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The McKay Correspondence, Tilting, and Rationality

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Abstract. We consider the problem of comparing t-structures under the derived McKay correspondence and for tilting equivalences. In low-dimensional cases, we relate the t-structures via torsion theories arising from additive functions on the triangulated category. As an application, we give a criterion for rationality for surfaces with a tilting bundle. We also show that every smooth projective surface that admits a full, strong, and exceptional collection of line bundles is rational.

1. Introduction

In this paper, we analyze two important families of derived equivalences: the derived McKay correspondence ([KV00; BKR01]) and (projective) tilting equivalences (see Section 5). In each of these families the derived equivalences have the form

$$\Phi\colon \mathbf{D}(X) \to \mathbf{D}(\mathbf{A}),$$

where X is a smooth quasi-projective variety, and $\mathbf{A}$ is the category of modules over a certain algebra. For our examples, we describe a precise relationship between $\Phi(\mathbf{coh}(X))$ and $\mathbf{A}$ in Theorems 4.5, 4.6, 4.7, and 5.5. The statements of these results are somewhat technical, so in this introduction, we present a toy example that has all the features of our general theorems.

Example. Consider $X = \mathbb{P}^1$, $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(1)$, and $\mathsf{A} = \operatorname{End}(\mathcal{O}, \mathcal{O}(1))$. The bundle $\mathcal{E}$ is an example of a tilting bundle, and the functor $\Phi(-) = \mathbf{R}\operatorname{Hom}(\mathcal{E}, -)$ is an example of a tilting equivalence between $\mathbf{D}(X)$ and $\mathbf{D}(\mathbf{A})$, where $\mathbf{A} = \mathbf{mod}\text{-}\mathsf{A}$. Any sheaf $\mathcal{F}$ on X can be written as a direct sum of sheaves of the form $\mathcal{O}_Z$ and $\mathcal{O}(n)$, where $Z \subset \mathbb{P}^1$ is a subscheme of finite length. Clearly $\Phi(\mathcal{O}_Z) \in \mathbf{A}$. If $n \geq -1$, then $\Phi(\mathcal{O}(n)) \in \mathbf{A}$, and otherwise it belongs to $\mathbf{A}[-1]$. The algebra A is the path algebra of the Kronecker quiver that has two vertices and two arrows from one to the other. To give a finite-dimensional (right) module over it is to give a pair of vector spaces (M_0, M_1) and a pair of linear maps $\phi_x, \phi_y\colon M_1 \to M_0$. Now, the interested reader can readily check that the modules $\Phi(\mathcal{O}_Z)$ and $\Phi(\mathcal{O}(n))$ for $n \geq -1$ are (up to isomorphism) precisely the indecomposable A-modules where $\dim(M_0) \geq \dim(M_1)$. (For one of these sheaves $\mathcal{F}$, $\Phi(\mathcal{F}) = M_0 \oplus M_1$ where $M_0 = \operatorname{Hom}(\mathcal{O}, \mathcal{F}) = \mathrm{H}^0(\mathcal{F})$ and $M_1 = \operatorname{Hom}(\mathcal{O}(1), \mathcal{F}) = \mathrm{H}^0(\mathcal{F}(-1))$.) On the other hand, the modules $\Phi(\mathcal{O}(n))\,[1]$ for $n < -1$ are the indecomposable A-modules where $\dim(M_1) > \dim(M_0)$. Let $\mathbf{F}$ and $\mathbf{T}$ denote the categories of modules whose indecomposable summands satisfy $\dim(M_0) \geq \dim(M_1)$ and

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$\dim(M_0) < \dim(M_1)$, respectively. Then we can express $\Phi(\mathbf{coh}(X))$ as an extension closure (2.8):

$$\Phi(\mathbf{coh}(X)) = \mathbf{F} * \mathbf{T}[-1].$$

Notice that the function $\theta(M) = \dim(M_1) - \dim(M_0)$ descends to a homomorphism $\theta\colon K_0(\mathbf{mod}\text{-}\mathsf{A}) \to \mathbb{Z}$ and that $\mathbf{T}$ and $\mathbf{F}$ are entirely described in terms of this function θ.

The formation of the subcategory $\mathbf{F} * \mathbf{T}[-1] \subset \mathbf{D}(\mathbf{A})$ is a very special case of a construction called tilting (not to be confused with "tilting" in "tilting equivalence"). Our analysis shows that in the examples we consider, the relation between $\mathbf{coh}(X)$ and $\mathbf{A}$ can be described in terms of tilting by using certain homomorphisms $K_0(\mathbf{A}) \to \mathbb{Z}$. We call these homomorphisms "weak central charges" since their definition is similar to but less restrictive than that of Bridgeland [Bri07]. Our results are parallel to a result obtained by Huybrechts [Huy08] for Fourier–Mukai equivalences between K3 surfaces.

We apply the results of our analysis of tilting equivalences to a rationality problem. In all known examples of tilting equivalences, X is a rational variety. An important open question in the area is whether a variety that has a tilting equivalence must be rational (or at least satisfy $\kappa(X) = -\infty$). We prove two results in this direction. Our first result (Theorem 5.5) is that the existence of a certain weak central charge on the derived category of a variety X with a tilting bundle implies that it satisfies $\kappa(X) = -\infty$. Our second rationality result is simple enough to state fully here:

THEOREM. *If a smooth projective surface X admits a tilting bundle that is a direct sum of line bundles, then X is rational.*

Conventions

We will abide by the following conventions. Let $\mathbf{A}$ be an Abelian category. Then $\mathbf{D}(\mathbf{A})$ is the bounded derived category of $\mathbf{A}$. For an Abelian group G, an *additive* function $\phi\colon \mathbf{A} \to G$ is a function of objects in $\mathbf{A}$ such that, for any short exact sequence

$$0 \to a \to b \to c \to 0,$$

we have $\phi(b) = \phi(a) + \phi(c)$.

All the categories with which we work are small categories. So for convenience and brevity, we use set-theoretic notation to work with torsion pairs and so on.

By a variety we mean an integral separated scheme of finite type over a field. Except for the material on the McKay correspondence (where we work over $\mathbb{C}$), we work over a general algebraically closed field $\mathbf{k}$ of characteristic zero.

2. Background

There are several standard notions from homological algebra and representation theory that we need to formulate our results. In this section, we collect definitions of t-structures, torsion pairs, and stability conditions.

Definition 2.1 ([BBD82], 1.3.1). Let $(\mathbf{D}^{\le 0}, \mathbf{D}^{\ge 0})$ be a pair of full subcategories of a triangulated category $\mathbf{D}$ and define $\mathbf{D}^{\le n} = \mathbf{D}^{\le 0}[-n]$ and $\mathbf{D}^{\ge n} = \mathbf{D}^{\ge 0}[-n]$. The pair $(\mathbf{D}^{\le 0}, \mathbf{D}^{\ge 0})$ is a *t-structure* if

1. $\mathbf{D}^{\le 0} \subset \mathbf{D}^{\le 1}$ and $\mathbf{D}^{\ge 1} \subset \mathbf{D}^{\ge 0}$,
2. for all $x \in \mathbf{D}^{\le 0}$ and $y \in \mathbf{D}^{\ge 1}$, $\mathrm{Hom}_{\mathbf{D}}(x, y) = 0$, and
3. for any $x \in \mathbf{D}$, there exists a triangle
$$x' \to x \to x'' \to x'[1]$$
where $x' \in \mathbf{D}^{\le 0}$ and $x'' \in \mathbf{D}^{\ge 1}$.

The *heart* of the t-structure is the full subcategory $\mathcal{A} = \mathbf{D}^{\le 0} \cap \mathbf{D}^{\ge 0} \subset \mathbf{D}$.

We work exclusively with *bounded* t-structures. A t-structure is bounded if every object of $\mathbf{D}$ is contained in $\mathbf{D}^{\le i} \cap \mathbf{D}^{\ge j}$ for some integers i, j.

Example 2.2 (Standard t-structure [BBD82], 1.3.2). If $\mathbf{A}$ is an Abelian category, then $\mathbf{D}(\mathbf{A})$ has a natural t-structure $(\mathbf{D}^{\le 0}, \mathbf{D}^{\ge 0})$ where $\mathbf{D}^{\le 0}$ and $\mathbf{D}^{\ge 0}$ are the full subcategories of complexes with cohomology supported in nonpositive and nonnegative degrees, respectively. In this case, the heart consists of those complexes whose cohomology is concentrated in degree zero. Thus, the heart of the standard t-structure is the essential image of the natural fully faithful functor $\mathbf{A} \to \mathbf{D}(\mathbf{A})$.

Convention 2.3. We reserve the font $\mathcal{H}^i$ to denote the cohomological functors associated with the standard t-structure on $\mathbf{D}(\mathbf{A})$.

Suppose that $(\mathbf{D}^{\le 0}, \mathbf{D}^{\ge 0})$ is a t-structure on a triangulated category $\mathbf{D}$ with heart $\mathcal{A}$. By [BBD82, 1.3.6], $\mathcal{A}$ is an Abelian category. Moreover, there is a theory of cohomologies of objects in $\mathbf{D}$ taking values in $\mathcal{A}$. The construction of this theory begins with [BBD82, 1.3.3], which states that the inclusions $\mathbf{D}^{\le 0} \subset \mathbf{D}$ and $\mathbf{D}^{\ge 0} \subset \mathbf{D}$ admit a right adjoint $\tau_{\le 0}$ and a left adjoint $\tau_{\ge 0}$, respectively.

Definition 2.4 ([BBD82], 1.3.6). The *cohomological functor* associated with $(\mathbf{D}^{\le 0}, \mathbf{D}^{\ge 0})$ is
$$\mathsf{H}^0 = \tau_{\ge 0}\tau_{\le 0} : \mathbf{D} \to \mathcal{A}.$$

We furthermore define $\mathsf{H}^i = \mathsf{H}^0 \circ [i]$ for $i \in \mathbb{Z}$. These functors have the property that for any triangle
$$x \to y \to z \to x[1]$$
in $\mathbf{D}$, the sequence
$$\mathsf{H}^0(x) \to \mathsf{H}^0(y) \to \mathsf{H}^0(z)$$
is exact in $\mathcal{A}$. Thus a triangle as before gives rise to a long exact sequence
$$\cdots \to \mathsf{H}^{-1}(z) \to \mathsf{H}^0(x) \to \mathsf{H}^0(y) \to \mathsf{H}^0(z) \to \mathsf{H}^1(x) \to \cdots.$$

Remark 2.5. The cohomology functors defined by the standard t-structure agree with those defined by formation of cohomology of complexes.

In this work, we are primarily interested in the situation where we have an equivalence of triangulated categories $\Phi\colon \mathbf{D} \to \mathbf{D}'$ and both source and target are equipped with t-structures. We study the exactness properties of Φ with respect to the t-structures. It turns out that these exactness properties can be related to structures in the hearts of our t-structures. Toward this end, consider the following well-known notion.

DEFINITION 2.6 ([Dic66]). Let $\mathbf{A}$ be an Abelian category. A *torsion pair* in $\mathbf{A}$ is a pair $(\mathbf{T}, \mathbf{F})$ of full subcategories satisfying

1. for any $a \in \mathbf{A}$, there exists a short exact sequence
$$0 \to a' \to a \to a'' \to 0$$
where $a' \in \mathbf{T}$ and $a'' \in \mathbf{F}$, and
2. for any $a' \in \mathbf{T}$ and $a'' \in \mathbf{F}$, $\mathrm{Hom}_{\mathbf{A}}(a', a'') = 0$.

The axioms of t-structures and torsion pairs are rather similar. In fact, work of Happel and an observation by Woolf provide a tight connection between t-structures and torsion pairs. A torsion pair in the heart of a t-structure can be used to define a new t-structure. Moreover, there is a condition that allows us to recognize when two t-structures are related by this construction.

DEFINITION 2.7 ([HRS96], §1, Proposition 2.1). Suppose $\mathbf{D}$ is a triangulated category with a t-structure having heart $\mathcal{A}$ and cohomological functor H^0. Let $(\mathbf{T}, \mathbf{F})$ be a torsion pair in $\mathcal{A}$. The *tilt of* $\mathcal{A}$ with respect to the torsion pair is the full subcategory
$$\mathcal{A}' = \{x \in \mathbf{D} : \mathsf{H}^0(x) \in \mathbf{T}, \mathsf{H}^{-1}(x) \in \mathbf{F},\ \text{and } \mathsf{H}^i(x) = 0 \text{ for } i \neq -1, 0\}.$$

REMARK 2.8. Observe that $\mathcal{A}'$ is the *extension-closure* $\mathbf{F}[1] * \mathbf{T}$, the collection of objects that fit into a triangle of the form
$$f \to x \to t \to f[1]$$
where $f \in \mathbf{F}[1]$ and $t \in \mathbf{T}$. (See [BBD82] for the properties of extension closures, including *associativity*.)

REMARK 2.9. In Definition 2.7, the tilted category $\mathcal{A}'$ contains $\mathbf{T}$ and $\mathbf{F}[1]$. In fact, $(\mathbf{F}[1], \mathbf{T})$ is a torsion pair in $\mathcal{A}'$, and the tilt of $\mathcal{A}'$ with respect to this torsion pair is $\mathcal{A}[1]$.

REMARK 2.10. In Definition 2.7, the tilted category $\mathcal{A}'$ is the heart of a bounded t-structure. (See [HRS96] for details.)

LEMMA 2.11 ([Woo10], Proposition 2.1). *Let* $\mathbf{D}$ *be a triangulated category with a* t*-structure having heart* $\mathcal{A}$ *and cohomological functors* H^i. *If* $\mathcal{A}' \subset \mathbf{D}$ *is the heart of a* t*-structure and* $\mathsf{H}^i(a) = 0$ *for all* $a \in \mathcal{A}'$ *and* $i \neq -1, 0$, *then the full* *subcategories*

$$\mathbf{T} = \mathcal{A} \cap \mathcal{A}',$$
$$\mathbf{F} = \mathcal{A} \cap (\mathcal{A}'[-1])$$

form a torsion pair $(\mathbf{T}, \mathbf{F})$ *in* $\mathcal{A}$, *and* $\mathcal{A}'$ *is the tilt of* $\mathcal{A}$ *with respect to this torsion pair.*

We need an extended version of this result, which shows when two hearts are related by two tilts:

LEMMA 2.12. *Let* $\mathbf{D}$ *be a triangulated category with a t-structure having heart* $\mathcal{A}$ *and cohomological functors* H^i. *Suppose* $(\mathbf{T}, \mathbf{F})$ *is a torsion pair in* A. *Let* $\mathcal{B} \subset \mathbf{D}$ *be the heart of a t-structure such that, for any object* $b \in \mathcal{B}$,

1. $\mathsf{H}^i(b) = 0$ *unless* $i = 0, 1, 2$,
2. $\mathsf{H}^0(b) \in \mathbf{F}$, *and* $\mathsf{H}^2(b) \in \mathbf{T}$.

Then the tilt $\mathcal{A}'$ *of* $\mathcal{A}$ *with respect to* $(\mathbf{T}, \mathbf{F})$ *is also a tilt of* $\mathcal{B}[1]$.

Proof. We view the $\mathcal{A}'$ as the extension-closure $\mathbf{F}[1] * \mathbf{T}$, as in Remark 2.8. Since $\mathcal{A}$ is the extension closure $\mathbf{T} * \mathbf{F}$, we see that the extension-closure $\mathcal{A}' * \mathcal{A}'[-1] = \mathbf{F}[1] * \mathcal{A} * \mathbf{T}[-1]$. This contains $\mathcal{B}[1]$ by hypothesis. Note that if $(\mathsf{H}')^i$ are the cohomological functors associated with the t-structure with heart $\mathcal{A}'[-1]$, then every object x of $\mathcal{A}' * \mathcal{A}'[-1]$ satisfies $(\mathsf{H}')^i(x) = 0$ unless $i = -1, 0$. So $\mathcal{B}[1]$ is a tilt of $\mathcal{A}'[-1]$ by Lemma 2.11, and therefore $\mathcal{A}'$ is a tilt of $\mathcal{B}[1]$ by Remark 2.9. $\square$

In later sections, we relate the derived equivalences in the derived McKay correspondence and the theory of tilting bundles to certain well-known tilting constructions for categories of sheaves and modules. We borrow a notion from the theory of Bridgeland stability conditions (see [Bri07]), suitably modified for our purposes.

DEFINITION 2.13. Let $\mathcal{A}$ be an Abelian category. A *weak central charge* on $\mathcal{A}$ is a homomorphism

$$\mathsf{Z}\colon K_0(\mathcal{A}) \to \mathbb{C}$$

that satisfies

1. $\psi(\mathsf{Z}(a)) \geq 0$, and
2. if $\psi(\mathsf{Z}(a)) = 0$, then $\theta(\mathsf{Z}(a)) \geq 0$,

where $\mathsf{Z} = -\theta + i \cdot \psi$.

A weak central charge can be used to define a torsion pair. The now ubiquitous construction described further essentially appeared first in [Bri08].

LEMMA 2.14. *Let* $\mathcal{A}$ *be a Noetherian, Abelian category with a weak central charge* $\mathsf{Z} = -\theta + i \cdot \psi$. *Assume that* ψ *takes values in a discrete additive subgroup of* $\mathbb{R}$.

Then the pair of full subcategories $(\mathbf{T}_{\mathsf{Z}}, \mathbf{F}_{\mathsf{Z}})$ *of* $\mathcal{A}$ *defined by*

$$\mathbf{T}_{\mathsf{Z}} = \{x \in \mathcal{A} : \text{ for all quotient objects } x \to x'', \text{ if } \psi(x'') > 0, \text{ then } \theta(x'') > 0\},$$

$$\mathbf{F}_{\mathsf{Z}} = \{x \in \mathcal{A} : \text{ for all nonzero subobjects } x' \subset x, \ \psi(x') > 0 \text{ and } \theta(x') \leq 0\}$$

is a torsion pair.

Proof. First, note that the full subcategory $\ker(\psi)$ is closed under extensions and quotients. Since $\mathcal{A}$ is Noetherian, [Pol07, Lemma 1.1.3] implies that $\ker(\psi)$ is the torsion part of a torsion pair. Given an object $x \in \mathbf{T}_{\mathsf{Z}}$, let

$$0 \to t \to x \to f \to 0$$

be the decomposition of x determined by this torsion pair. Since $\psi(t) = 0$ and $\psi(f) > 0$ when $f \neq 0$, we see that $\theta(x) \geq 0$ and $\theta(x) = 0$ if and only if $\psi(x) = 0$.

Next, observe that, by construction, $\mathbf{T}_{\mathsf{Z}}$ is closed under quotients. Let

$$0 \to x' \to x \to x'' \to 0$$

be an exact sequence where $x', x'' \in \mathbf{T}_{\mathsf{Z}}$. Suppose that $x \to z$ is a quotient and consider the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & x' \arrow[r] \arrow[d, two heads] & x \arrow[r] \arrow[d, two heads] & x'' \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & z' \arrow[r] & z \arrow[r] & z'' \arrow[r] & 0
\end{tikzcd}$$

where z' is the image of x' in z and z'' is the cokernel of $z' \to z$. The objects z', z'' belong to $\mathbf{T}_{\mathsf{Z}}$. Hence $\theta(z) = \theta(z') + \theta(z'') > 0$ unless $\psi(z') = \psi(z'') = 0$. However, in this case, $\psi(z) = 0$. We conclude that $\mathbf{T}_{\mathsf{Z}}$ is closed under extensions. A second application of [Pol07, Lemma 1.1.3] yields that $\mathbf{T}_{\mathsf{Z}}$ is the torsion part of some torsion pair.

It remains to show that $\mathbf{F}_{\mathsf{Z}}$ is the right orthogonal to $\mathbf{T}_{\mathsf{Z}}$. Clearly, $\mathbf{F}_{\mathsf{Z}}$ is contained in the right orthogonal to $\mathbf{T}_{\mathsf{Z}}$. Suppose that x is an object such that $\mathrm{Hom}_{\mathcal{A}}(y, x) = 0$ for any $y \in \mathbf{T}_{\mathsf{Z}}$. Every object in $\ker(\psi)$ belongs to $\mathbf{T}_{\mathsf{Z}}$. Therefore, ψ is positive on nonzero subobjects of x. Suppose that x admits a nonzero subobject $x' \subset x$ such that $\theta(x') > 0$. Since ψ takes values in a discrete subset of $\mathbb{R}$, we may take such an x' that minimizes ψ among all possible choices. Since x' does not belong to $\mathbf{T}_{\mathsf{Z}}$, it must admit a quotient $x' \to x''$ such that $\theta(x'') < 0$. Since θ is nonnegative on objects where ψ is zero, we see that $\psi(x'') > 0$. Put $y = \ker(x' \to x'')$ and observe that $\psi(y) < \psi(x')$ and $\theta(y) > \theta(x') > 0$, a contradiction. □

DEFINITION 2.15. Let $\mathbf{D}$ be a triangulated category equipped with a t-structure having heart $\mathcal{A}$. Let $\mathcal{A}'$ be the heart of a second t-structure. We say that $\mathcal{A}'$ is a *Harder–Narasimhan (HN) tilt* of $\mathcal{A}'$ if it is the tilt of $\mathcal{A}$ with respect to the torsion pair $(\mathbf{T}_{\mathsf{Z}}, \mathbf{F}_{\mathsf{Z}})$ associated with some weak central charge Z on $\mathcal{A}$.

The notion of Harder–Narasimhan tilt makes sense in a very general setting, abstracting the well-known structure found in categories of representations of an algebra or coherent sheaves on a variety. In these two examples, the torsion pairs associated with (certain) weak central charges can be made much more explicit. Aside from Remark 2.17 further, we will not need any details from the following two examples, though they are useful for grounding the discussion.

EXAMPLE 2.16 (King semistability). Let A be a finite-dimensional associative algebra and consider the category $\mathcal{A}$ of finitely generated A-modules. The Grothendieck group $K_0(\mathcal{A})$ is a free Abelian group in which the collection of simple A-modules forms a basis. Every finitely generated A-module has finite length. Let ℓ be the length function on $\mathcal{A}$. If θ is another additive function, then $\mathsf{Z} = -\theta + i\ell$ is a weak central charge. Given the additive function θ, there is a standard notion of semistability due to King, based on considerations from geometric invariant theory. (For details, see [Kin94].) The *slope* of an A-module M is

$$\mu(M) = \theta(M)/\ell(M).$$

An A-module is *θ-semistable in the sense of King* if for every submodule $M' \subset M$, $\mu(M') \leq \mu(M)$. Any finite-dimensional A-module M admits a unique strictly increasing filtration

$$0 = M_{-1} \subset M_0 \subset \cdots \subset M_n = M$$

such that the A-modules M_i/M_{i-1} are θ-semistable and

$$\mu(M_i/M_{i-1}) > \mu(M_{i+1}/M_i).$$

This filtration is the *Harder–Narasimhan filtration* of M. We refer to the subquotients M_i/M_{i-1} of M as its Harder–Narasimhan factors. The torsion pair $(\mathbf{T}_{\mathsf{Z}}, \mathbf{F}_{\mathsf{Z}})$ in $\mathcal{A}$ has a description in terms of Harder–Narasimhan filtrations: an A-module M belongs to $\mathbf{T}_{\mathsf{Z}}$ or $\mathbf{F}_{\mathsf{Z}}$ if and only if the slopes of its Harder–Narasimhan factors are all positive or all nonpositive, respectively.

REMARK 2.17. Suppose that $M \in \mathbf{F}_{\mathsf{Z}}$. If $\theta(M) = 0$, then M has to be semistable. Otherwise, it would have subquotients on which θ is negative.

EXAMPLE 2.18 (Mumford–Takemoto semistability). Suppose that X is a projective variety of dimension n with an ample line bundle $\mathcal{L}$. Let $\mathcal{F}$ be a nonzero torsion-free sheaf and define its slope to be

$$\mu(\mathcal{F}) = \frac{c_1(\mathcal{F}) \cdot c_1(\mathcal{L})^{n-1}}{\mathrm{rk}(\mathcal{F})}.$$

A torsion-free sheaf $\mathcal{F}$ is *semistable in the sense of Mumford–Takemoto* if for any nonzero subsheaf $\mathcal{F}' \subset \mathcal{F}$ such that $\mathrm{rk}(\mathcal{F}') < \mathrm{rk}(\mathcal{F})$, we have $\mu(\mathcal{F}') \leq \mu(\mathcal{F})$. There is a notion of Harder–Narasimhan filtrations of torsion-free sheaves. For an additive function θ of the form

$$\theta(-) = c_1(-) \cdot c_1(\mathcal{L})^{n-1} + s \cdot \mathrm{rk}(-) \quad (s \in \mathbb{R}),$$

$\mathsf{Z} = -\theta + i \cdot \mathrm{rk}$ is a weak central charge, and the torsion pair $(\mathbf{T}_{\mathsf{Z}}, \mathbf{F}_{\mathsf{Z}})$ has a description in terms of Harder–Narasimhan filtrations analogous to that in the previous example.

We end our discussion of background material by proving a persistence result for Noetherianity under Harder–Narasimhan tilts.

LEMMA 2.19. *Let $\mathcal{A}$ be a Noetherian, Abelian category, and $\mathsf{Z} = -\theta + i \cdot \psi$ a weak central charge with discrete image such that, for any $f \in \mathbf{F}_{\mathsf{Z}}$ and t such that $\mathsf{Z}(t) = 0$, $\operatorname{Ext}^1(t, f) = 0$. Then the HN tilt $\mathcal{A}'$ of $\mathcal{A}$ is Noetherian.*

Proof. Let $a \in \mathcal{A}'$ and suppose that (a_i) is an ascending chain of subobjects of a. Recall that $(\mathbf{F}_{\mathsf{Z}}[1], \mathbf{T}_{\mathsf{Z}})$ is a torsion pair in $\mathcal{A}'$. Then (a_i') is an ascending chain of subobjects in a', where a_i' and a' are the $\mathbf{F}_{\mathsf{Z}}[1]$-parts of a_i and a, respectively. Since $a_i' \to a'$ are monomorphisms in $\mathcal{A}'$, the induced map $\mathcal{H}^{-1}(a_i') \to \mathcal{H}^{-1}(a')$ is a monomorphism in $\mathcal{A}$ having cokernel in $\mathbf{F}_{\mathsf{Z}}$. Since $\mathcal{A}$ is Noetherian, the chain $(\mathcal{H}^{-1}(a_i'))$ stabilizes. So for $0 \ll i \le j$, $\mathcal{H}^{-1}(a_i') \to \mathcal{H}^{-1}(a_j')$ is an isomorphism. Thus $a_i' \to a_j'$ is an isomorphism. Discarding an initial segment of our chain, we may assume that $a_i' = a_j'$ for all i, j. Furthermore, in this setting, (a_i) stabilizes if and only if (a_i/a_0') stabilizes in a/a_0'. We reduce to the case where $a_i \in \mathbf{T}_{\mathsf{Z}}$ for all i.

Consider the exact sequence

$$0 \to \mathcal{H}^{-1}(a) \to \mathcal{H}^{-1}(a/a_i) \to \mathcal{H}^0(a_i) \to \mathcal{H}^0(a) \to \mathcal{H}^0(a/a_i) \to 0.$$

The images of $\mathcal{H}^0(a_i) \to \mathcal{H}^0(a)$ eventually stabilize to some object, which we denote b. Write $c = \mathcal{H}^{-1}(a)$. Then, for $j > i \gg 0$, we obtain the diagram

$$\begin{tikzcd} 0 \arrow[r] & c \arrow[r] \arrow[d, "="] & \mathcal{H}^{-1}(a/a_i) \arrow[r] \arrow[d, "f_{ij}"] & \mathcal{H}^0(a_i) \arrow[r] \arrow[d, "g_{ij}"] & b \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & c \arrow[r] & \mathcal{H}^{-1}(a/a_j) \arrow[r] & \mathcal{H}^0(a_j) \arrow[r] & b \arrow[r] & 0 \end{tikzcd}$$

We can deduce from the snake lemma that $\ker(f_{ij}) \cong \ker(g_{ij})$ and $\operatorname{coker}(f_{ij}) \cong \operatorname{coker}(g_{ij})$. It follows that $\ker(g_{ij}) \in \mathbf{F}_{\mathsf{Z}}$ and $\operatorname{coker}(f_{ij}) \in \mathbf{T}_{\mathsf{Z}}$. Therefore,

$$\begin{aligned} \theta(\mathcal{H}^{-1}(a/a_j)) &= \theta(\mathcal{H}^{-1}(a/a_i)) - \theta(\ker(f_{ij})) + \theta(\operatorname{coker}(f_{ij})) \\ &\ge \theta(\mathcal{H}^{-1}(a/a_i)). \end{aligned}$$

Since the objects $\mathcal{H}^{-1}(a/a_j)$ are in $\mathbf{F}_{\mathsf{Z}}$, θ is nonpositive on them. We conclude that eventually $\theta(\mathcal{H}^{-1}(a/a_i))$ is independent of i. It follows by additivity of θ in the above diagram that $\theta(\mathcal{H}^0(a_i))$ is also eventually independent of i. So for $j > i \gg 0$, $\theta(\ker(g_{ij})) = \theta(\operatorname{coker}(g_{ij}))$. Since $\ker(g_{ij}) \in \mathbf{F}_{\mathsf{Z}}$ and $\operatorname{coker}(g_{ij}) \in \mathbf{T}_{\mathsf{Z}}$, it must be the case that $\theta(\ker(g_{ij})) = \theta(\operatorname{coker}(g_{ij})) = 0$, and thus $\psi(\operatorname{coker}(g_{ij})) = 0$. If $\psi(\operatorname{coker}(g_{ij})) = 0$, then $\psi(\mathcal{H}^0(a_i)) \ge \psi(\mathcal{H}^0(a_j))$. So for sufficiently large i, j, $\psi(\mathcal{H}^0(a_i)) = \psi(\mathcal{H}^0(a_j))$. In this case, $\psi(\ker(f_{ij})) = 0$, and therefore $\ker(f_{ij}) = 0$ since $\ker(f_{ij}) \in \mathbf{F}_{\mathsf{Z}}$.

In summary, after discarding a finite initial segment of the our chain, we may assume that $\mathcal{H}^0(a/a_i) \to \mathcal{H}^0(a/a_j)$ is a monomorphism whose cokernel is in the kernel of Z. By our Ext assumption, f_{ij} is split. Finally, we note that in the following diagram, where $\mathcal{H}^{-1}(a/a_j) \cong \mathcal{H}^{-1}(a/a_i) \oplus w_{ij}$, the object w_{ij} is a summand of $\mathcal{H}^0(a_j)$:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{H}^{-1}(a/a_i)/c \arrow[r] \arrow[d] & \mathcal{H}^0(a_i) \arrow[r] \arrow[d] & b \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathcal{H}^{-1}(a/a_i)/c \oplus w_{ij} \arrow[r] & \mathcal{H}^0(a_j) \arrow[r] & b \arrow[r] & 0
\end{tikzcd}$$

Indeed, the lower row is the pushout of the upper row. Since $w_{ij} \in \mathbf{F}_{\mathsf{Z}}$ and $\mathcal{H}^0(a_j) \in \mathbf{T}_{\mathsf{Z}}$, it follows that $w_{ij} = 0$. Therefore f_{ij} is and isomorphism, and hence so is g_{ij}. $\square$

REMARK 2.20. Abramovich and Polishchuk [AP06] prove that the hearts associated with a discrete Bridgeland stability condition are Noetherian. Bayer, Macri, and Toda [BMT14] prove the above result in the case where $\mathcal{A}$ is the category of coherent sheaves on a smooth projective variety for a certain class of Z.

3. Main Lemmas

In this section, we collect some results abstracting the behavior of the families of equivalences under study. Given X a quasiprojective variety and $W \subset X$ a closed subset, we denote by $\mathbf{D}_W(X)$ the full triangulated subcategory of $\mathbf{D}(X)$ consisting of those objects whose cohomology sheaves are supported on W. Note that $\mathbf{D}_X(X) = \mathbf{D}(X)$.

Consider an equivalence $\Phi : \mathbf{D}_W(X) \to \mathbf{D}(\mathbf{A})$. We adopt the simplified notation $\Phi^i(\mathcal{F}^\bullet) = \mathcal{H}^i(\Phi(\mathcal{F}^\bullet))$ for $\mathcal{F}^\bullet \in \mathbf{D}(X)$.

DEFINITION 3.1. We say that Φ is *left exact* if $\Phi^i(\mathcal{F}) = 0$ for all $i < 0$ and $\mathcal{F} \in \mathbf{coh}(X)$.

DEFINITION 3.2. Let $\Phi : \mathbf{D}_W(X) \to \mathbf{D}(\mathbf{A})$ be a left exact functor.

1. We say that Φ *satisfies Grothendieck vanishing (GV)* if for all $\mathcal{F} \in \mathbf{coh}(X)$, $\Phi^i(\mathcal{F}) = 0$ whenever $i > \dim(\mathcal{F})$, where $\dim(\mathcal{F})$ is the dimension of the support of $\mathcal{F}$.
2. We say that Φ *satisfies Serre vanishing (SV)* if for any ample line bundle $\mathcal{L}$ on X and sheaf $\mathcal{F} \in \mathbf{coh}(X)$, there is n_0 such that if $n > n_0$, then $\Phi^i(\mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for $i > 0$.

For example, if $W = X$ and $\Phi = \mathbf{R}\mathrm{Hom}(\mathcal{E}, -)$ for a vector bundle $\mathcal{E}$, then Φ satisfies both (GV) and (SV) by the corresponding vanishing theorems in sheaf cohomology. The properties (GV) and (SV) can be thought of as expressing compatibility between the functor Φ and the geometry of X. In particular, if a left exact functor Φ satisfies (GV), then $\Phi(\mathcal{O}_x) \in \mathbf{A}$ for any closed point $x \in X$.

We are now ready to give the main technical results of the paper. Our goal is to understand how the functor Φ mixes the standard t-structures on $\mathbf{D}(\mathbf{A})$ and $\mathbf{D}_W(X)$. Ultimately, we would like to recover a description of which complexes of objects in $\mathbf{A}$ correspond to sheaves on X, and vice versa. We do so by giving a procedure for constructing $\mathbf{coh}_W(X)$ in $\mathbf{D}(\mathbf{A})$ using weak central charges and HN tilts. The first step is to establish criteria for when a torsion pair on $\mathbf{A}$ captures information about sheaves on X.

LEMMA 3.3. *Let $\Phi\colon \mathbf{D}_W(X) \to \mathbf{D}(\mathbf{A})$ be a left exact equivalence that satisfies (GV) and (SV). Suppose that $(\mathbf{T}, \mathbf{F})$ is a torsion pair in $\mathbf{A}$ such that*

1. *$\Phi(\mathcal{O}_x) \in \mathbf{F}$ for every point $x \in W$, and*
2. *for every surjection $\Phi^0(\mathcal{F}) \twoheadrightarrow a$ where $\mathcal{F}$ is a zero-dimensional sheaf supported on W and $0 \neq a \in \mathbf{F}$, there exist a point $x \in W$ and surjection $a \twoheadrightarrow \Phi^0(\mathcal{O}_x)$.*

Then, for any sheaf $\mathcal{E}$ supported on W, we have $\Phi^0(\mathcal{E}) \in \mathbf{F}$, and if $m = \dim(\mathcal{E}) > 0$, then $\Phi^m(\mathcal{E}) \in \mathbf{T}$.

Proof. The case where $\dim(W) = 0$ is trivial, so assume that $\dim(W) > 0$. Let $\mathcal{E}$ be a sheaf of dimension m supported on W. Choose a sufficiently positive ample line bundle $\mathcal{L}$ on X such that $\mathcal{L}$ is base point free and $\Phi^j(\mathcal{E} \otimes \mathcal{L}) = 0$ for $j > 0$. For a general section $\sigma \in H^0(X, \mathcal{L})$ viewed as a morphism $\mathcal{O}_X \to \mathcal{L}$, we derive an exact sequence in $\mathbf{coh}_W(X)$:

$$0 \to \mathcal{E} \to \mathcal{E} \otimes \mathcal{L} \to \mathcal{F}_\sigma \to 0.$$

We start with the statement that $\Phi^m(\mathcal{E}) \in \mathbf{T}$. We proceed by induction on the dimension.

The base case is where $\dim(\mathcal{E}) = 1$. We have the following exact sequence in $\mathbf{A}$:

$$0 \to \Phi^0(\mathcal{E}) \to \Phi^0(\mathcal{E} \otimes \mathcal{L}) \to \Phi^0(\mathcal{F}_\sigma) \to \Phi^1(\mathcal{E}) \to 0.$$

For any $x \in W$, we may choose σ such that the support of $\mathcal{F}_\sigma$ has dimension 0 and is disjoint from x. By (GV), $\Phi(\mathcal{F}_\sigma) = \Phi^0(\mathcal{F}_\sigma)$ and $\Phi(\mathcal{O}_x) = \Phi^0(\mathcal{O}_x)$, and therefore $\mathrm{Hom}(\Phi^0(\mathcal{F}_\sigma), \Phi^0(\mathcal{O}_x)) = \mathrm{Hom}(\mathcal{F}_\sigma, \mathcal{O}_x) = 0$. Since $\Phi^1(\mathcal{E})$ is a quotient of $\Phi^0(\mathcal{F}_\sigma)$, we must have $\mathrm{Hom}(\Phi^1(\mathcal{E}), \Phi^0(\mathcal{O}_x)) = 0$. Condition (2) then implies that $\Phi^1(\mathcal{E})$ has no nonzero quotient in $\mathbf{F}$, and so $\Phi^1(\mathcal{E}) \in \mathbf{T}$.

For the inductive step, suppose $\dim(\mathcal{E}) = m > 1$ and choose a section $\sigma \in H^0(X, \mathcal{L})$ as before such that $\mathcal{F}_\sigma$ has dimension $m - 1$. By vanishing, $\Phi^{m-1}(\mathcal{F}_\sigma) \cong \Phi^m(\mathcal{E})$, and by hypothesis $\Phi^{m-1}(\mathcal{F}_\sigma) \in \mathbf{T}$.

We now prove that $\Phi^0(\mathcal{E}) \in \mathbf{F}$ by induction on the dimension. If $\dim(\mathcal{E}) = 0$, then $\mathcal{E}$ is an iterated extension of sheaves of the form $\mathcal{O}_x$, so $\Phi^0(\mathcal{E})$ is an iterated extension of objects in $\mathbf{F}$, and thus $\Phi^0(\mathcal{E}) \in \mathbf{F}$.

Suppose that $\dim(\mathcal{E}) > 0$. Choose $\sigma \in \mathrm{H}^0(\mathcal{L})$ such that the zero locus of σ contains no component of the support of $\mathcal{E}$. We derive the exact sequence

$$0 \to \Phi^0(\mathcal{E} \otimes \mathcal{L}^{-1}) \to \Phi^0(\mathcal{E}) \to \Phi^0(\mathcal{F}_\sigma \otimes \mathcal{L}^{-1}),$$

where $\dim(\mathcal{F}_\sigma) = \dim(\mathcal{E}) - 1$. By induction, $\Phi^0(\mathcal{F}_\sigma \otimes \mathcal{L}^{-1}) \in \mathbf{F}$, so any monomorphism from an object in $\mathbf{T}$ to $\Phi^0(\mathcal{E})$ must factor through $\Phi^0(\mathcal{E} \otimes \mathcal{L}^{-1})$. Let b be the torsion part of $\Phi^0(\mathcal{E})$ with respect to $(\mathbf{T}, \mathbf{F})$. Then the canonical morphism $b \to \Phi^0(\mathcal{E})$ factors through $\Phi^0(\mathcal{E} \otimes \mathcal{L}^{\otimes -j})$ for all positive j. Now let

$\Psi\colon \mathbf{D}(\mathbf{A}) \to \mathbf{D}_W(X)$ be the inverse equivalence to Φ. Since Φ is left exact, Ψ is right exact. Thus $\Psi(b)$ is supported in nonpositive degrees, so any map from $\Psi(b)$ to a sheaf must factor through a map $\Psi(b) \to \mathcal{H}^0(\Psi(b))$.

Consider the map $\mathcal{H}^0(\Psi(b)) \to \mathcal{E}$ that factors through $\mathcal{E} \otimes \mathcal{L}^{\otimes -j} \to \mathcal{E}$ for all positive j. Let $\mathcal{G}$ be the image of $\mathcal{H}^0(\Psi(b))$ in $\mathcal{E}$. Then the inclusion of $\mathcal{G}$ also factors through $\mathcal{E} \otimes \mathcal{L}^{\otimes -j} \to \mathcal{E}$. We will show that $\dim(\mathcal{G}) = 0$.

Let x be a point in the vanishing locus of σ and choose an open affine set $\mathbb{S}\mathrm{pec}\, R$ containing x such that $\mathcal{L}$ restricts to the trivial bundle on $\mathbb{S}\mathrm{pec}\, R$. Fixing a trivialization of $\mathcal{L}$ over R, the map $\mathcal{L}^{-1} \to \mathcal{O}$ corresponds to multiplication by an element $f \in R$.

The map $\mathcal{G} \to \mathcal{E}$ restricts to a map of finitely generated R-modules $\tau : N \to M$ such that, for all positive integers j, the map τ factors through multiplication by f^j. By the Krull intersection theorem [Eis95, Theorem 5.4] there is $\alpha \in R$ such that $1 + \alpha f$ annihilates N. Since f vanishes at x, we must have that $1 + \alpha f$ does not vanish at x. Since a result $\mathcal{G}$ restricts to the zero sheaf on an open set containing x, x is not in the support of $\mathcal{G}$. The support of $\mathcal{G}$ is a closed subset that does not intersect an ample divisor, so $\dim(\mathcal{G}) = 0$.

Applying the equivalence Φ we see that $b \to \Phi^0(\mathcal{E})$ factors through $\Phi^0(\mathcal{G}) \to \Phi^0(\mathcal{E})$. Since $\Phi^0(\mathcal{G})$ is in $\mathbf{F}$, $b \to \Phi^0(\mathcal{E})$ is the zero map. Therefore $\Phi^0(\mathcal{E})$ is in $\mathbf{F}$. $\square$

Next, we apply Lemma 3.3 to the case of a torsion pair arising from a weak central charge. Let $\mathsf{Z} = -\theta + i \cdot \psi$ be a weak central charge on a category $\mathbf{A}$. We say that an object $x \in \mathbf{A}$ is ψ-torsion if $\psi(x) = 0$, and that x is ψ-free if x has no nonzero ψ-torsion subobject.

LEMMA 3.4. *Let $\Phi\colon \mathbf{D}_W(X) \to \mathbf{D}(\mathbf{A})$ be a left exact equivalence that satisfies (GV) and (SV). Let $\mathsf{Z} = -\theta + i \cdot \psi$ be a weak central charge on $\mathbf{A}$ such that ψ takes values in a finitely generated additive subgroup of $\mathbb{R}$. Suppose that, for every point $x \in W$,*

1. *$\Phi^0(\mathcal{O}_x)$ is ψ-free,*
2. *$\theta(\Phi^0(\mathcal{O}_x)) = 0$ and $\theta(a) > 0$ for every proper nonzero quotient $\Phi^0(\mathcal{O}_x) \twoheadrightarrow a$, where a is ψ-free,*
3. *for any object $a \in \mathbf{A}$ with $\mathsf{Z}(a) = 0$, $\mathrm{Ext}^1(a, \Phi^0(\mathcal{O}_x)) = 0$.*

Then, for a sheaf $\mathcal{E}$ supported on W, we have $\Phi^0(\mathcal{E}) \in \mathbf{F}_{\mathsf{Z}}$, and if $m = \dim(\mathcal{E}) > 1$, $\Phi^m(\mathcal{E}) \in \mathbf{T}_{\mathsf{Z}}$.

Proof. We apply Lemma 3.3 to the torsion pair $(\mathbf{T}_{\mathsf{Z}}, \mathbf{F}_{\mathsf{Z}})$. By (GV) and assumptions (1) and (2), the first condition of Lemma 3.3 is already satisfied, so we proceed to check the second.

Suppose $\mathcal{F}$ is a finite-length sheaf supported on W, and there is an epimorphism $\Phi^0(\mathcal{F}) \twoheadrightarrow a$ such that $0 \neq a \in \mathbf{F}_{\mathsf{Z}}$. We must produce a point $x \in W$ and an epimorphism $a \twoheadrightarrow \Phi^0(\mathcal{O}_x)$. To this end, we will show inductively that a is an iterated extension of objects of the form $\Phi^0(\mathcal{O}_x)$, where x ranges over points of W

in the support of $\mathcal{F}$. By (GV) this is equivalent to the statement that $a = \Phi^0(\mathcal{G})$, where $\mathcal{G}$ is a finite-length sheaf supported on W.

We proceed by induction on the length of $\mathcal{F}$. In the base case, $\mathcal{F} = \mathcal{O}_x$, and the only nonzero quotient of $\Phi^0(\mathcal{O}_x)$ that is in $\mathbf{F}_{\mathsf{Z}}$ is $\Phi^0(\mathcal{O}_x)$ itself.

Now consider $\mathcal{F}$ of length greater than 1 and suppose $\Phi^0(\mathcal{F}) \twoheadrightarrow a$ with $a \in \mathbf{F}_{\mathsf{Z}}$. Since $\mathcal{F}$ is an iterated extension of objects of the form $\Phi^0(\mathcal{O}_x)$, and $\operatorname{Hom}(\Phi^0(\mathcal{F}), a) \neq 0$, we must have $\operatorname{Hom}(\Phi^0(\mathcal{O}_x), a) \neq 0$ for some x in the support of $\mathcal{F}$. So there exists a nonzero map $\mu : \Phi^0(\mathcal{O}_x) \to a$. Since $a \in \mathbf{F}_{\mathsf{Z}}$, so is the image of μ. But the only nonzero quotient of $\Phi^0(\mathcal{O}_x)$ that is in $\mathbf{F}_{\mathsf{Z}}$ is $\Phi^0(\mathcal{O}_x)$ itself, so μ is a monomorphism.

Let b be the cokernel. Then there is an epimorphism $\Phi^0(\mathcal{F}/\mathcal{O}_x) \to b$. The additivity of θ implies that $\theta(b) \leq 0$ since $\theta(\Phi^0(\mathcal{O}_x)) = 0$ and $\theta(a) \leq 0$. Now b has a decomposition via the torsion pair $(\mathbf{T}_{\mathsf{Z}}, \mathbf{F}_{\mathsf{Z}})$:

$$0 \to b_T \to b \to b_F \to 0.$$

The object b_F is a quotient of $\Phi^0(\mathcal{F}/\mathcal{O}_x)$, so it is an extension of objects of the form $\Phi^0(\mathcal{O}_x)$. Again, since $\theta(\Phi^0(\mathcal{O}_x)) = 0$ for all x, we deduce then that $\theta(b_F) = 0$, and thus b_T must satisfy $\theta(b_T) \leq 0$. We find that $\mathsf{Z}(b_T) = 0$, so $\operatorname{Ext}^1(b_T, \Phi^0(\mathcal{O}_x)) = 0$ by condition (3). This implies that pulling back the exact sequence

$$0 \to \Phi^0(\mathcal{O}_x) \to a \to b \to 0$$

along $b_T \to b$ yields a split exact sequence and thus a map $b_T \to a$ through which $b_T \to b$ factors. On the other hand, $\operatorname{Hom}(b_T, a) = 0$ since b_T is torsion. The map $b_T \to b$ must be zero, and thus $b_T = 0$. $\square$

The final result of this section will allow us to determine when a t-structure on $\mathbf{D}_W(X)$ is an HN tilt of the standard t-structure with respect to a weak central charge. Let I be a (nonempty) union of irreducible components of W, and let $\eta_1, \dots, \eta_p$ be the generic points of these components. Then, for any sheaf $\mathcal{F}$ supported on W, we define

$$\operatorname{rk}_I(\mathcal{F}) = \sum_{i=1}^{p} \operatorname{length}_{\mathcal{O}_{X,\eta_i}}(\mathcal{F}_{\eta_i}).$$

We say that a sheaf $\mathcal{F}$ is *torsion on I* if $\operatorname{rk}_I(\mathcal{F}) = 0$. We say that $\mathcal{F}$ is *torsion-free on I* if it has no nonzero torsion on I subsheaf. Note that the classes of torsion on I and torsion-free on I sheaves form a torsion pair in $\mathbf{coh}_W(X)$, so it makes sense to talk about the torsion-free on I part of a sheaf supported on W.

LEMMA 3.5. *Let $\mathcal{B}$ be the heart of a t-structure on $\mathbf{D}_W(X)$ such that $\mathcal{H}^i(x) = 0$ for $x \in \mathcal{B}$ unless $i = -1, 0$. Suppose that $\mathsf{Z} = -\theta + i \cdot \operatorname{rk}_I$ is a weak central charge. The subcategory $\mathcal{B}$ is an HN tilt of the standard t-structure on $\mathbf{D}_W(X)$ for the weak central charge Z if and only if*

1. *every sheaf which is torsion on I belongs to $\mathcal{B}$,*
2. *θ is nonnegative on $\mathcal{B}$, and*
3. *if $\mathcal{F} \in \mathbf{coh}_W(X) \cap \mathcal{B}$ is a sheaf such that $\theta(\mathcal{F}) = 0$, then $\mathcal{F}$ is torsion on I.*

Proof. The subcategory $\mathcal{B}$ is a tilt of the standard t-structure by Lemma 2.11. It is an HN tilt with respect to Z if and only if $\mathbf{T}_{\mathsf{Z}} = \mathbf{coh}_W(X) \cap \mathcal{B}$.

A sheaf $\mathcal{F}$ is torsion on I if and only if it is rk_I-torsion. The first two conditions imply that θ is nonnegative on rk_I-torsion sheaves. The second and third conditions together imply that if $\mathcal{F} \in \mathbf{coh}_W(X) \cap \mathcal{B}$, then either $\theta(\mathcal{F}) > 0$, or $\mathrm{rk}_I(\mathcal{F}) = 0$. But if $\theta(\mathcal{F}) > 0$, then we must also have $\theta > 0$ for the I-torsion-free part of $\mathcal{F}$ since $\mathbf{coh}_W(X) \cap \mathcal{B}$ is closed under quotients. Thus $\mathcal{F}$ is in $\mathbf{T}_{\mathsf{Z}}$.

Conversely, suppose $\mathcal{F} \in \mathbf{T}_{\mathsf{Z}}$. The t-structure β defines a torsion pair on $\mathbf{coh}_W(X)$. Let $\mathcal{F}'$ be the free part of $\mathcal{F}$ with respect to this torsion pair. The first condition implies that $\mathcal{F}'$ is torsion-free on I. The second implies that $\theta(\mathcal{F}') \leq 0$. But by the definition of $\mathbf{T}_{\mathsf{Z}}$ we must then have $\mathcal{F}' = 0$. So $\mathcal{F} \in \mathcal{B}$. □

To illustrate how we will apply these results, let $X = W$ be a smooth projective surface. Suppose that we have a left exact equivalence $\Phi : \mathbf{D}(X) \to \mathbf{D}(\mathbf{A})$ that satisfies (GV) and (SV) and that we have a torsion pair $(\mathbf{T}, \mathbf{F})$ in $\mathbf{A}$ that satisfies the hypotheses of Lemma 3.3.

The equivalence Φ lets us view t-structures on $\mathbf{D}(X)$ as t-structures on $\mathbf{D}(\mathbf{A})$. In particular, we can transport the standard t-structure on $\mathbf{D}(X)$ to $\mathbf{D}(\mathbf{A})$. Using the torsion pair $(\mathbf{T}, \mathbf{F})$, we may form the tilt $\mathcal{B}$ of $\mathbf{A} \subset \mathbf{D}(\mathbf{A})$. Lemmas 3.3 and 2.11 imply that $\mathcal{B}$ and $\mathbf{coh}(X)$ are related by a tilt. Moreover, we find that torsion sheaves on X actually belong to $\mathcal{B}$.

We can build the Φ-image of the standard t-structure in $\mathbf{D}(X)$ from the standard t-structure on $\mathbf{D}(\mathbf{A})$ by tilting twice. However, we want to give conditions such that these two tilts are HN tilts with respect to a weak central charge. Suppose now that θ is an additive function on $\mathbf{A}$ such that θ is nonnegative on $\mathbf{T}$ and nonpositive on $\mathbf{F}$. Via Φ, we may view θ as an additive function on $\mathbf{coh}(X)$. Then $\mathsf{Z} = -\theta + i\,\mathrm{rk}$ is a weak central charge on $\mathbf{coh}(X)$.

Corollary 3.6. *With the setup of the preceding paragraphs, the category $\mathcal{B}$ is an HN tilt of* $\mathbf{coh}(X)$ *with respect to* Z *if and only if any sheaf $\mathcal{F}$ on X that satisfies the three conditions $\theta(\mathcal{F}) = 0$, $\Phi^2(\mathcal{F}) = 0$, and $\Phi^1(\mathcal{F}) \in \mathbf{T}$ is a torsion sheaf.*

Proof. Follows from Lemma 3.5. □

In our applications, the category $\mathbf{A}$ will usually be of finite length, and we will obtain HN tilts using the length function.

4. The McKay Correspondence

Let $G \subset \mathrm{SL}_n(\mathbb{C})$ be a nontrivial finite subgroup, so that G acts on $\mathbb{C}^n$. Though the quotient $\mathbb{C}^n/G$ is singular, there is an approach to resolving the singularities on $\mathbb{C}^n/G$ by considering a parameter space of G-equivariant sheaves on $\mathbb{C}^n$ of length $|G|$. When this procedure is successful, we expect a tight connection between the representation theory of G and the geometry of such a resolution of $\mathbb{C}^n/G$.

DEFINITION 4.1.

1. A *G-constellation* on $\mathbb{C}^n$ is a G-equivariant finite-length sheaf $\mathcal{F}$ such that $\mathrm{H}^0(\mathcal{F}) \cong \mathbb{C}G$ as representations of G.
2. A *G-cluster* is a finite G-invariant subscheme $Z \subset \mathbb{C}^n$ such that $\mathrm{H}^0(\mathcal{O}_Z) \cong \mathbb{C}G$ as representations of G.

There is a fine moduli space of G-clusters, called the G-Hilbert scheme and denoted $G\text{-Hilb}(\mathbb{C}^n)$, which arises as a component of the fixed points of the Hilbert scheme under the action of G [IN00; Nak01]. There is a natural morphism $\sigma : G\text{-Hilb}(\mathbb{C}^n) \to \mathbb{C}^n/G$, which can be thought of as sending scheme theoretic G-orbits to set-theoretic G-orbits. This morphism σ is birational and in dimensions 2 and 3 is a crepant resolution of singularities [BKR01].

More generally, we can consider parameter spaces of G-constellations. To produce a separated moduli space it is necessary to impose a notion of stability for G-constellations, generalizing King stability.

Let $\theta\colon K_0(\mathbb{C}G) \to \mathbb{Z}$ be a homomorphism such that $\theta(\mathbb{C}G) = 0$. Then we say that a G-constellation $\mathcal{F}$ is *θ-semistable* if for every proper, nonzero quotient $\mathcal{F} \twoheadrightarrow \mathcal{F}''$, $\theta(\mathcal{F}'') \geq 0$. For each θ, there is a fine moduli space $\mathcal{M}_\theta$ of θ-stable G-constellations [CI04]. There is an open cone of choices for θ such that the θ-stable G-constellations are exactly the G-clusters and $\mathcal{M}_{\theta_0} \cong G\text{-Hilb}$ [CI04, §2.3]. For a given group G, one such example is $\theta_0\colon K_0(\mathbb{C}G) \to \mathbb{Z}$, defined on irreducible representations by

$$\theta_0(V) = \begin{cases} -\dim(V)^2 & \text{if } V \text{ is nontrivial,} \\ |G| - 1 & \text{if } V \text{ is trivial.} \end{cases}$$

We denote the category of G-equivariant coherent sheaves on $\mathbb{C}^n$ by $\mathbf{coh}(\mathbb{C}^n)^G$ and its bounded derived category by $\mathbf{D}(\mathbb{C}^n)^G$. Of course, $\mathbf{coh}(\mathbb{C}^n)^G$ may be interpreted as the category of coherent sheaves on the stack quotient $[\mathbb{C}^n/G]$. We can view the stack quotient $[\mathbb{C}^n/G]$ as a tautological crepant resolution of the categorical quotient $\mathbb{C}^n/G$. One aspect of the (mostly conjectural) derived McKay correspondence is that $\mathbf{D}(\mathbb{C}^n)^G$ should be equivalent to the bounded derived category of any geometric crepant resolution of $\mathbb{C}^n/G$. Regardless of whether $\mathcal{M}_\theta$ is a crepant resolution, there is a commutative diagram of schemes

$$\begin{tikzcd} \mathbb{C}^n \times \mathcal{M}_\theta \arrow[r, "p"] \arrow[d, "\pi"] & \mathcal{M}_\theta \arrow[d] \\ \mathbb{C}^n \arrow[r] & \mathbb{C}^n/G, \end{tikzcd}$$

and a sheaf $\mathcal{E}_\theta \in \mathbf{coh}(\mathbb{C}^n \times \mathcal{M}_\theta)^G$, called the universal G-constellation, where G acts on $\mathbb{C}^n \times \mathcal{M}_\theta$ via the first factor. Then we consider the functor

$$\Phi = \mathbf{R}\pi_*(\mathcal{E}_\theta \otimes p^*(-))\colon \mathbf{D}(\mathcal{M}_\theta) \to \mathbf{D}(\mathbb{C}^n)^G.$$

For any point $p \in \mathcal{M}_\theta$, Φ sends $\mathcal{O}_p$ to the corresponding G-constellation [BKR01; CI04].

Kapranov and Vasserot [KV00] proved that Φ is an equivalence in dimension $n=2$, and Bridgeland, King, and Reid [BKR01] proved that it is an equivalence in dimension $n=3$ for the G-Hilbert scheme $\mathcal{M}_{\theta_0}$. It is expected that all other crepant resolutions of $\mathbb{C}^n/G$ result from varying the parameter θ and that Φ is then an equivalence; Craw and Ishii [CI04] proved both when $n=3$ and G is Abelian. In dimension $n=2$ the minimal resolution is unique, so there is no need to consider other values of θ.

We will work in the setting where $n \leq 3$, $\mathcal{M}_\theta$ is a crepant resolution of $\mathbb{C}^n/G$, and Φ is an equivalence. Our goal is to give a description of the category of sheaves on $\mathcal{M}_\theta$ as an iterated tilt of G-equivariant modules on $\mathbb{C}^n$. We will require the fact that Φ satisfies the standard vanishing theorems.

LEMMA 4.2. *The equivalence Φ is left exact and satisfies both* (*GV*) *and* (*SV*).

Proof. The equivalence Φ has Fourier–Mukai kernel $\mathcal{E}_\theta$, which is flat over $\mathcal{M}_\theta$. Thus Φ is left exact. The property (GV) follows from the usual vanishing for the higher direct image functors. Let $\mathcal{L}$ be an ample line bundle on $\mathcal{M}_\theta$. Then $p^*\mathcal{L}$ is relatively ample over $\mathbb{C}^n$, and (SV) follows from the usual vanishing theorems. $\square$

DEFINITION 4.3. Suppose that $\theta\colon K_0(\mathbb{C}G) \to \mathbb{Z}$ is a homomorphism such that $\theta(\mathbb{C}G) = 0$. Let $\mathbf{T}_\theta$ be the full subcategory

$$\mathbf{T}_\theta = \{\mathcal{F}\colon \text{ for all nonzero finite-length } G\text{-equivariant quotients } \mathcal{F} \twoheadrightarrow \mathcal{G}, \\ \text{we have } \theta(\mathrm{H}^0(\mathcal{G})) > 0\}.$$

Since $\mathbf{T}_\theta$ is closed under quotients, by [Pol07, Lemma 1.1.3] there is a unique category $\mathbf{F}_\theta$ such that $(\mathbf{T}_\theta, \mathbf{F}_\theta)$ forms a torsion pair.

Consider a sheaf $\mathcal{F}$ and point p such that $\mathcal{F}_p \neq 0$ and the stabilizer subgroup G_p is trivial. Let $G \cdot p$ be the orbit of p and $\mathcal{O}_{G\cdot p}$ its structure sheaf. The category of G-equivariant coherent sheaves supported on the free orbit $G \cdot p$ has a unique simple object, $\mathcal{O}_{G\cdot p}$. Therefore $\mathrm{H}^0(\mathcal{F} \otimes \mathcal{O}_{G\cdot p}) = \mathbb{C}G^{\oplus N}$ for some N. It follows that $\theta(\mathcal{F} \otimes \mathcal{O}_{G\cdot p}) = 0$. Hence, if $\mathcal{F} \in \mathbf{T}_\theta$, then $\mathcal{F}$ is supported on the locus in $\mathbb{C}^n$, where the action of G is not free.

REMARK 4.4. Suppose the action of G on $\mathbb{C}^n$ is free away from the origin. Let $\mathcal{F}$ be a G-equivariant sheaf on $\mathbb{C}^n$ and suppose $\mathcal{F}' \subset \mathcal{F}$ is the maximal subsheaf supported on 0. It follows from the preceding discussion that $\mathcal{F} \in \mathbf{F}_\theta$ if and only if for any subobject $\mathcal{G} \subset \mathcal{F}'$, $\theta(\mathrm{H}^0(\mathcal{G})) \leq 0$.

Suppose that $n=2$. Then $\mathbb{C}^2/G$ is a Kleinian singularity, and $\mathcal{M}_\theta$ is a crepant resolution if and only $\mathcal{M}_\theta$ is isomorphic to $G\text{-}\mathrm{Hilb}(\mathbb{C}^2)$. In this situation, $\mathbf{T}_\theta$ consists entirely of G-equivariant sheaves supported at the origin. Using the length function ℓ on the category of G-equivariant sheaves supported at the origin, we can construct HN filtrations. Then $\mathbf{T}_\theta$ consists of those G-equivariant sheaves supported at the origin whose HN factors all have positive slope.

THEOREM 4.5. *Suppose $n = 2$ and that θ lies in the same GIT cone as θ_0. Under the equivalence $\Phi\colon \mathbf{D}(G\text{-}\mathrm{Hilb}(\mathbb{C}^2)) \to \mathbf{D}(\mathbb{C}^2)^G$, $\mathbf{coh}(G\text{-}\mathrm{Hilb}(\mathbb{C}^2))[1]$ is identified with the tilt of $\mathbf{coh}(\mathbb{C}^2)^G$ with respect to $(\mathbf{T}_\theta, \mathbf{F}_\theta)$.*

Proof. By Lemma 4.2 Φ is left exact and satisfies (SV) and (GV). Let $E \subset G\text{-}\mathrm{Hilb}(\mathbb{C}^2)$ be the exceptional divisor of the map $G\text{-}\mathrm{Hilb}(\mathbb{C}^2) \to \mathbb{C}^2/G$. Then Φ restricts to an equivalence $\Phi_0\colon \mathbf{D}_E(G\text{-}\mathrm{Hilb}(\mathbb{C}^2)) \to \mathbf{D}_0(\mathbb{C}^2)^G = \mathbf{D}(\mathbf{coh}_0(\mathbb{C}^2)^G)$.

The torsion pair $(\mathbf{T}_\theta, \mathbf{F}_\theta)$ on $\mathbf{coh}(\mathbb{C}^2)^G$ induces a torsion pair $(\mathbf{T}_\theta, \mathbf{F}_\theta \cap \mathbf{coh}_0(\mathbb{C}^2)^G)$ on the category $\mathbf{coh}_0(\mathbb{C}^2)^G$ of G-equivariant sheaves supported at the origin. Both θ and the length ℓ induce additive $\mathbb{Z}$-valued functions on $K_0(\mathbf{coh}_0(\mathbb{C}^2)^G)$, and since $\ell(\mathcal{F}) = 0$ if and only if $\mathcal{F} = 0$, we have a weak central charge

$$\mathsf{Z} = -\theta + i\ell.$$

As a result, Z defines a torsion pair $(\mathbf{T}_\mathsf{Z}, \mathbf{F}_\mathsf{Z})$ on $\mathbf{coh}_0(\mathbb{C}^2)^G$. But $\mathbf{T}_\mathsf{Z} = \mathbf{T}_\theta$, so $\mathbf{F}_\mathsf{Z} = \mathbf{F}_\theta \cap \mathbf{coh}_0(\mathbb{C}^2)^G$. We next apply Lemma 3.4. Conditions 1 and 3 hold because there is no nonzero element of $\mathbf{coh}_0(\mathbb{C}^2)^G$ with $\ell = 0$. Condition 2 holds because, for every point $x \in G\text{-}\mathrm{Hilb}(\mathbb{C}^2)$, $\Phi(\mathcal{O}_x)$ is a G-cluster, and G-clusters are θ stable. We conclude that if $\mathcal{F}$ is a sheaf supported on E, then $\Phi^0(\mathcal{F}) \in \mathbf{F}_\theta$ and $\Phi^1(\mathcal{F}) \in \mathbf{T}_\theta$.

For any $\mathcal{F} \in \mathbf{coh}(G\text{-}\mathrm{Hilb}(\mathbb{C}^2))$, $\Phi^i(\mathcal{F}) = 0$ unless $i = 0, 1$. Thus Lemma 2.11 implies that there is a torsion pair $(\mathbf{T}, \mathbf{F})$ in $\mathbf{coh}(\mathbb{C}^2)^G$ whose tilt is $\Phi(\mathbf{coh}(G\text{-}\mathrm{Hilb}(\mathbb{C}^2)))[1]$. Then $\mathbf{T} \subset \mathbf{coh}_0(\mathbb{C}^2)^G$. We compute (abusing notation by dropping the reference to Φ):

$$\begin{aligned}
\mathbf{T} &= \mathbf{coh}(\mathbb{C}^2)^G \cap \mathbf{coh}(G\text{-}\mathrm{Hilb}(\mathbb{C}^2))[1] \\
&= \mathbf{coh}_0(\mathbb{C}^2)^G \cap \mathbf{coh}(G\text{-}\mathrm{Hilb}(\mathbb{C}^2))[1] \\
&= \mathbf{coh}_0(\mathbb{C}^2)^G \cap \mathbf{coh}_E(G\text{-}\mathrm{Hilb}(\mathbb{C}^2))[1] \\
&= \mathbf{T}_\theta,
\end{aligned}$$

where the third equality follows from the fact that Φ identifies $\mathbf{D}_E(G\text{-}\mathrm{Hilb}(\mathbb{C}^2))$ with $\mathbf{D}_0(\mathbb{C}^2)^G$. □

We now turn to dimension three. If $\mathbb{C}^3/G$ has an isolated singularity, then $\mathbf{T}_\theta$ once again consists of G-equivariant sheaves supported at the origin, and the torsion class $\mathbf{T}_\theta$ can be described in terms of HN filtrations. However, if the singularity is not isolated, then the torsion pair can be much more complicated. Let $\Phi\colon \mathbf{D}(\mathcal{M}_\theta) \to \mathbf{D}(\mathbb{C}^3)^G$ be the functor defined before whose Fourier–Mukai kernel is the universal G-constellation, and $E \subset \mathcal{M}_\theta$ be the part of the exceptional locus of $\mathcal{M}_\theta \to \mathbb{C}^3/G$ lying over $0 \in \mathbb{C}^3/G$. Assume that $\mathcal{M}_\theta$ is a crepant resolution of $\mathbb{C}^3/G$ and that Φ is an equivalence. Then we can define the restricted functor

$$\Phi_0\colon \mathbf{D}_E(\mathcal{M}_\theta) \to \mathbf{D}_0(\mathbb{C}^n)^G,$$

which is also an equivalence since $\mathcal{F}$ is supported on E if and only if $\Phi^i(\mathcal{F})$ is supported on the origin for all i [BKR01, §9].

As in the dimension 2 case, we can define a weak central charge $\mathsf{Z}_G = -\theta + i \cdot \ell$ on $\mathbf{coh}_0(\mathbb{C}^3)^G$, where ℓ is the length function, and we view the stability parameter θ as a function on $K_0(\mathbf{coh}_0(\mathbb{C}^3)^G)$. Via the equivalence Φ, we can also view θ as a function on $K_0(\mathbf{coh}_E(\mathcal{M}_\theta))$. Let I be the union of all irreducible components of E of dimension 2. Then we can define an additive function $\mathsf{Z}_{\mathcal{M}} = \theta + i \cdot \mathrm{rk}_I$ on $K_0(\mathbf{coh}_E(\mathcal{M}_\theta))$.

THEOREM 4.6. *The functor Φ identifies the tilt of $\mathbf{coh}(\mathbb{C}^3)^G$ with respect to $(\mathbf{T}_\theta, \mathbf{F}_\theta)$ with a (possibly trivial) tilt of $\mathbf{coh}(\mathcal{M}_\theta)[1]$.*

THEOREM 4.7. *$\mathsf{Z}_{\mathcal{M}}$ is a weak central charge on $\mathbf{coh}_E(\mathcal{M}_\theta)$, and the restricted equivalence*

$$\Phi_0 \colon \mathbf{D}_E(\mathcal{M}_\theta) \to \mathbf{D}_0(\mathbb{C}^3)^G$$

identifies the HN tilts of $\mathbf{coh}_E(\mathcal{M}_\theta)[1]$ and $\mathbf{coh}_0(\mathbb{C}^3)^G$ with respect to $\mathsf{Z}_{\mathcal{M}}$ and Z_G, respectively.

Proof of Theorem 4.7. We will apply Lemma 3.4. Since Φ is left exact and satisfies (GV) and (SV) by Lemma 4.2, the same is true for Φ_0. Observe that $\mathcal{F} \in \mathbf{coh}_0(\mathbb{C}^3)^G$ satisfies $\ell(\mathcal{F}) = 0$ if and only if $\mathcal{F} = 0$. So conditions (1) and (3) of Lemma 3.4 are vacuous here. Condition (2) holds because $\Phi(\mathcal{O}_p)$ is a θ-stable G-equivariant sheaf. (Finite-length G-equivariant sheaves are θ-(semi)stable if they satisfy King's criterion from Example 2.16 in the category of finite length G-equivariant sheaves.) We conclude that, for any $\mathcal{F} \in \mathbf{coh}_E(\mathcal{M}_\theta)$, $\Phi^0(\mathcal{F}) \in \mathbf{F}_{\mathsf{Z}_G}$ and $\Phi^2(\mathcal{F}) \in \mathbf{T}_{\mathsf{Z}_G}$.

The function rk_I is nonnegative on $\mathbf{coh}_E(\mathcal{M}_\theta)$, so to prove the first statement, it remains to show that for a sheaf $\mathcal{F}$ with $\mathrm{rk}_I(\mathcal{F}) = 0$, $\theta(\mathcal{F}) \leq 0$. If $\mathrm{rk}_I(\mathcal{F}) = 0$, then $\mathcal{F}$ has dimension at most 1, so $\Phi^2(\mathcal{F}) = 0$, and by Lemma 3.4, $\Phi^1(\mathcal{F}) \in \mathbf{T}_{\mathsf{Z}_G}$. On $\mathbf{F}_{\mathsf{Z}_G}$, $\theta \leq 0$, and $\theta \geq 0$ on $\mathbf{T}_{\mathsf{Z}_G}$, so

$$\theta(\mathcal{F}) = \theta(\Phi^0(\mathcal{F})) - \theta(\Phi^1(\mathcal{F})) \leq 0.$$

Let $\mathcal{A}_G$ be the heart of the t-structure on $\mathbf{coh}_0(\mathbb{C}^3)^G$ obtained by tilting the standard t-structure using $(\mathbf{T}_{\mathsf{Z}_G}, \mathbf{F}_{\mathsf{Z}_G})$. Since Φ^i are the cohomology functors for $\mathbf{coh}_0(\mathbb{C}^3)^G$, by Lemma 2.12, $\mathcal{A}_G$ is a tilt of $\Phi(\mathbf{coh}_E(\mathcal{M}_\theta)[1])$.

Now we will apply Lemma 3.5 to $\mathcal{A}_G[-1]$ to show that it is the HN tilt of $\Phi(\mathbf{coh}_E(\mathcal{M}_\theta))$ with respect to $\mathsf{Z}_{\mathcal{M}}$. If $\mathcal{F}$ is a sheaf on E with $\mathrm{rk}_I(\mathcal{F}) = 0$, then $\dim(\mathcal{F}) \leq 1$. Hence $\Phi^2(\mathcal{F}) = 0$, $\Phi^1(\mathcal{F}) \in \mathbf{T}_{\mathsf{Z}_G}$, and $\Phi^0(\mathcal{F}) \in \mathbf{F}_{\mathsf{Z}_G}$. Hence (1) of Lemma 3.5 is satisfied. Condition (2) is satisfied because $\theta \geq 0$ on objects of $\mathbf{T}_{\mathsf{Z}_G}$, $\theta \leq 0$ on objects of $\mathbf{F}_{\mathsf{Z}_G}$, and every object of $\mathcal{A}_G[-1]$ is an extension of an object of $\mathbf{F}_{\mathsf{Z}_G}$ by one of $\mathbf{T}_{\mathsf{Z}_G}[-1]$.

Finally, we check condition (3). Assume that $\mathcal{F}$ is a sheaf on $\mathcal{M}$ supported on E such that $\theta(\mathcal{F}) = 0$ and $\Phi(\mathcal{M}) \in \mathcal{A}_G[-1]$. Then $\Phi^0(\mathcal{F}) \in \mathbf{F}_{\mathsf{Z}_G}$, $\Phi^1(\mathcal{F}) \in \mathbf{T}_{\mathsf{Z}_G}$, and $\Phi^2(\mathcal{F}) = 0$. Therefore $\theta(\mathcal{F}) = \theta(\Phi^0(\mathcal{F})) - \theta(\Phi^1(\mathcal{F}))$. Since θ is nonpositive on $\mathbf{F}_{\mathsf{Z}_G}$ and positive on nonzero objects in $\mathbf{T}_{\mathsf{Z}_G}$, it must be $\theta(\Phi^0(\mathcal{F})) = 0$ and $\Phi^1(\mathcal{F}) = 0$. Moreover, $\Phi^0(\mathcal{F})$ is θ-semistable, and therefore there are finitely many $p \in E$ such that $\mathrm{Hom}(\Phi^0(\mathcal{F}), \Phi^0(\mathcal{O}_p)) = \mathrm{Hom}(\mathcal{F}, \mathcal{O}_p) \neq 0$. This implies that $\mathcal{F}$ is a sheaf of finite length. Thus, $\mathcal{A}_G[-1]$ is the HN tilt of $\Phi(\mathbf{coh}_E(\mathcal{M}_\theta))$ with respect to $\mathsf{Z}_{\mathcal{M}}$. To perform this tilt in the shifted category, we need only reverse the sign of $\mathsf{Z}_{\mathcal{M}}$. □

Proof of Theorem 4.6. We first establish that, for any sheaf $\mathcal{F}$ on $\mathcal{M}$, $\Phi^0(\mathcal{F}) \in \mathbf{F}_\theta$. We must check the conditions of Lemma 3.3. For any point $x \in \mathcal{M}_\theta$, $\Phi^0(\mathcal{O}_x)$ is a stable G-constellation. By θ-stability no subobject of $\Phi^0(\mathcal{O}_x)$ is in $\mathbf{T}_\theta$, and so $\Phi^0(\mathcal{O}_x)$ is in $\mathbf{F}_\theta$. This is condition (1). Now for condition (2). Let $\mathcal{G}$ be a zero-dimensional sheaf on $\mathcal{M}_\theta$. We proceed by induction on the length of $\mathcal{G}$. The base case is established by θ-stability of $\Phi^0(\mathcal{O}_x)$. For the inductive step, $\Phi^0(\mathcal{G})$ is an extension of stable G-constellations, and we may write

$$0 \to B \to \Phi^0(\mathcal{G}) \to C \to 0,$$

where B and C are extensions of stable G constellations of strictly smaller length. Then for any surjection $\Phi^0(\mathcal{G}) \twoheadrightarrow a$ with $a \in \mathbf{F}_\theta$, we get an exact sequence

$$0 \to a_B \to a \to a_C \to 0,$$

where a_B is the image of B in a, and $C \twoheadrightarrow a_C$. We must have by θ-stability that $\theta(a_B) = \theta(a) = \theta(a_C) = 0$. Also, C has no quotient with $\theta > 0$, so neither does a_C. Thus $a_C \in \mathbf{F}_\theta$. So if $a_C \neq 0$, there is a surjection from a to a θ-stable G-constellation by induction. But if $a_C = 0$, then $a_B = a$, and we may apply induction. So Lemma 3.3 applies and $\Phi^0(\mathcal{F}) \in \mathbf{F}_\theta$.

It remains to show that for any sheaf $\mathcal{F}$, $\Phi^2(\mathcal{F}) \in \mathbf{T}_\theta$. For a sheaf $\mathcal{F}$ that is supported on the preimage E of 0, this follows from the previous proof. Otherwise, choose a divisor D on $\mathcal{M}_\theta$ that is relatively ample over $\mathbb{C}^3/G$ and is supported on the exceptional divisors. Then, by the negativity lemma [KM98, Lemma 3.39], $-D$ is effective. Choose n sufficiently large such that $\Phi^2(\mathcal{F}(nD))$ vanishes. Then $\Phi^2(\mathcal{F}) \cong \Phi^2(\mathcal{G})$, where $\mathcal{G}$ is supported on the exceptional divisors. But the only divisors that contribute to Φ^2 are the divisors whose image is the origin in $\mathbb{C}^3$. The theorem follows by Lemma 2.12. □

REMARK 4.8. Suppose that $G \subset \mathrm{SL}_2(\mathbb{C})$. We can think of G as a finite subgroup of $\mathrm{SL}_3(\mathbb{C})$, for example, by using a splitting $\mathbb{C}^3 = \mathbb{C}^2 \oplus \mathbb{C}$ and having G act trivially on the second factor. Then $\mathbb{C}^3/G \cong \mathbb{C}^2/G \times \mathbb{C}$ is a transverse singularity with crepant resolution $\mathcal{M}_\theta \cong G\text{-}\mathrm{Hilb}(\mathbb{C}^2) \times \mathbb{C}$. Note that the fiber dimension of $\mathcal{M}_\theta \to \mathbb{C}^3/G$ is at most one. Thus, for any sheaf $\mathcal{F}$ on $\mathcal{M}_\theta$, $\Phi^i(\mathcal{F}) = 0$ for $i \neq 0, 1$. So $\Phi(\mathbf{coh}(\mathcal{M}_\theta))[1]$ is a tilt of $\mathbf{coh}(\mathbb{C}^3)^G$, and by Theorem 4.6, it must be the tilt with respect to $(\mathbf{T}_\theta, \mathbf{F}_\theta)$.

REMARK 4.9. In the three-dimensional derived McKay correspondence, the t-structure induced by the standard one on $\mathbf{D}(\mathcal{M}_\theta)$ under Φ can be very nontrivial. One guess for how to describe this t-structure explicitly would be to adapt the construction of perverse (coherent) sheaves and attempt to define the induced t-structure by restricting the possible cohomologies. This is especially appealing in light of the results in [CCL12]. However, it turns out that this is not generally the right description. Consider $G = \mu_3$, the center of $\mathrm{SL}_3(\mathbb{C})$. Then $X = G\text{-}\mathrm{Hilb}(\mathbb{C}^3)$

is naturally isomorphic to the blow-up of $\mathbb{C}^3/G$ at the singular point. It can then be identified with the total space of $\omega_{\mathbb{P}^2}$. We will show that there do not exist full subcategories $\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2$ of $\mathbf{coh}(\mathbb{C}^3)^G$ such that $\mathcal{F}^\bullet \in \mathbf{D}(\mathbb{C}^3)^G$ has the form $F^\bullet = \Phi(\mathcal{G})$ if and only if $\mathcal{H}^i(F^\bullet) \in \mathbf{A}_i$ for $i = 0, 1, 2$ and $\mathcal{H}^i(F^\bullet) = 0$ for $i \neq 0, 1, 2$. If this were the case, then the full subcategory $\Phi(\mathbf{coh}(X))$ would be closed under taking cohomology sheaves in the sense that if $F^\bullet \in \Phi(\mathbf{coh}(X))$, then $\mathcal{H}^0(F^\bullet)$, $\mathcal{H}^1(F^\bullet)[-1]$, and $\mathcal{H}^2(F^\bullet)[-2]$ all belong to $\Phi(\mathbf{coh}(X))$ as well.

Viewing X as the total space of $\omega_{\mathbb{P}^2}$, let $E \cong \mathbb{P}^2$ be the zero section. Then if $\mathcal{E}$ is the universal G-cluster on $\mathbb{C}^3 \times X$, then we identify $\mathrm{pr}_{X*}\mathcal{E}$ as $\pi^*(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2))$, where $\pi : X \to \mathbb{P}^2$ is the line bundle structure map. Now let $p \in E$ be a point, and $\mathcal{I}_p \subset \mathcal{O}_E$ the ideal sheaf on E of p. From the exact sequence

$$0 \to \mathcal{I}_p(-3) \to \mathcal{O}_E(-3) \to \mathcal{O}_p \to 0$$

we see that $\Phi^0(\mathcal{I}_p(-3)) = 0$,whereas $\Phi^1(\mathcal{I}_p(-3)) = \Phi^0(\mathcal{O}_p)$. If $\Phi(\mathbf{coh}(X))$ were closed under taking cohomology, then it would have to contain both $\Phi^0(\mathcal{O}_p)$ and $\Phi^0(\mathcal{O}_p)[-1]$. This is impossible since $\Phi(\mathbf{coh}(X)) \cap \Phi(\mathbf{coh}(X))[-1] = \{0\}$.

5. Tilting Equivalences

Let X be a variety.

DEFINITION 5.1. A *tilting bundle* $\mathcal{E}$ on X is a vector bundle such that

1. $\mathrm{Ext}^i(\mathcal{E}, \mathcal{E}) = 0$ for $i > 0$.
2. The zero sheaf is the only sheaf $\mathcal{F}$ such that $\mathrm{Ext}^i(\mathcal{E}, \mathcal{F}) = 0$ for all i.

A tilting bundle gives rise to a pair of inverse equivalences

$$\mathbf{D}(X) \underset{-\otimes^{\mathbf{L}}\mathcal{E}}{\overset{\mathbf{R}\,\mathrm{Hom}(\mathcal{E},-)}{\rightleftarrows}} \mathbf{D}(\mathsf{A})$$

where $\mathsf{A} = \mathrm{End}(\mathcal{E})$ and $\mathbf{D}(\mathsf{A})$ is the bounded derived category of finite-dimensional A-modules. (See [Kel07] for a general discussion of tilting equivalences.) We aim to understand the relationship between the standard hearts of the categories under this equivalence. Next, we introduce the notion of a full strong exceptional sequence to provide a source of tilting bundles.

DEFINITION 5.2. An object $\mathcal{F} \in \mathbf{D}(X)$ is *exceptional* if $\mathrm{End}(\mathcal{F}) = \mathbf{k}$ and $\mathrm{Ext}^i(\mathcal{F}, \mathcal{F}) = 0$ for $i \neq 0$. A *full exceptional sequence* is a sequence $\mathcal{F}_1, \ldots, \mathcal{F}_n$ such that each $\mathcal{F}_i$ is exceptional, $\mathrm{Ext}^i(\mathcal{F}_j, \mathcal{F}_k) = 0$ whenever $j > k$, and the smallest thick subcategory of $\mathbf{D}(X)$ containing $\mathcal{F}_1, \ldots, \mathcal{F}_n$ is $\mathbf{D}(X)$. Finally, a full exceptional sequence $\mathcal{F}_1, \ldots, \mathcal{F}_n$ is *strong* if in addition $\mathrm{Ext}^i(\mathcal{F}_j, \mathcal{F}_k) = 0$ for all $i \neq 0$.

REMARK 5.3. If $\mathcal{F}_1, \ldots, \mathcal{F}_n$ is a full strong exceptional sequence consisting of vector bundles, then $\mathcal{E} = \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_n$ is a tilting bundle. See [Bon89] for a discussion of tilting in this special case.

We will investigate the structure of these equivalences in the case where X is a surface. It is known that every rational surface admits a tilting bundle [HP14, Theorem 1.1]. However, the converse is a well-known open question:

OPEN QUESTION. Is every a smooth projective surface which admits a tilting bundle rational?

Let X be a smooth projective surface with a tilting bundle $\mathcal{E}$ and set $\mathsf{A} = \mathrm{End}(\mathcal{E})$. Write $\Phi\colon \mathbf{D}(X) \to \mathbf{D}(\mathsf{A})$ for $\mathbf{R}\,\mathrm{Hom}(\mathcal{E}, -)$. Denote the length function on $\mathbf{mod}$-A by ℓ. Consider a weak central charge $\mathsf{Z} = -\theta + i\ell$ on $\mathbf{mod}$-A. We will assume that $\mathcal{E}$ does not have repeated indecomposable summands, so that every simple A-module is one-dimensional. Then the isomorphism classes of one-dimensional simple modules are in bijection with indecomposable idempotents of A and form a basis for $K_0(\mathsf{A})$, the Grothendieck group of finite-dimensional modules. Let $e_1, \ldots, e_m$ be the indecomposable idempotents of A, and $S_1, \ldots, S_m$ the corresponding simple modules. Then given a finite-dimensional A-module M, the class of M in $K_0(\mathsf{A})$ is $\sum_{i=1}^m \dim_{\mathbf{k}}(Me_i)[S_i]$. The tuple $(\dim_{\mathbf{k}}(Me_i))$ is called the dimension vector of M. So we can regard Z as a complex-valued function on the set of integral dimension vectors. Observe that since $\mathcal{E}$ is a vector bundle, the class of $\mathcal{O}_p$ in $K_0(X)$ is independent of p.

DEFINITION 5.4. Let Z be a weak central charge such that $\theta = 0$ on the class of $\mathcal{O}_p \in X$. We say that Z is *compatible* with X if for each point $p \in X$, $\Phi(\mathcal{O}_p)$ is a θ-stable representation of A.

THEOREM 5.5. *Suppose X is a smooth projective surface with a tilting bundle $\mathcal{E}$. If* $\mathbf{mod}$-A *admits a weak central charge $\mathsf{Z}_{\mathsf{A}} = -\theta + i \cdot \ell$ compatible with X, then X is rational. Moreover, Φ identifies $\mathcal{A}_X$ and $\mathcal{A}_{\mathsf{A}}[-1]$ where $\mathcal{A}_X$ and $\mathcal{A}_{\mathsf{A}}$ are the HN tilts of* $\mathbf{coh}(X)$ *and* $\mathbf{mod}$-A *with respect to $\mathsf{Z}_X = \theta + i \cdot \mathrm{rk}$ and Z_{A}, respectively.*

Proof. We first prove the second claim. Since the tilting bundle is flat, the tilting equivalence Φ is left exact; since $\mathcal{E}$ is a sheaf, Φ satisfies (GV) and (SV). So we will apply Lemma 3.4 with $Z = X$. Now, since $\ell(x) = 0$ if and only if $x = 0$, conditions (1) and (3) of the lemma are vacuous. Condition (2) holds by assumption in this case. Hence $\Phi^0(\mathcal{F}) \in \mathbf{F}_{\mathsf{Z}}$ and $\Phi^2(\mathcal{F}) \in \mathbf{T}_{\mathsf{Z}}$ for any coherent sheaf $\mathcal{F}$ on X. Next, let $\mathcal{A}$ be the shift of the HN tilt of $\mathbf{mod}$-A with respect to Z_{A}, where objects have standard cohomology in degrees 0 and 1. By Lemma 2.12 $\mathcal{A}$ is a tilt of $\Phi(\mathbf{coh}(X)[1])$. We now apply Lemma 3.5 to $\mathcal{A}$ to see that it is the HN tilt of $\Phi(\mathbf{coh}(X))$ with respect to Z_X. Since a torsion sheaf has dimension at most one, if $\mathcal{F}$ is torsion, then $\Phi(\mathcal{F}) \in \mathcal{A}$ by Lemma 3.4. By construction, θ is nonpositive on $\mathcal{A}$. So we have to check that if $\Phi(\mathcal{F}) \in \mathcal{A}$ and $\theta(\mathcal{F}) = 0$, then $\mathcal{F}$ is torsion. In fact, if $\Phi(\mathcal{F}) \in \mathcal{A}$, then $\theta(\mathcal{F}) = \theta(\Phi^0(\mathcal{F})) - \theta(\Phi^1(\mathcal{F}))$. Now, since $\Phi^0(\mathcal{F}) \in \mathbf{F}_{\mathsf{Z}}$, $\theta(\Phi^0(\mathcal{F})) \leq 0$ and since $\Phi^1(\mathcal{F}) \in \mathbf{T}_{\mathsf{Z}}$, if $\Phi^1(\mathcal{F}) \neq 0$, we have $\theta(\Phi^1(\mathcal{F})) > 0$. So we conclude that $\Phi^1(\mathcal{F}) = 0$ and $\theta(\Phi^0(\mathcal{F})) = 0$. By Remark 2.17, the only objects of $\mathbf{F}_{\mathsf{Z}}$ on which θ takes the value 0 are semistable, which implies that $\Phi^0(\mathcal{F})$ is semistable. Therefore there are only finitely many

$p \in X$ such that $\mathrm{Hom}(\Phi^0(\mathcal{F}), \Phi^0(\mathcal{O}_p)) = \mathrm{Hom}(\mathcal{F}, \mathcal{O}_p)$ is nonzero. Hence $\mathcal{F}$ has finite support and so is torsion.

We now establish the rationality of X. Let $K_0(X)_{\leq 0}$ be the subgroup of $K_0(X)$ generated by the classes of sheaves of finite length. Since $\theta(\mathcal{O}_p) = 0$ for any point $p \in X$, θ defines a function on $K_0(X)/K_0(X)_{\leq 0}$. We observe that if D is effective, then $[\mathcal{E} \otimes \mathcal{O}(D)] = [\mathcal{E}] + [\mathcal{E} \otimes \mathcal{O}_D(D)]$, and in $K_0(X)/K_0(X)_{\leq 0}$, $[\mathcal{E} \otimes \mathcal{O}_D] = \mathrm{rk}(\mathcal{E})[\mathcal{O}_D]$. Now the map $D \mapsto [\mathcal{O}_D]$ defines an injective homomorphism $\mathrm{Cl}(X) \to K_0(X)/K_0(X)_{\leq 0}$. Likewise $\mathcal{E} \otimes [\mathcal{O}_D(D)] - \mathcal{E} \otimes [\mathcal{O}_D]$ belongs to $K_0(X)_{\leq 0}$, so $\theta(\mathcal{E} \otimes \mathcal{O}_D(D)) = \theta(\mathcal{E} \otimes \mathcal{O}_D)$. We see that $\alpha(D) = \theta(\mathcal{E} \otimes \mathcal{O}(D)) - \theta(\mathcal{E})$ defines a group homomorphism $\alpha\colon \mathrm{Cl}(X) \to \mathbb{Z}$.

Now we observe that for any effective divisor D, $\Phi(\mathcal{E} \otimes \mathcal{O}_D(D))$ belongs to $\mathcal{A}$ because $\mathcal{O}_D(D)$ is supported in codimension one. Hence $\theta(\mathcal{E} \otimes \mathcal{O}_D) \leq 0$. On the other hand, $\Phi(\mathcal{E}) = \Phi^0(\mathcal{E}) = \mathsf{A}[0]$ and $\Phi(\mathcal{E} \otimes \omega_X) = \Phi^2(\mathcal{E} \otimes \omega_X)[-2] = \mathsf{A}^\vee[-2]$, by Serre duality. Therefore $\Phi(\mathcal{E}) \in \mathbf{F}_{\mathsf{Z}}$ and $\Phi(\mathcal{E} \otimes \omega_X) \in \mathbf{T}_{\mathsf{Z}}[2]$. Hence $\theta(\mathcal{E}) < 0$, whereas $\theta(\mathcal{E} \otimes \omega_X) \geq 0$. We conclude that $\alpha(\omega_X) > 0$.

Since $K_0(X) \cong K_0(\mathsf{A})$, it is free of finite rank, and since all of the objects $\Phi(\mathcal{O}_x)$ have the same dimension vector, $K_0(X)_{\leq 0} = \mathbb{Z} \cdot [\mathcal{O}_p]$ for any point $p \in X$. Thus, $\mathrm{NS}(X) \cong K_0(X)_{\leq 1}/K_0(X)_{\leq 0} \otimes_{\mathbb{Z}} \mathbb{R}$, where $K_0(X)_{\leq 1}$ is the subgroup generated by sheaves of dimension at most 1. We extend α to a linear map $\mathrm{NS}(X) \to \mathbb{R}$. Now note that $\alpha \leq 0$ on effective divisors but $\alpha(\omega_X) > 0$. Thus the canonical divisor of X is not in the closure of the cone of effective divisors, and thus no multiple of it has a nonzero section. So the Kodaira dimension of X is $-\infty$. Next, note that $\mathcal{O}_X$ is a summand of the sheaf of endomorphisms $\mathcal{E}nd(\mathcal{E})$ via the trace map. Since $\mathrm{H}^i(\mathcal{E}nd(\mathcal{E})) = 0$ for $i > 0$, we observe that $H^1(X, \mathcal{O}_X) = 0$. By the Enriques–Kodaira classification of surfaces [BPvdV84, §VI, Theorem 1.1], X is rational. $\square$

REMARK 5.6. It is possible to obtain more general results, in exchange for less control over the torsion pairs that appear. Let $\mathbf{A}$ be a Noetherian Abelian category of homological dimension 2 with a tilting object T. Denote by Φ the tilting equivalence between $\mathbf{D}(\mathbf{A})$ and $\mathbf{D}(\mathrm{End}(T))$. Lo [Lo15] proves that $\phi(\mathbf{A})$ and $\mathbf{mod}\text{-}\mathrm{End}(T)$ are related by two tilts.

It is not known if a compatible weak central charge always exists for a given tilting bundle. Bergman and Proudfoot studied the problem in [BP08] with the aim of giving a GIT construction of any variety that admits a tilting bundle. They use the term "great", where we use the term compatible. In general the question of whether a set of modules can be made stable simultaneously is very subtle. For example, this can be impossible if we consider partial tilting bundles, that is, vector bundles satisfying 1 but not 2 in Definition 5.1.

EXAMPLE 5.7 (Lutz Hille). Let $B_q\mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at q, and let X be the blow-up of $B_q\mathbb{P}^2$ at a point on the exceptional divisor E_1 of $B_q\mathbb{P}^2 \to \mathbb{P}^2$. Let $f\colon X \to B_q\mathbb{P}^2$ be the blowing-up map, E_2 the exceptional divisor, and E_1' the strict transform of E_1. Then the cohomology class in $\mathrm{H}^1(\mathcal{O}(E_1'))$ defines an exact sequence

$$0 \to f^*\mathcal{O}(E_1) \to \mathcal{E} \to \mathcal{O}(E_2) \to 0.$$

The vector bundle $\mathcal{E}$ satisfies $\operatorname{Ext}^i(\mathcal{E},\mathcal{E}) = 0$ for $i = 1, 2$ and $\operatorname{End}(\mathcal{E}) \cong \mathbf{k}[x]/(x^2)$. The algebra $\mathbf{k}[x]/(x^2)$ has one simple module, and therefore the only module that is ever θ-stable for some θ is the simple module. Indeed, every other module admits a nontrivial endomorphism that is not an automorphism.

If $\mathcal{E}$ is a tilting bundle, then this type of pathology cannot occur. Indeed, for p, $q \in X$,

$$\operatorname{Hom}_{\mathsf{A}}(\operatorname{Hom}(\mathcal{E},\mathcal{O}_p), \operatorname{Hom}(\mathcal{E},\mathcal{O}_q)) = \begin{cases} \mathbf{k}, & p = q, \\ 0, & p \neq q. \end{cases}$$

Hence there is no proper quotient module of $\operatorname{Hom}(\mathcal{E},\mathcal{O}_p)$ that ever appears as a submodule of $\operatorname{Hom}(\mathcal{E},\mathcal{O}_q)$ for any q. For a discussion of stable quiver representations with many interesting examples including modules with trivial endomorphism ring that cannot be made stable, see [Rei08].

We now turn to the tilting bundles constructed by Hille and Perling to show that these bundles do fit into the framework of Theorem 5.5. They are defined inductively starting with a full strong exceptional sequence of line bundles on a minimal rational surface. We will describe some of the features of their construction and refer the reader to [HP14] for details. Suppose that X is a smooth rational surface and

$$X = X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$$

is a sequence of blow-ups along smooth centers recovering X from a minimal rational surface X_0. Hille and Perling use this data to construct tilting bundles $\mathcal{E}_i$ on X_i. For each i, let E_i be the exceptional divisor of f_i. Then

$$\operatorname{Ext}^2(\mathcal{O}(E_i), f_i^*\mathcal{E}_{i-1}) = 0 \quad \text{and} \quad \operatorname{Ext}^\bullet(f_i^*\mathcal{E}_{i-1}, \mathcal{O}(E_i)) = 0,$$

but $\operatorname{Ext}^1(\mathcal{O}(E_i), f_i^*\mathcal{E}_{i-1}) \neq 0$. More precisely, Hille and Perling construct $\mathcal{E}_{i-1}$ so that it has a unique indecomposable summand $\mathcal{E}'_{i-1}$ such that $\operatorname{Ext}^1(\mathcal{O}(E_i), f_i^*\mathcal{E}'_{i-1}) \neq 0$ and moreover this Ext group is one-dimensional. So there is also a unique extension

$$0 \to f^*\mathcal{E}'_{i-1} \to \mathcal{F}_i \to \mathcal{O}(E_i) \to 0.$$

Then they put $\mathcal{E}_i = f^*\mathcal{E}_{i-1} \oplus \mathcal{F}_i$ and show that it is a tilting bundle on X_i.

THEOREM 5.8. *Let X be a rational surface, and let $\mathcal{E}$ be one of Hille and Perling's tilting bundles. Then $\mathsf{A} = \operatorname{End}(\mathcal{E})$ admits a compatible weak central charge.*

Proof. Our approach is based on an idea of Bergman and Proudfoot (see [BP08]). Let $\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_m$ be the decomposition of $\mathcal{E}$ into indecomposable summands, and let $e_1, \ldots, e_m \in \mathsf{A}$ be the corresponding projectors. Suppose that M is an A-module such that Me_1 is one-dimensional and generates M. Then M is stable with respect to θ defined by

$$\theta(S_1) = \dim_{\mathbf{k}}(M) - 1 \quad \text{and} \quad \theta(S_i) = -1 \quad (i = 2, \ldots, m).$$

Indeed, since Me_1 generates M, it will generate every quotient. So if $M \twoheadrightarrow M''$ is a nonzero quotient, then

$$\begin{aligned}\theta(M'') &= \dim_{\mathbf{k}}(M) + 1 - \sum_{i=2}^{m} \dim_{\mathbf{k}}(M'' e_i) \\ &> \dim_{\mathbf{k}}(M) + 1 - \sum_{i=2}^{m} \dim_{\mathbf{k}}(M e_i) = 0.\end{aligned}$$

Suppose X is minimal. According to [HP14], $\mathcal{E}$ is a direct sum of line bundles. It is straightforward to verify that there is line bundle summand $\mathcal{L}$ of $\mathcal{E}$ such that, for each point $p \in X$, $\operatorname{Hom}(\mathcal{E}, \mathcal{O}_p)$ is generated by $\operatorname{Hom}(\mathcal{L}, \mathcal{O}_p)$.

Now, we proceed by induction on the Picard rank. Suppose $f : X \to X'$ is the blow-up of a single point and that $\mathcal{E} = f^*\mathcal{E}_{X'} \oplus \mathcal{F}$ as before, where there are an indecomposable summand $\mathcal{E}'_{X'}$ of $\mathcal{E}_{X'}$ and an exact sequence

$$0 \to f^*\mathcal{E}'_{X'} \to \mathcal{F} \to \mathcal{O}(E) \to 0,$$

where E is the exceptional divisor of f.

Notice that since $\mathcal{O}(E)|_E \cong \mathcal{O}_E(-1)$, we see that $\operatorname{Hom}(f^*\mathcal{E}_{X'}, \mathcal{O}) \to \operatorname{Hom}(f^*\mathcal{E}_{X'}, \mathcal{O}(E))$ is an isomorphism. Hence any map $f^*\mathcal{E}_{X'} \to \mathcal{F}$ has to factor through $f^*\mathcal{E}'_{X'} \to \mathcal{F}$ along E. For $p \in E$, we have the exact sequence

$$0 \to \operatorname{Hom}(\mathcal{O}(E), \mathcal{O}_p) \to \operatorname{Hom}(\mathcal{F}, \mathcal{O}_p) \to \operatorname{Hom}(f^*\mathcal{E}'_{X'}, \mathcal{O}_p) \to 0.$$

Thus the one-dimensional subspace $\operatorname{Hom}(\mathcal{O}(E), \mathcal{O}_p) \subset \operatorname{Hom}(\mathcal{E}, \mathcal{O}_p)$ is in fact an A-submodule.

Let $g : X' \to X_0$ be the map to a minimal rational surface used to construct $\mathcal{E}_{X'}$. By induction, there is a line bundle summand $\mathcal{L}$ of $\mathcal{E}_{X_0}$ such that $\operatorname{Hom}(g^*\mathcal{L}, \mathcal{O}_q)$ generates $\operatorname{Hom}(\mathcal{E}_{X'}, \mathcal{O}_q)$ for all $q \in X'$. Then for any point $p \in X$, the submodule of $\operatorname{Hom}(\mathcal{E}, \mathcal{O}_p)$ generated by $\operatorname{Hom}(f^*g^*\mathcal{L}, \mathcal{O}_p)$ contains the submodule $\operatorname{Hom}(f^*\mathcal{E}_{X'}, \mathcal{O}_p)$. Let M be the cokernel of $\operatorname{Hom}(f^*g^*\mathcal{L}, \mathcal{O}_p) \otimes_{\mathbf{k}} \mathsf{A} \to \operatorname{Hom}(\mathcal{E}, \mathcal{O}_p)$. If $M = 0$, we are done.

Otherwise, choose a nonzero element γ of M. This element must lift to an element of $\operatorname{Hom}(\mathcal{F}, \mathcal{O}_p)$ that restricts to 0 in $\operatorname{Hom}(f^*\mathcal{E}'_{X'}, \mathcal{O}_p)$. But this implies that the submodule of M generated by γ is isomorphic to the simple A module $\operatorname{Hom}(\mathcal{O}(E), \mathcal{O}_p)$. Since this holds for every element, M is isomorphic to a finite direct sum of modules of this form. Hence M admits a one-dimensional quotient, which must be isomorphic to the one-dimensional submodule $\operatorname{Hom}(\mathcal{O}(E), \mathcal{O}_q) \subset \operatorname{Hom}(\mathcal{E}, \mathcal{O}_q)$ for any point $q \in E$. Therefore, if $M \neq 0$, then there are a point $q \in E$, not equal to p, and a nonzero A-module map

$$\Phi_{\mathcal{E}}(\mathcal{O}_p) = \operatorname{Hom}(\mathcal{E}, \mathcal{O}_p) \to \Phi_{\mathcal{E}}(\mathcal{O}_q) = \operatorname{Hom}(\mathcal{E}, \mathcal{O}_q).$$

However, since $\mathcal{E}$ is a tilting bundle, $\operatorname{Hom}(\Phi_{\mathcal{E}}(\mathcal{O}_p), \Phi_{\mathcal{E}}(\mathcal{O}_q)) = 0$ if $p \neq q$. We conclude that $\operatorname{Hom}(\mathcal{E}, \mathcal{O}_p)$ is always generated by the one-dimensional space $\operatorname{Hom}(f^*g^*\mathcal{L}, \mathcal{O}_p)$, and therefore there exists a weak central charge Z on **mod**-A compatible with the tilting equivalence. $\square$

We conclude with a result of independent interest. If X is a surface with a tilting bundle $\mathcal{E}$ that decomposes as a direct sum of line bundles, then we can prove directly that it is rational.

THEOREM 5.9. *Let X be a smooth projective surface with a tilting bundle that is a direct sum of line bundles. Then X is rational.*

Proof. Denote the tilting bundle $\mathcal{E} = \bigoplus_{i=1}^{r} \mathcal{O}(D_i)$. Since $\mathcal{E}$ is a tilting bundle, $\mathrm{H}^1(\mathcal{O}(D_j - D_k)) = \mathrm{H}^2(\mathcal{O}(D_j - D_k)) = 0$ for all j, k. Suppose that i, j are such that $h^0(\mathcal{O}(D_i - D_j)) \neq 0$ and let $D \in |D_i - D_j|$. Write $D = \sum_{i=1}^{m} C_i$, where C_i are distinct irreducible, but not necessarily reduced, curves. Since $h^i(\mathcal{O}_X) = h^i(\mathcal{O}(D)) = 0$ for $i > 0$, we see that $h^1(\mathcal{O}_D) = 0$. Now, consider the exact sequence

$$0 \to \mathcal{O}_D \to \bigoplus_{i=1}^{m} \mathcal{O}_{C_i} \to \mathcal{G} \to 0.$$

Since $\dim(\mathcal{G}) = 0$, we find that $h^1(\mathcal{O}_{C_i}) = 0$. So the arithmetic genus of C_i is zero. Thus, if $\overline{C}_i \subset C_i$ is the reduced induced subscheme, then $\overline{C}_i$ also has arithmetic genus zero and is thus rational. Now, if $h^0(\mathcal{O}(D_i - D_j)) > 1$, then D must have a moving component, and X is covered by rational curves. By the classification of surfaces [BPvdV84, §VI, Theorem 1.1], X is rational, or X is a blow up of a ruled surface over a curve C. Then Orlov's theorem on blowups [Orl92, Theorem 4.3] implies that that the map $K_0(C) \to K_0(X)$ induced by derived pullback is injective. However, since $K_0(X)$ is torsion-free and $K_0(C)$ has torsion unless $C = \mathbb{P}^1$, we find that X is a blowup of a rational ruled surface and hence X is rational.

So it remains to show that, for some pair i, j, $h^0(\mathcal{O}(D_j - D_i)) > 1$. By a Riemann–Roch computation due to Hille and Perling [HP11, Lemma 3.3], if $h^0(\mathcal{O}(D_j - D_i)), h^0(\mathcal{O}(D_k - D_j)) > 0$, then

$$h^0(\mathcal{O}(D_k - D_i)) = h^0(\mathcal{O}(D_k - D_j)) + h^0(\mathcal{O}(D_j - D_i)).$$

Suppose, for contradiction, that, for all i, j, we have $h^0(\mathcal{O}(D_j - D_i)) \leq 1$. If $h^0(\mathcal{O}(D_j - D_i)) \neq 0$, then for all k, $h^0(\mathcal{O}(D_k - D_j)) = 0$. Let Q be the quiver with vertices $\{1, \ldots, r\}$ and with a single edge $i \to j$ whenever $h^0(\mathcal{O}(D_j - D_i)) = 1$. Then Q has no paths of length 2. Now, we note that $\mathrm{End}(\mathcal{E})$ is isomorphic to a quotient of the path algebra $\mathbf{k}Q$, where the kernel is contained in the span of the paths of length at least two. Hence $\mathrm{End}(\mathcal{E}) \cong \mathbf{k}Q$. However, the global dimension of $\mathbf{k}Q$ is 1 (see [Sch14, Theorem 2.15]), whereas the minimum possible global dimension of $\mathrm{End}(\mathcal{E})$ is two (see [BF12, Theorem 3.4]). So we see that, for at least one pair i, j, $h^0(\mathcal{O}(D_j - D_i)) > 1$, and hence X is rational. $\square$

REMARK 5.10. There is another result in this direction. Bondal and Polishchuk [BP93] have shown that if X is a smooth n-dimensional projective variety that admits a full exceptional collection of length $n+1$ (the minimum possible length), then X is a Fano variety.

REMARK 5.11. The McKay correspondence may be viewed as a tilting equivalence. (For a similar treatment, see [Aus86].) Let $G \subset \mathrm{SL}_n(\mathbb{C})$ be a finite subgroup. Then G acts on $S = \mathbb{C}[x_1, \ldots, x_n]$, and we can form the twisted group ring $S \rtimes G$. The category of G-equivariant sheaves on $\mathbb{C}^n$ is naturally equivalent to the category of left modules over $S \rtimes G$. The equivariant sheaf corresponding to the free module $S \rtimes G$ is $\mathcal{O} \otimes \mathbb{C}G$. We can check, for $n = 2, 3$, the inverse equivalence $\mathbf{D}(\mathbb{C}^n)^G \cong \mathbf{D}(\mathcal{M}_\theta)$ carries $\mathcal{O} \otimes \mathbb{C}G$ to $\mathcal{F} = \mathrm{pr}_{\mathcal{M}_\theta *}\mathcal{E}$, where $\mathcal{E}$ is the universal θ-stable G-constellation. Indeed, $\mathrm{pr}^*_{[\mathbb{C}^n/G]}(S \rtimes G) = \mathcal{O}_{\mathbb{C}^n \times \mathcal{M}_\theta} \rtimes G = \mathcal{O}_{[\mathbb{C}^n \times \mathcal{M}_\theta / G]}$ for the action of G on the first factor. Therefore, $\mathrm{pr}^*_{[\mathbb{C}^n/G]}(S \rtimes G) \otimes \mathcal{E} \cong \mathcal{E}$. Since $\mathcal{E}$ is flat over $\mathcal{M}_\theta$ we see that $\mathcal{F}$ is a tilting bundle on $\mathcal{M}_\theta$. The dual of a tilting bundle is also a tilting bundle. So we can interpret Φ as $\mathbf{R}\mathrm{Hom}(\mathcal{F}^\vee, -)$.

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