{ε}. Proceeding by induction, the same holds after arbitrarily many contractions. If G does not admit such divisor, then by degree consideration it admits a degree 3 divisor of rank at least 1 with *precisely* one D_v among the configurations on the second row of the same figure. Let ε be the topological edge of shortest length within this particular D_v. If ε is part of the cycle (resp. is a side of the triangle), then the closure $G \backslash G_{v_0}$ is unaffected after contracting along ε, thus remaining of rank at least 1. Otherwise, we consider D'_v obtained by setting $a = 0$ as before. Note that G^{ε} now admits D with all D_v among the first row. □

All degenerations of graphs in Section 4.2.1 admit a degree 3 divisor of rank at least 1 according to Proposition 4.1. If G belongs to any of the homeomorphic families from Sections 4.2.1, then G^{ε} satisfies the conditions of Lemma 4.1. Degenerations of the remaining homeomorphic families are considered independently. We note that the divisors presented further all satisfy the conditions of Proposition 4.1.

Let G belong to the first family. If ε is the middle (vertical) topological edge, then G^{ε} has $(2, 2)$-bridge decomposition. Otherwise, G^{ε} is as shown further. For each case, we present a degree 3 divisor of rank at least 1. See Figure 12

Let G belong to the second family. If ε participates in a cycle, then G^{ε} has at least two loops, and Lemma 4.3 applies. If ε is part of the fourth configuration in Figure 4, then the arguments from Section 4.2.3 are still valid. There is one remaining choice for ε, and G^{ε} is shown in Figure 13. The divisor presented is of rank at least 1.

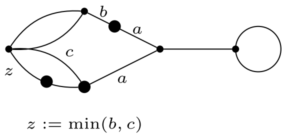


Figure 13 Edge contraction of the second family

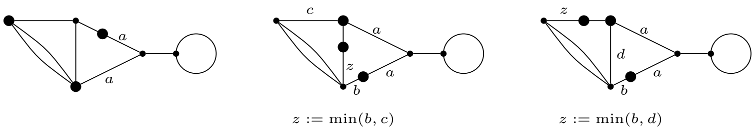


Figure 14 Edge contraction of the third family

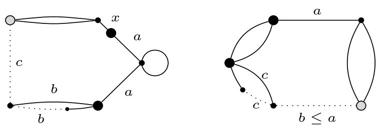


Figure 15 Possible G^{ε} for loop of loops in genus 4

Let G belong to the third family. Arguing as in the previous case, there is a single possibility for ε that needs to be examined. The contracted graph G^{ε} is shown in Figure 14. For each case, the divisor presented is of rank at least 1.

Finally, let G be a loop of loops, that is, the fourth family. There are two possibilities for ε: either it participates in a loop, or it connects two loops. Both cases are considered in Figure 15. The last chip is placed at the distance $\min(x, b + c)$ from the gray vertex in the first case and at the distance $\min(a, b + c)$ in the second. Both divisors depicted have rank at least 1, which can be verified using Dhar's burning algorithm.

The graph on the left in Figure 15 can also be obtained as a degeneration from the second family, and the one on the right as a degeneration from the last family in Section 4.2.1. Therefore, the gonality conjecture holds for all of degenerations of the loop of loops. We have exhausted all graphs of genus 4 and thus confirmed the gonality conjecture in genus 4. Combined with Proposition 3.1, we deduce the following:

THEOREM 4.6. *The Brill–Noether existence conjecture holds for all graphs of genus* 4.

5. Graphs of Genus 5

In this section, we prove the Brill–Noether existence for graphs of genus 5. In light of Corollary 3.1, it suffices to construct a divisor D of degree 4 and $\mathrm{rk}(D) \geq 1$ for

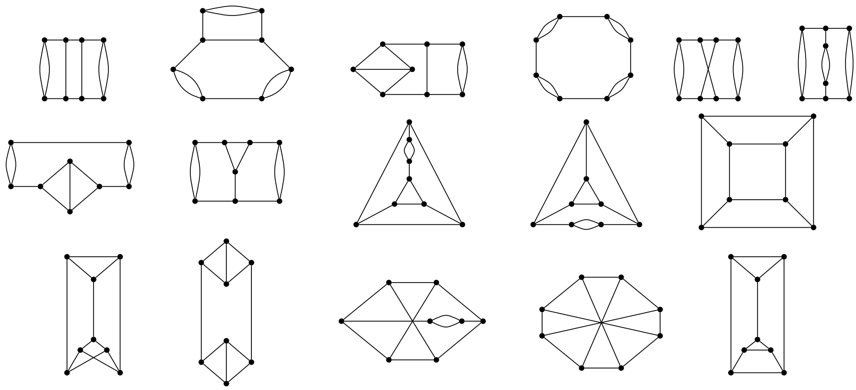


Figure 16 The topological types of trivalent genus 5 graphs with no bridges or loops

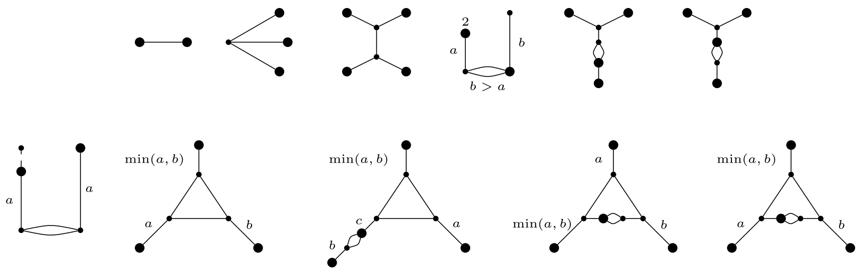


Figure 17 Possible divisors D_v on graphs G_v. Topological edges are allowed to have arbitrary length, unless otherwise indicated

every genus 5 graph G. As in genus 4, we begin with topologically trivalent graphs and extend the constructions to general graphs of genus 5 via edge contractions.

5.1. Topologically Trivalent Graphs

By applying Lemmas 4.2 and 4.3 we study only topologically trivalent graphs of genus 5 that have no bridges or loops. The graphs are depicted in Figure 16.

The following result, in the spirit of Lemma 4.4 and Proposition 4.1, not only produces degree 4 divisors of rank at least 1 but also deals with multiple edge contractions.

Proposition 5.1. *Let G be a graph of genus* 5, *and let $D \in \mathrm{Div}_+(G)$ be of degree* 4. *For any $v \in V(G) \setminus \mathrm{supp}(D)$, let G_v and D_v be defined as in Lemma* 4. *If all G_v and D_v are among the following, then*

(1) $\mathrm{rk}(D) \geq 1$, *and*
(2) *for any set of topological edges $\varepsilon_1, \ldots, \varepsilon_k$ of G, the graph $G^{\varepsilon_1, \ldots, \varepsilon_k}$ admits a degree 4 divisor of rank at least* 1.

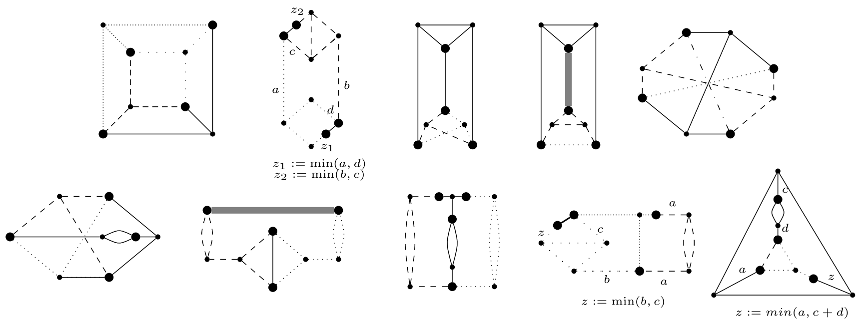


Figure 18 Topological types of genus 5 graphs with a divisor of degree 4 and rank at least 1, independent of edge lengths. The edges highlighted in gray belong to more than one of the subgraphs G_v

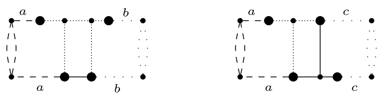


Figure 19 Degree 4 divisors on the first genus 5 family broken up into the cases (1) $b < c$ and (2) $b \geq c$

Proof. The proof of part (1) is similar to that of Lemma 4.4, and the proof of part (2) is similar to that of Proposition 4.1. □

We apply Proposition 5.1 to produce degree 4 divisors of rank at least 1 for the remaining families. For each graph G, the subgraphs G_v and their corresponding divisors D_v as defined before will be drawn in different edge patterns (dotted, dashed, etc.). Only two families do not fall into the scope of Proposition 5.1, and for them, we explicitly produce divisors of desired degree and rank. In Section 4.3, we deal with edge contractions performed on graphs from these two families.

5.1.1. Straightforward Cases. Many of the topological types of genus 5 graphs from Figure 16 admit, for all edge lengths, a degree 4 divisor D satisfying Proposition 5.1. These graphs and their divisors are depicted in Figure 18.

5.1.2. First Family. For this family, we consider two cases. In both cases, the depicted divisor has rank at least 1 according to Proposition 5.1, and depicted in Figure 19.

5.1.3. Second Family. Place the first three chips as shown below and the fourth at the distance $\min(a, c + d)$ from the grey vertex along the dashed arrow. See Figure 20.

5.1.4. Fourth Family. There are two possible constructions for $D \in \mathrm{Div}(G)$ depending on the relative position of the two longest topological edges. In the first case, the longest two topological edges are adjacent, and we may assume that $b \geq a \geq \max(c, d)$. The last chip is placed at the distance $\min(d + e, x)$ from the

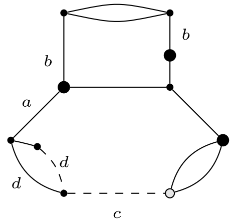


Figure 20 The degree 4 divisor is obtained by placing a fourth chip at the distance $\min(a, c + d)$ from the grey vertex along the dashed edge

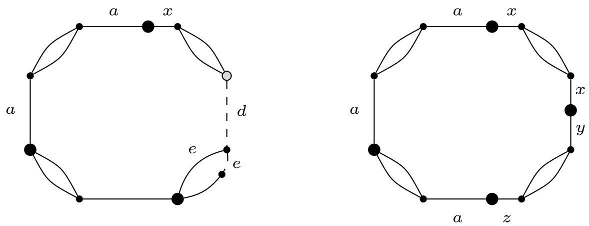


Figure 21 A configuration of 4 chips having rank at least 1 on the loops of loops

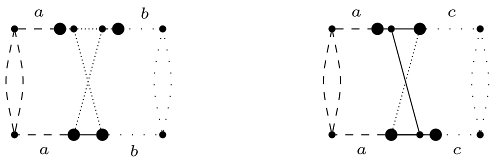


Figure 22 Degree 4 configurations on the sixth family having rank at least 1, depending on the edge lengths as above

gray vertex, as indicated by the dashed line. In the second case, the longest two topological edges are not adjacent, and suppose $b \geq c \geq a \geq d$. Note that in this case $|y - z| \leq a + x = b$. Verifying that this divisor has rank at least 1 is done as in Section 4.2.5. See Figure 21

5.1.5. Sixth Family. For this family, we consider three cases. For each case, the depicted divisor further has rank at least 1 according to Proposition 5.1. See Figure 22.

5.1.6. Seventh Family. For this family, we consider three cases. For each case, the depicted divisor further has rank at least 1 according to Proposition 5.1. Note that the last divisor has two chips placed on the same vertex. See Figure 23.

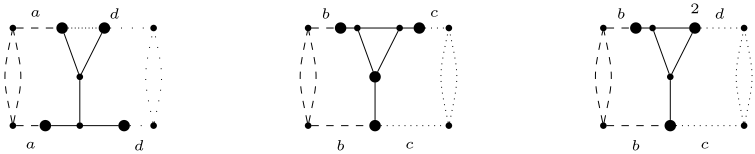


Figure 23 Degree 4 divisors on the seventh genus 5 family broken up into cases (1) $a \leq b, d \leq c$, (2) $b \leq a, c \leq d$, and (3) $b \leq a, d \leq c$

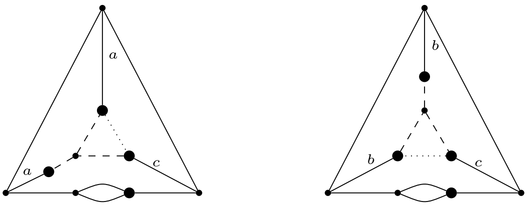


Figure 24 Degree 4 divisors on the ninth genus 5 family broken up into the cases (1) $a := \min(a, b, c)$ and (2) $b := \min(a, b, c)$

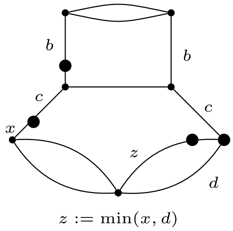


Figure 25 Edge contraction of second family and a degree 4 divisor of rank at least 1

5.1.7. Ninth Family. For this family, we consider two cases. In both cases, the divisor shown further has rank at least 1 according to Proposition 5.1. See Figure 24.

5.2. General Graphs of Genus 5 Via Edge Contractions

Since Proposition 5.1 allows for multiple contractions, we only need examine the families for which it does not apply. These are the second and the fourth.

5.2.1. Edge Contractions to the Second Family. Examining both cases in Section 5.1.3, we need to consider only contractions of the uppermost edge connecting the two loops. We perform the contraction and place the chips as shown further. The new divisor satisfies Proposition 5.1 and thus remains of rank at least 1 under repeated edge contractions. See Figure 25.

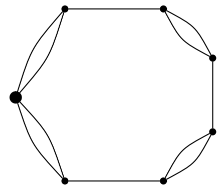


Figure 26 Contraction of a topological edge in loops of loops

5.2.2. Edge Contractions of the Fourth Family. Note that edge contraction of any topological edge participating in a cycle produces a loop, and then the existence of the desired divisor follows from the bridge lemma. Therefore, we can contract only a topological edge connecting two loops, as illustrated further. We place the chip as shown further, and the remaining three chips can be placed according to the construction of Section 4.2.5 as if the cycle of two loops were one loop. See Figure 26.

6. Graphs of High Genus

In this final section, we record some infinite families of graphs of increasing genus for which the existence conjecture holds in rank 1. The main results of this section are Theorem 6.2 and Proposition 6.2.

6.1. Complete and Complete k-Partite Graphs

Suppose G is a graph homeomorphic to K_n, the complete graph on n vertices. We can place one chip on all but one of its topological vertices and obtain a divisor D of rank at least one. Note further that $\deg(D) \leq \lfloor \frac{g(K_n)+3}{2} \rfloor$, where $g(K_n) = \frac{(n-1)(n-2)}{2}$ is the genus of K_n.

Proposition 6.1. *Let $n_1 \leq \cdots \leq n_s$ be integers. Suppose G is a graph homeomorphic to the complete s-partite graph $K_{n_1,\ldots,n_s}$. Then G admits a divisor D of degree $\sum_{i=1}^{s-1} n_i$ and rank at least one. Furthermore, the gonality of G is precisely $\sum_{i=1}^{s-1} n_i$.*

Proof. Let $\{V_l\}_{l=1}^{s}$ with $|V_l| = n_l$ partition the set of topological vertices so that two vertices in V_i and V_j are connected along a topological edge if and only if $i \neq j$. Then, consider the divisor

$$D = \sum_{v \in V_1 \cup \cdots \cup V_{s-1}} (v).$$

It has $\deg(D) = \sum_{i=1}^{s-1} n_i$ and rank at least one by running Dhar's burning algorithm. That this is the gonality follows from [12, Thm. 2]. □

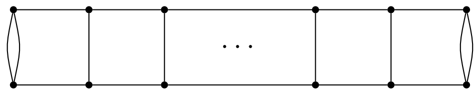


Figure 27 Ladder graph of genus g

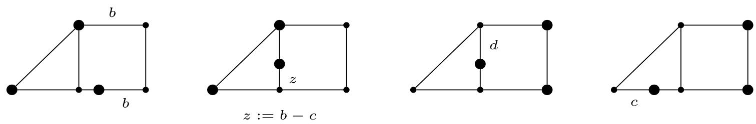


Figure 28 Divisors with desired properties for all edge lengths

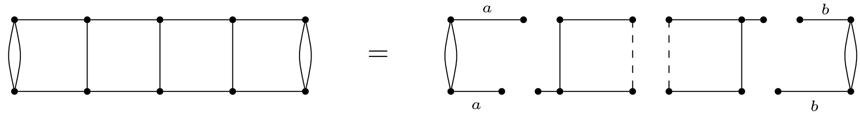
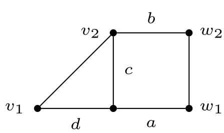


Figure 29 Decomposition of a genus 6 ladder graph. The dashed edges are identified

6.2. Ladder Graph

Let G be homeomorphic to the genus g ladder graph from Figure 27. In this section, we show that G supports a divisor of degree $\lfloor (g+3)/2 \rfloor$ and rank at least 1. Note that the genus g ladder graph has $g-3$ vertical edges, 2 cycles, and $g-2$ cells, where we do not count the two end cycles as cells.

LEMMA 6.1. *Let G be the graph shown further with edge lengths $a, b, c, d \in \mathbb{N}$. Denote by v_1, v_2 and w_1, w_2 the leftmost and rightmost pairs of vertices, respectively. Then there exists $D \in \mathrm{Div}(G)$ of $\deg(D) = 3$ and $\mathrm{rk}(D) \geq 1$ such that $[D - v_1 - v_2]$ and $[D - w_1 - w_2]$ are effective.*



Proof. There are four cases to be examined. The order in which they appear in Figure 28 from left to right is: (1) $a > b$; (2) $a \leq b < a + \min(c, d)$; (3) $a + \min(c, d) < b$ and $c \leq d$; and (4) $a + \min(c, d) < b$ and $d < c$. Verifying that each divisor has the desired properties follows by running Dhar's burning algorithm. □

The motivation behind this result comes from the decomposition of graphs homeomorphic to the genus 6 ladder graph shown in Figure 29 and Figure 30.

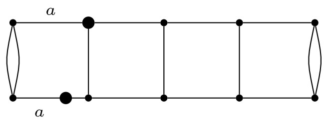


Figure 30 Ladder graph of genus 6

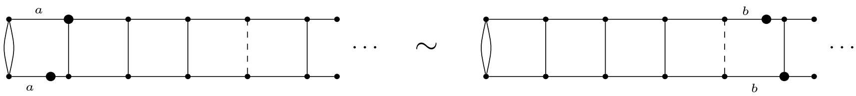


Figure 31 A chip configuration on a cluster of the ladder

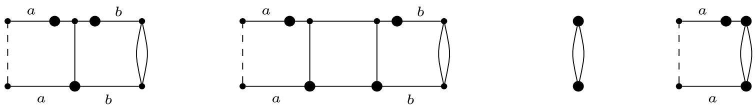


Figure 32 Remaining cells for ladder graph of genus $g = 4k + t$, $t \in \{0, 1, 2, 3\}$

In light of Lemma 6.1, the two middle components allow us to perform chip firing moves and advance chips from left to right. More precisely, the lemma asserts that we can place two additional chips on the divisor depicted further and obtain a divisor of rank at least 1.

Therefore, we can place two more chips somewhere on the first four cells so that the two divisors further are equivalent.

Let us call a *cluster* each configuration of four consecutive cells. Let us also index the cells from left to right, so that the leftmost is numbered 1, the one on its right is numbered 2, and so on. The observation before Figure 31 shows that we can place two chips in each cluster spanning cells numbered from $4m + 1$ to $4m + 4$, where $0 \leq m \leq \lfloor \frac{g-2}{4} \rfloor$, and can chip fire to a configuration with two chips in the cell numbered with $4\lfloor \frac{g-2}{4} \rfloor + 1$. To finish the argument, we only need to examine four cases depending on the residue g modulo 4.

Suppose $g = 4k$. Then $\lfloor (g+3)/2 \rfloor = 2k + 1$, and G has $4k - 2$ cells. We place two chips in each of the $\lfloor (4k - 2)/4 \rfloor = k - 1$ clusters. We place the remaining $2k + 1 - 2(k - 1) = 3$ chips as depicted in the leftmost graph in Figure 32, which portrays only the remaining cells that are not part of any cluster. We analogously deal with the cases $g = 4k + t$ for $t \in \{1, 2, 3\}$. The placement of the last chips for these cases are shown in Figure 32. The dashed line indicates the rightmost edge of the last cluster.

To summarize, we have obtained the following:

THEOREM 6.2. *Given a graph G, which is homeomorphic to the genus g ladder graph, there exists a divisor D on G of degree $\lfloor (g + 3)/2 \rfloor$ and rank at least* 1.

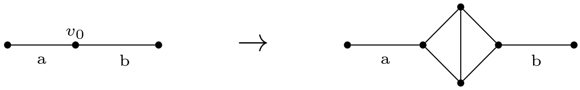


Figure 33 Kite insertion at v_0

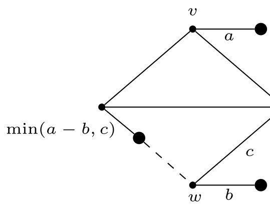


Figure 34 Placing the additional chip on the kite

6.3. Inserting Kites to Graphs

In this section, we use our knowledge of graphs and their divisors for genera up to 5 to produce graphs of arbitrary high genus for which the gonality conjecture holds. We do so by inserting a kite graph on appropriately chosen vertices as depicted in Figure 33.

PROPOSITION 6.2. *Let G be a genus g graph, and let $D \in \mathrm{Div}_+(G)$ be of degree at most $\lfloor (g+3)/2 \rfloor$. Suppose D_v and G_v, defined as in Proposition 4.1, are among the ones in Figure 17, such that no two configurations from the second row share a common vertex. Then, for any bivalent $v_0 \in V(G)$ with D_{v_0} belonging to the first row of the same figure, we can insert a kite at v_0, and the newly obtained genus $(g+2)$ graph (as well as any of its contractions) admits a divisor of degree at most $\lfloor (g+5)/2 \rfloor$ and rank at least 1.*

Proof. Inserting a kite at a vertex v_0 increases the genus by 2, so it is enough to place one additional chip on a vertex in the kite and show that the newly obtained divisor is of rank at least 1. We place the last chip as shown in Figure 34, assuming that $a \geq b$.

The most delicate part is calculating the lengths a and b since there might be some trivalent vertices through which the chips pass before reaching the kite endpoints at v or w. Since these distances should be independent of the size of the kite inserted, we compute them as follows. Starting with G and D as before, we insert a kite at $v_0 \in V(G) \setminus \mathrm{supp}(D)$ with all edge lengths longer than the sum of the lengths of the elongated edges of G. We then place two chips, one at each vertex v and w, and run Dhar's burning algorithm for the divisor $D-(u)$. Here D is viewed as a divisor on the new graph, and u is any of the trivalent vertices of the kite, different from v and w (see Figure 35). Record at which run of the algorithm a second chip reaches v and w, respectively. These numbers are a and b. Both numbers are well-defined by our choice of edge lengths of the kite.

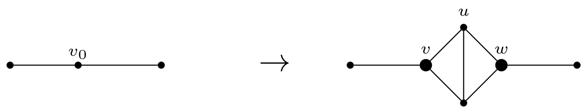


Figure 35 Chip configurations for the insertion of a kite

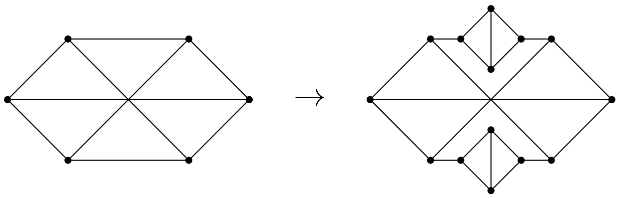


Figure 36 The bipartite $K_{3,3}$ with two kites inserted

The newly obtained divisor has rank at least 1. Indeed, $G \backslash G_{v_0}$ remains unaffected by the kite insertion and the new divisor on G', the graph obtained from G_{v_0} by inserting kite at v_0, is also of rank at least 1 as can be seen by running Dhar's burning algorithm. The edge contractions are dealt with as in the proof of Proposition 5.1. $\square$

REMARK 6.1. The same holds for any v_0 with D_{v_0} belonging to the second row as long as v_0 lies on one of the topological edges within the triangle. In this case, however, we are not always guaranteed the existence of divisors with prescribed rank and degree for its contractions. The ideas in the proof of this proposition can be modified to allow kite insertions in other families of graphs. The authors did not pursue these ideas.

For instance, as a consequence of Proposition 6.2, we can insert two kites to the bipartite graph $K_{3,3}$ and obtain a graph of genus 8 that, for any edge lengths, admits a degree 5 divisor of rank at least 1. See Figure 36.

Kite graphs are not the only ones we can insert. With similar arguments, we obtain the following:

PROPOSITION 6.3. *Let g be an even integer. Let G be a genus g graph, and let $D \in \mathrm{Div}_+(G)$ be of degree at most $\lfloor (g+3)/2 \rfloor$. Suppose D_v and G_v, defined as in Proposition 4.1, are among the ones in Figure 17, such that no two configurations from the second row share a common vertex. Then, for any bivalent $v_0 \in V(G)$ with D_{v_0} belonging to the first row of the same figure, we can insert a cycle at v_0, and the newly obtained genus $(g+1)$ graph (as well as any of its contractions) admits a divisor of degree at most $\lfloor (g+4)/2 \rfloor$ and rank at least* 1.

Proof. Note that $\lfloor (g+4)/2 \rfloor = \lfloor (g+3)/2 \rfloor + 1$ for even g. Thus we have one additional chip to place. We place it on one of the endpoints of the inserted cycle and obtain a divisor with prescribed degree and rank. The details are omitted. $\square$

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