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Lifting Homeomorphisms and Cyclic Branched Covers of Spheres

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Abstract. We characterize the cyclic branched covers of the 2-sphere where every homeomorphism of the sphere lifts to a homeomorphism of the covering surface. This answers the question that appeared in an early version of the erratum of Birman and Hilden [2].

1. Introduction

Let Σ be a closed orientable surface. Let $p : \Sigma \to \Sigma_0$ be a cyclic branched covering space of the sphere Σ_0. We assume that all homeomorphisms of Σ_0 preserve the set of branch points. For brevity, we say that a homeomorphism f of Σ_0 *lifts* if there exists a homeomorphism $\widetilde{f}$ of Σ such that $p\widetilde{f} = fp$.

The 2-fold cover $\Sigma \to \Sigma_0$ induced by the hyperelliptic involution of Σ is of interest in the study of low-dimensional topology (see, e.g., Brendle, Margalit, and Putman [3], Johnson and Schmoll [6], and Morifuji [7]) and algebraic geometry (see, e.g., Gorchinskiy and Viviani [4] and Hidalgo [5]). In this 2-fold covering space, every homeomorphism of Σ_0 lifts. However, it is not true in general that every homeomorphism of Σ_0 lifts to a homeomorphism of the covering surface.

We answer the question that appeared in an early version of [2].

Question (Birman–Hilden). Let Σ be a closed orientable surface, and Σ_0 a 2-sphere. For which cyclic branched covering spaces of the sphere $\Sigma \to \Sigma_0$ does every homeomorphism of Σ_0 lift?

Let Σ_0° denote a sphere with punctures, and $x_0 \in \Sigma_0^\circ$. Each characteristic subgroup of $\pi_1(\Sigma_0^\circ, x_0)$ corresponds to a branched covering space of Σ_0 where every homeomorphism lifts. However, cyclic subgroups of $\pi_1(\Sigma_0^\circ, x_0)$ are rarely characteristic in $\pi_1(\Sigma_0^\circ, x_0)$. In fact, in the case of the hyperelliptic involution, the corresponding subgroup of $\pi_1(\Sigma_0^\circ, x_0)$ is not characteristic, but all homeomorphisms of Σ_0 lift.

Let A be a finite Abelian group. An *admissible k-tuple* is a tuple $(a_1, \dots, a_k) \in (A \setminus \{0\})^k$ such that $\sum_{i=1}^k a_i = 0$ and $\{a_1, \dots, a_k\}$ is a generating set for A. An admissible k-tuple defines a cyclic covering space over a punctured sphere as follows.

Let $\Sigma_{0,k}$ be a sphere with k punctures, and fix an enumeration of the punctures. Let x_i be the homology class of a loop surrounding the ith puncture that is oriented counterclockwise. The homomorphism $\phi : H_1(\Sigma_{0,k}; \mathbb{Z}) \to A$ defined by

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$\phi(x_i) = a_i$ is surjective and therefore determines a regular cover of $\Sigma_{0,k}$ with deck group A. By filling in the punctures we obtain a regular branched cover of Σ_0.

THEOREM 1.1. *Let A be a finite cyclic group, and $(a_1, \dots, a_k)$ an admissible k-tuple. Let $\Sigma \to \Sigma_0$ be the cyclic branched cover of the sphere with deck group A and k branch points defined by the admissible tuple. Every homeomorphism of Σ_0 lifts if and only if one of the following is true*:

- *There is an isomorphism $\delta : A \to \mathbb{Z}/n\mathbb{Z}$ with $k \equiv 0 \mod n$ such that $\delta(a_i) = 1$ for all i.*
- *$k = 2$, and there is an isomorphism $\delta : A \to \mathbb{Z}/n\mathbb{Z}$ for some $n \geq 3$ such that $\delta(a_1) = 1$ and $\delta(a_2) = -1$.*

The 2-fold cover induced by the hyperelliptic involution is defined by the admissible tuple $(1, \dots, 1)$ with deck group equal to $\mathbb{Z}/2\mathbb{Z}$. Our theorem provides an alternative proof that for the cover induced by the hyperelliptic involution, every homeomorphism lifts.

Superelliptic Curves: Choose distinct points $z_1, \dots, z_t \in \mathbb{C}$. Any cyclic branched cover of the sphere can be modeled by an irreducible plane curve C of the form

$$y^n = (x - z_1)^{a_1} \cdots (x - z_t)^{a_t} \tag{1}$$

for some $n \geq 2$ and integers $1 \leq a_i \leq n - 1$. Indeed, let $\widetilde{C}$ be the normalization of the projective closure of the affine curve C. Projection onto the x axis gives an n-sheeted cyclic branched covering $\widetilde{C} \to \mathbb{P}^1$ that is branched at each z_i and possibly at infinity. There is branching at infinity if and only if $\sum_{i=1}^{t} a_i \not\equiv 0 \mod n$.

A cyclic branched covering space defined by (1) has a deck group $A \cong \mathbb{Z}/n\mathbb{Z}$. Such a cover is defined by the admissible tuple $(a_1, \dots, a_t)$ if there is no branching at infinity. If there is a branch point at infinity, then the cover is defined by $(a_1, \dots, a_t, -\sum_{i=1}^{t} a_i)$ [8, Ch. 2]. The irreducibility of C ensures that $\{a_i\}$ form a generating set for A. We have an immediate corollary of Theorem 1.1.

COROLLARY 1.2. *Let $\widetilde{C} \to \mathbb{P}^1$ be a cyclic branched cover defined by an irreducible superelliptic curve as in equation* (1). *Then every homeomorphism of $\mathbb{P}^1$ lifts if and only if one of the following is true*:

- *$a_1 = \cdots = a_t$ and $t \equiv 0$ or $-1 \mod n$,*
- *$n \geq 3$ and $t = 1$, or*
- *$n \geq 3$, $t = 2$, and $a_1 \equiv -a_2 \mod n$.*

A cover where not all homeomorphisms lift: We may represent a genus 2 surface Σ_2 as a 10-gon with sides identified as in Figure 1. A counterclockwise rotation by $2\pi/5$ of the 10-gon induces an action of $\mathbb{Z}/5\mathbb{Z}$ on Σ_2. The resulting quotient space is homeomorphic to a sphere. The quotient map is a branched covering map $\Sigma_2 \to \Sigma_0$ branched at three points. The preimages of the branch points in the 10-gon are the center and two distinct orbits of vertices under rotation. The cover corresponds to the admissible tuple $(1, 1, 3)$ with the deck group equal to $\mathbb{Z}/5\mathbb{Z}$. Therefore, by Theorem 1.1, not all homeomorphisms of Σ_0 lift.

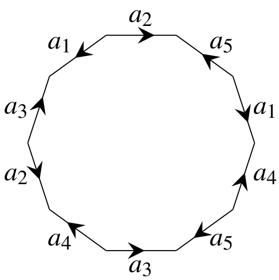

Figure 1 A rotation by $2\pi/5$ generates an action of $\mathbb{Z}/5\mathbb{Z}$ on Σ_2

Application to the mapping class group: Let X be an orientable surface possibly with marked points. The mapping class group of X, denoted $\mathrm{Mod}(X)$, is the group of orientation-preserving homeomorphisms of X that preserve the set of marked points up to isotopies that preserve the set of marked points.

Let Σ be a surface, and $\Sigma \to \Sigma_0$ be a cyclic branched cover with deck group A. The symmetric mapping class group of Σ, denoted $\mathrm{SMod}(\Sigma)$, is the subgroup of $\mathrm{Mod}(\Sigma)$ comprised of isotopy classes of fiber-preserving homeomorphisms of Σ. By work of Birman and Hilden [1, Theorem 3], the quotient $\mathrm{SMod}(\Sigma)/A$ is isomorphic to a finite-index subgroup of $\mathrm{Mod}(\Sigma_0)$. When the conditions of Theorem 1.1 are satisfied, $\mathrm{SMod}(\Sigma)/A$ is isomorphic to $\mathrm{Mod}(\Sigma_0)$. This gives us the following corrected statement of Theorem 5 in [1].

THEOREM 1.3. *Let A be a finite cyclic group, and $(a_1, \dots, a_k)$ an admissible k-tuple. Let $\Sigma \to \Sigma_0$ be the cyclic branched cover of the sphere with deck group A and k branch points defined by the admissible tuple. The quotient* $\mathrm{SMod}(\Sigma)/A$ *is isomorphic to* $\mathrm{Mod}(\Sigma_0)$ *if there is an isomorphism $\delta : A \to \mathbb{Z}/n\mathbb{Z}$ with $k \equiv 0 \mod n$ such that $\delta(a_i) = 1$ for all i and (n, k) is not equal to* $(2, 2)$, $(2, 4)$, *or* $(3, 3)$.

Theorem 5 from [1] requires that Σ is hyperbolic. In the statement of Theorem 1.3, we have excluded the cases from Theorem 1.1 where Σ is not hyperbolic.

2. Proof of Main Theorem

Let $\Sigma_{0,k}$ be a 2-sphere with k punctures. Any finite sheeted regular cover with base space $\Sigma_{0,k}$ and Abelian deck group A is determined by a surjective homomorphism $\phi : H_1(\Sigma_{0,k}; \mathbb{Z}) \to A$. The homomorphism ϕ is unique up to an automorphism of A.

Reduction to the unbranched case: A branched cover $\Sigma \to \Sigma_0$ with k branch points induces a cover of $\Sigma_{0,k}$ by removing the branch points and their preimages in Σ. Conversely, a cover of $\Sigma_{0,k}$ can be completed to a branched cover where each puncture is filled with a branch point. We will restrict homeomorphisms of Σ_0 to $\Sigma_{0,k}$ and extend homeomorphisms of $\Sigma_{0,k}$ to homeomorphisms of Σ_0 when it is convenient.

Fix an enumeration of the punctures and let x_i be the homology class of a loop surrounding the ith puncture that is oriented counterclockwise. Each $x_i \in H_1(\Sigma_{0,k}; \mathbb{Z})$ is supported on a neighborhood of the ith puncture. Let f be a homeomorphism of $\Sigma_{0,k}$. The automorphism f_* of $H_1(\Sigma_{0,k}, \mathbb{Z})$ is determined by the permutation induced by f on the punctures of $\Sigma_{0,k}$. Indeed, let $\sigma \in S_k$ be the permutation induced by f. If f is orientation preserving, then $f_*(x_i) = x_{\sigma(i)}$. If f is orientation reversing, then $f_*(x_i) = -x_{\sigma(i)}$.

Recall that an admissible k-tuple defines a surjective homomorphism $\phi : H_1(\Sigma_{0,k}; \mathbb{Z}) \to A$ by $\phi(x_i) = a_i$ and therefore defines a branched cover of Σ_0. Another admissible k-tuple $(a'_1, \ldots, a'_k)$ determines an equivalent covering space if and only if there is an automorphism $\psi \in \operatorname{Aut}(A)$ such that $\psi(a_i) = a'_i$ for all i.

LEMMA 2.1. *Let A be a finite Abelian group, and $(a_1, \ldots, a_k)$ an admissible k-tuple. Let $\Sigma \to \Sigma_{0,k}$ be the covering space defined by this tuple. Let f be a homeomorphism of $\Sigma_{0,k}$, and let $\sigma \in S_k$ be the permutation of the punctures induced by f. The homeomorphism f lifts if and only if there is an automorphism $\psi \in \operatorname{Aut}(A)$ such that $\psi(a_i) = a_{\sigma(i)}$ for all i.*

Proof. Let $\phi : H_1(\Sigma_{0,k}; \mathbb{Z}) \to A$ be the homomorphism defining the cover and defined by $\phi(x_i) = a_i$. Let f_* be the automorphism of $H_1(\Sigma_{0,k}; \mathbb{Z})$ induced by f. We use the following facts:

(1) The equality $f_*(\ker(\phi)) = \ker(\phi)$ holds if and only if $\ker(\phi f_*) = \ker(\phi)$.
(2) Let $f, g : G \to A$ be surjective homomorphisms. Then $\ker(f) = \ker(g)$ if and only if $f = \xi g$ for some $\xi \in \operatorname{Aut}(A)$.

We omit the proofs of these facts.

The homeomorphism f lifts if and only if $f_*(\ker(\phi)) = \ker(\phi)$. Therefore by fact (1), f lifts if and only if $\ker(\phi f_*) = \ker(\phi)$. By fact (2), f lifts if and only if there exists an automorphism $\psi \in \operatorname{Aut}(A)$ such that $\phi f_* = \psi\phi$. If f is orientation preserving, then $a_{\sigma(i)} = \phi f_*(x_i) = \psi\phi(x_i) = \psi(a_i)$ for all i, and the result follows.

The map $a \mapsto -a$ is an automorphism of an Abelian group. If f is orientation reserving, then we compose $a \mapsto -a$ with the automorphism ψ in the orientation-preserving case. □

LEMMA 2.2. *Let A be a finite cyclic group, and let $(a_1, \ldots, a_k)$ be an admissible tuple. For every permutation $\sigma \in S_k$, there exists $\psi \in \operatorname{Aut}(A)$ such that $\psi(a_i) = a_{\sigma(i)}$ for all i if and only if one of the following is true*:

- *There is an isomorphism $\delta : A \to \mathbb{Z}/n\mathbb{Z}$ with $k \equiv 0 \mod n$ such that $\delta(a_i) = 1$ for all i.*
- *$k = 2$, and there is an isomorphism $\delta : A \to \mathbb{Z}/n\mathbb{Z}$ for some $n \geq 3$ such that $\delta(a_1) = 1$ and $\delta(a_2) = -1$.*

Proof. For the forward direction, first suppose that all a_i are equal. Then each a_i is a generator of A, and there is an isomorphism $\delta : A \to \mathbb{Z}/n\mathbb{Z}$ such that $\delta(a_i) = 1$ for all i. The condition $\sum_{i=1}^{k} a_i = 0$ implies that $k \equiv 0 \mod n$.

Suppose now that the a_i are not all equal. Then they must all be distinct. Indeed, assume to the contrary that there exist three distinct elements a_p, a_q, a_r of the admissible tuple such that $a_p = a_q \neq a_r$. Let $\sigma \in S_k$ be the transposition that switches q and r. By assumption there exists $\psi \in \mathrm{Aut}(A)$ such that $\psi(a_i) = a_{\sigma(i)}$ for all i. Therefore $a_p = \psi(a_p) = \psi(a_q) = a_r$, which is a contradiction.

We may therefore assume the a_i are distinct. Then there is a subgroup of $\mathrm{Aut}(A)$ isomorphic to the symmetric group S_k. Since the automorphism group of a cyclic group is abelian, it must be that $k = 2$. Since $k = 2$, we have that $a_1 = -a_2$ with a_1 a generator of A. Since a_1 and a_2 are distinct, $|A| \geq 3$. Therefore the map $\delta : A \to \mathbb{Z}/n\mathbb{Z}$ with $\delta(a_1) = 1$ and $\delta(a_2) = -1$ is an isomorphism when $n = |A|$.

For the converse, we must write down an appropriate automorphism for each permutation $\sigma \in S_k$. In the case that $\delta(a_i) = 1$ for all i, the identity automorphism suffices for all permutations. In the case where $k = 2$, $\delta(a_1) = 1$, and $\delta(a_2) = -1$, the automorphism $a \mapsto -a$ of A suffices for the nontrivial permutation. □

We now prove the main result.

Proof of Theorem 1.1. Any permutation of the branch points can be induced by a homeomorphism of the sphere. Therefore, the result follows by combining Lemmas 2.1 and 2.2. □

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