Michigan Math. J. 66 (2017), 745–767

Classification Problem of Holomorphic Isometries of the Unit Disk Into Polydisks

Shan Tai Chan

Abstract. We study the classification problem of holomorphic isometric embeddings of the unit disk into polydisks as in [Ng10; Ch16a]. We give a complete classification of all such holomorphic isometries when the target is the 4-disk Δ^4. Moreover, we classify those holomorphic isometric embeddings with certain prescribed sheeting numbers. In addition, we prove that a known example in the space $\mathbf{HI}_k(\Delta, \Delta^{qk}; q)$ is globally rigid for any integers $k, q \geq 2$, which generalizes Theorem 1.1 in [Ch16a].

1. Introduction

In 2011, Mok [Mok11, pp. 262–263] raised a question about the structure of the space $\mathbf{HI}_k(\Delta, \Delta^p)$ of holomorphic isometric embeddings from $(\Delta, k\, ds^2_\Delta)$ to $(\Delta^p, ds^2_{\Delta^p})$, where ds^2_Δ (resp. $ds^2_{\Delta^p}$) denotes the Bergman metric on the open unit disk Δ in $\mathbb{C}$ (resp. the open unit polydisk Δ^p in $\mathbb{C}^p$), and $k > 0$ is a real constant. More precisely, Mok [Mok11] asked whether all holomorphic isometries from $(\Delta, k\, ds^2_\Delta)$ to $(\Delta^p, ds^2_{\Delta^p})$ are parameterized by the qth root embeddings for $q \leq p$, the diagonal embeddings, and automorphisms of Δ and Δ^p. This is precisely Problem 5.1.2 in [Mok11, pp. 262–263], which we call the classification problem of holomorphic isometric embeddings of the unit disk into polydisks (or simply the classification problem). Note that such a real constant k is indeed a positive integer satisfying $1 \leq k \leq p$ by [Ng10, p. 2909]. Ng [Ng10] has provided a complete description of $\mathbf{HI}_k(\Delta, \Delta^p)$ for $p = 2, 3$ and solved the classification problem affirmatively for the space $\mathbf{HI}(\Delta, \Delta^p)$ when $p = 2$ or 3. Given any $f \in \mathbf{HI}_k(\Delta, \Delta^p)$, we call a map given by $F = \Psi \circ f \circ \psi$ a reparameterization of f, where Ψ, ψ are some automorphisms of Δ^p, Δ, respectively. In the case where $k = p$, Ng [Ng08; Ng10] showed that any $f \in \mathbf{HI}_p(\Delta, \Delta^p)$ is given by $f(z) = (z, \dots, z)$ up to reparameterizations. The general case where $f \in \mathbf{HI}(\Delta, \Delta^p)$ for some $p \geq 4$ remains unknown. Recently, the author [Ch16a] has proven that any $f \in \mathbf{HI}_1(\Delta, \Delta^p; p)$ is the pth root embedding up to reparameterizations, where $p \geq 2$ is an integer. In particular, the 4th root embedding in $\mathbf{HI}_1(\Delta, \Delta^4; 4)$ is globally rigid in the sense of [Mok11, p. 261] (cf. [Ch16a]). One of the main objectives of this paper is to provide a complete description of $\mathbf{HI}_k(\Delta, \Delta^4)$ so that the classification problem of all holomorphic isometric embeddings from $(\Delta, k\, ds^2_\Delta)$ to $(\Delta^4, ds^2_{\Delta^4})$ will be solved as follows:

Received May 27, 2016. Revision received June 13, 2017.

THEOREM 1.1. *Let $f \in \mathbf{HI}_k(\Delta, \Delta^4)$ be a holomorphic isometric embedding such that all component functions of f are nonconstant.*

(1) *If $k = 1$, then f is one of the following up to reparameterizations*:
  (a) *the 4th root embedding $F_4 : \Delta \to \Delta^4$,*
  (b) $(\alpha_1, \alpha_2 \circ \beta_1, \alpha_3 \circ (\beta_2 \circ \beta_1), \beta_3 \circ (\beta_2 \circ \beta_1))$, *where* $(\alpha_j, \beta_j) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$ *for* $j = 1, 2, 3$,
  (c) $(\alpha_1, h^2 \circ \alpha_2, h^3 \circ \alpha_2, h^4 \circ \alpha_2)$, *where* $(\alpha_1, \alpha_2) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$ *and* $(h^2, h^3, h^4) \in \mathbf{HI}_1(\Delta, \Delta^3; 3)$,
  (d) $(\beta_1, \alpha_1 \circ \beta_2, \alpha_2 \circ \beta_2, \beta_3)$, *where* $(\beta_1, \beta_2, \beta_3) \in \mathbf{HI}_1(\Delta, \Delta^3; 3)$ *and* $(\alpha_1, \alpha_2) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$,
  (e) $(\alpha_1 \circ \alpha_2, \beta_1 \circ \alpha_2, \alpha_3 \circ \beta_2, \beta_3 \circ \beta_2)$, *where* $(\alpha_j, \beta_j) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$ *for* $j = 1, 2, 3$.

(2) *If $k = 2$, then $f(z)$ is one of the following up to reparameterizations*:
  (a) $(\alpha_1(z), \beta_1(z), \alpha_2(z), \beta_2(z))$, *where* $(\alpha_j, \beta_j) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$ *for* $j = 1, 2$.
  (b) $(z, \alpha_1(z), (\alpha_2 \circ \beta_1)(z), (\beta_2 \circ \beta_1)(z))$, *where* $(\alpha_j, \beta_j) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$ *for* $j = 1, 2$.
  (c) $(z, \alpha_1(z), \alpha_2(z), \alpha_3(z))$, *where* $(\alpha_1, \alpha_2, \alpha_3) \in \mathbf{HI}_1(\Delta, \Delta^3; 3)$.

(3) *If $k = 3$, then $f(z) = (z, z, \alpha(z), \beta(z))$ up to reparameterizations, where* $(\alpha, \beta) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$.

(4) *If $k = 4$, then $f(z) = (z, z, z, z)$ is the diagonal embedding up to reparameterizations.*

REMARK. In fact, this theorem says that all holomorphic isometric embeddings $f : (\Delta, k\, ds^2_\Delta) \to (\Delta^4, ds^2_{\Delta^4})$ with the isometric constant k are parameterized by the diagonal embeddings, automorphisms of Δ (resp. Δ^4), and the pth root embeddings up to reparameterizations for $2 \le p \le 4$.

Moreover, we will show that it is possible to provide a complete description of all holomorphic isometric embeddings with certain prescribed sheeting numbers. In addition, we prove that a known example in the space $\mathbf{HI}_k(\Delta, \Delta^{qk}; q)$ is globally rigid for any integers $k, q \ge 2$, which generalizes Theorem 1.1 in [Ch16a].

1.1. Preliminary

Let $\Delta \subset \mathbb{C}$ be the open unit disk with the Poincaré metric $ds^2_\Delta = 2\,\mathrm{Re}(g\, dz \otimes d\bar{z})$, where $g = -2 \frac{\partial^2}{\partial z \partial \bar{z}} \log(1 - |z|^2)$. For any integer $p \ge 2$, let $\Delta^p = \{(z_1, \dots, z_p) \in \mathbb{C}^p \mid |z_j| < 1, 1 \le j \le p\}$ be the polydisk, which is viewed as p copies of Δ. Moreover, Δ^p is equipped with the Kähler metric $ds^2_{\Delta^p}$, which is the product metric induced from the Poincaré metric ds^2_Δ. More precisely, we take the real analytic function $-2 \sum_{j=1}^p \log(1 - |z_j|^2)$ as a Kähler potential for $ds^2_{\Delta^p}$ (see [Ng10, p. 2908]). Let $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere.

Let $f : (\Delta, k\, ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$ be a holomorphic isometric embedding with the isometric constant k and the global sheeting number n (see [Ng10, pp. 2908–2909]). In this paper, all holomorphic isometric embeddings

$$f = (f^1, \ldots, f^p) : (\Delta, k\, ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$$

are assumed to be *genuine,* i.e., all component functions of f are nonconstant, as mentioned in [Ng08, p. 7]. We may always assume that $f(0) = \mathbf{0}$ after composing with some $\Psi \in \mathrm{Aut}(\Delta^p)$. In [Ng10], we have the functional equation

$$\prod_{\mu=1}^{p} (1 - |f^\mu(z)|^2) = (1 - |z|^2)^k \quad \forall z \in \Delta$$

and the polarized functional equation

$$\prod_{\mu=1}^{p} (1 - f^\mu(z)\overline{f^\mu(w)}) = (1 - z\overline{w})^k \quad \forall z, w \in \Delta.$$

Let $V \subset \mathbb{P}^1 \times (\mathbb{P}^1)^p$ be the irreducible projective-algebraic curve such that $\mathrm{Graph}(f) \subset V$ as obtained in [Ng10, Proposition 4.2]. From [Ng10, p. 2911], $V_j := P_j(V)$ is a projective-algebraic curve containing the graph of f^j, where $P_j : V \to \mathbb{P}^1 \times \mathbb{P}^1$ is defined by $P_j(z, w_1, \ldots, w_p) = (z, w_j)$, $1 \le j \le p$. Let $\pi : V \to \mathbb{P}^1$ be the finite branched covering given by $\pi(z, w_1, \ldots, w_p) = z$, and $\pi_j : V_j \to \mathbb{P}^1$ be defined by $\pi_j(z, w_j) = z$, $1 \le j \le p$. Recall that f has the global sheeting number equal to n or, equivalently, π is an n-sheeted branched covering. In addition, the sheeting number s_j of a component function f^j of f is defined so that $\pi_j : V_j \to \mathbb{P}^1$ is an s_j-sheeted branched covering, $j = 1, \ldots, p$. Moreover, Ng [Ng10, p. 2913] has shown that there is a rational function $R_j : \mathbb{P}^1 \to \mathbb{P}^1$ such that $R_j(f^j(z)) = z$ for $z \in \Delta$ and $R_j(\frac{1}{\overline{w}}) = 1/\overline{R_j(w)}$, so that $R_j(\partial\Delta) \subset \partial\Delta$ for $1 \le j \le p$, which is indeed obtained from the s_j-sheeted branched covering π_j such that R_j is of degree s_j. We refer the readers to [Ng10, pp. 2910–2913] for details.

Given any bounded symmetric domains $D \Subset \mathbb{C}^n$ and $\Omega \Subset \mathbb{C}^N$, Mok [Mok11] has introduced the space $\mathbf{HI}(D, \Omega)$ of all holomorphic isometries from $(D, \lambda\, ds^2_D)$ to (Ω, ds^2_Ω) for some real constant $\lambda > 0$, where ds^2_D and ds^2_Ω denote the Bergman metrics of D and Ω, respectively. In particular, in the case where $D = \Delta$ and $\Omega = \Delta^p$, we also have the spaces $\mathbf{HI}_k(\Delta, \Delta^p)$, $\mathbf{HI}_k(\Delta, \Delta^p; n)$, and $\mathbf{HI}_k(\Delta, \Delta^p; n; s_1, \ldots, s_p)$ so as to specify the isometric constant k, the sheeting number s_j of each component function of the isometries, $1 \le j \le p$, and the global sheeting number n (see [Mok11, p. 263]).

Let V' be a smooth irreducible algebraic curve, and Y be a compact Riemann surface. If $\pi' : V' \to Y$ is a finite branched covering, then, for each point $y \in Y$, denote by $v(\pi', x)$ the ramification index of π' at x and by $b(\pi', y)$ the branching order of π' at y in the sense of [GH78, p. 217], where $x \in \pi'^{-1}(y)$. From [Ng08; Ng10; Ch16a], for $f \in \mathbf{HI}_1(\Delta, \Delta^p; n; s_1, \ldots, s_p)$, we denote all branches of f^j

over Δ by f_l^j, all branches of f^j over $\mathcal{O} := \mathbb{P}^1 \setminus \overline{\Delta}$ by $f_{l,-}^j$, $1 \le l \le s_j$, and $f_1^j := f^j$, $1 \le j \le p$.

Let $\mathcal{H} := \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$ be the the upper half-plane, and $\mathcal{H}^p := \{(\tau_1, \dots, \tau_p) \in \mathbb{C}^p \mid \operatorname{Im}\tau_j > 0, 1 \le j \le p\}$ for $p \ge 1$. Denote by $ds^2_{\mathcal{H}}$ the Poincaré metric on $\mathcal{H}$, so that $(\mathcal{H}, ds^2_{\mathcal{H}})$ is of constant Gaussian curvature -1, i.e, $ds^2_{\mathcal{H}} = 2\operatorname{Re}(d\tau \otimes d\overline{\tau}/(2(\operatorname{Im}\tau)^2))$. Moreover, $\mathcal{H}^p$ is equipped with the Kähler metric $ds^2_{\mathcal{H}^p}$, which is the product metric induced from the Poincaré metric $ds^2_{\mathcal{H}}$. Mok [Mok12] has defined a map $\rho_p : \mathcal{H} \to \mathcal{H}^p$ $(p \ge 2)$ by $\rho_p(\tau) = (\tau^{1/p}, \gamma\tau^{1/p}, \dots, \gamma^{p-1}\tau^{1/p})$, where $\gamma := e^{i\pi/p}$ and $\tau^{1/p} = r^{1/p}e^{i\theta/p}$ for $\tau = re^{i\theta}$, $0 < \theta < \pi$. From [Mok12], the map $\rho_p : (\mathcal{H}, ds^2_{\mathcal{H}}) \to (\mathcal{H}^p, ds^2_{\mathcal{H}^p})$ is a nonstandard (i.e., not totally geodesic) holomorphic isometric embedding. Then, the pth root embedding $F_p : (\Delta, ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$ can be defined from ρ_p via the Cayley transform $\iota : \mathcal{H} \to \Delta$, $\tau \mapsto \frac{\tau - i}{\tau + i}$, and target automorphisms (see [Ch16a]). When $p = 2$ (resp. $p = 3$), F_p is called the square-root embedding (resp. cube-root embedding).

We denote by Σ_p the symmetric group on p elements. Moreover, we say that two holomorphic maps $G_1, G_2 : D \to \Omega$ between bounded symmetric domains D and Ω are congruent to each other if $G_1 = \phi \circ G_2 \circ \psi$ for some $\phi \in \operatorname{Aut}(\Omega)$ and $\psi \in \operatorname{Aut}(D)$.

2. General Properties of Holomorphic Isometries in $\mathbf{HI}_k(\Delta, \Delta^p)$

2.1. *Special Branching Behavior of Certain Holomorphic Isometries in* $\mathbf{HI}_k(\Delta, \Delta^p)$

For holomorphic isometric embeddings $f \in \mathbf{HI}_k(\Delta, \Delta^p)$ with certain branching behaviour, we will prove that the classification problem of such isometries can be reduced to that of holomorphic isometric embeddings in $\mathbf{HI}_k(\Delta, \Delta^{p-1})$.

LEMMA 2.1. *Let $g : \Delta \to \Delta$ be a component function of a holomorphic isometric embedding $f = (f^1, \dots, f^p) \in \mathbf{HI}_k(\Delta, \Delta^p)$ satisfying $f(0) = \mathbf{0}$. Suppose that there is $\varphi \in \operatorname{Aut}(\mathbb{P}^1)$ such that $\varphi \circ g$ is also a component function of f, where $\varphi(z) := \frac{az+b}{cz+d}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} u_3 & 0 \\ -\det \mathbf{U} & u_1 \end{pmatrix}$ for some unitary matrix $\mathbf{U} = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}$ satisfying $u_1, u_3 \in \mathbb{C} \setminus \{0\}$. Then, we have*

$$(1 - |g(z)|^2)(1 - |\varphi(g(z))|^2) = 1 - |h(z)|^2,$$

where $h : \Delta \to \mathbb{C}$ is a holomorphic function defined by

$$h(z) := \frac{g(z) - u_4(g(z))^2}{u_1 - (\det \mathbf{U})g(z)}.$$

Proof. We may assume without loss of generality that $g = f^1$ and $\varphi \circ g = f^2$. Then, $R_1(f^1(z)) = z = R_2(f^2(z)) = R_2(\varphi(f^1(z)))$ so that R_1 and $R_2 \circ \varphi$ are meromorphic functions on $\mathbb{P}^1$ satisfying $R_1|_{U'} = (R_2 \circ \varphi)|_{U'}$, where U' is the image of f^1 in $\mathbb{P}^1$, which is an open subset by the Open Mapping Theorem for holomorphic functions. In particular, $R_1 = R_2 \circ \varphi$ by the Identity Theorem. We compute

$$\begin{aligned}&u_1 h(z) + u_2 f^1(z)(\varphi\circ f_1)(z)\\ =&\frac{u_1 f^1(z) - u_1u_4(f^1(z))^2}{u_1-(\det\mathbf{U})f^1(z)} + u_2\frac{u_3(f^1(z))^2}{u_1-(\det\mathbf{U})f^1(z)}\\ =&f^1(z)\end{aligned}$$

and

$$\begin{aligned}&u_3 h(z) + u_4 f^1(z)(\varphi\circ f_1)(z)\\ =&\frac{u_3 f^1(z) - u_3u_4(f^1(z))^2}{u_1-(\det\mathbf{U})f^1(z)} + u_4\frac{u_3(f^1(z))^2}{u_1-(\det\mathbf{U})f^1(z)}\\ =&\frac{u_3 f^1(z)}{u_1-(\det\mathbf{U})f^1(z)} = \varphi(f^1(z)).\end{aligned}$$

Thus, we have

$$\begin{pmatrix} f^1(z) \\ \varphi(f^1(z))\end{pmatrix} = \mathbf{U}\cdot\begin{pmatrix} h(z) \\ f^1(z)\varphi(f^1(z))\end{pmatrix}.$$

Actually, we also need to show that $f^1(z) \neq u_1/\det\mathbf{U}$ for $z\in\overline{\Delta}$ so as to ensure that h is holomorphic. Suppose that $f^1(z_0) = u_1/\det\mathbf{U}$ for some $z_0\in\overline{\Delta}$. Then, $\varphi(f^1(z_0)) = \infty$. This would imply that $\infty = R_2(\infty) = R_2(\varphi(f^1(z_0))) = R_1(f^1(z_0)) = z_0$ by [Ng10, p. 2913] and the fact that $R_2\circ\varphi = R_1$, which is a contradiction. Thus, $f^1(z)\neq u_1/\det\mathbf{U}$ for $z\in\overline{\Delta}$ so that the function h is holomorphic on Δ and continuous on $\overline{\Delta}$, i.e., the extension $\widetilde{h}:\overline{\Delta}\to\overline{\Delta}$ of h is continuous. Now, we have

$$|f^1(z)|^2 + |\varphi(f^1(z))|^2 = |h(z)|^2 + |f^1(z)\varphi(f^1(z))|^2$$

for $z\in\Delta$ because $\mathbf{U}$ is an unitary matrix and thus $\mathbf{U}$ preserves the Euclidean norm of holomorphic mappings. The result follows. □

THEOREM 2.2. *Let* $f = (f^1,\ldots,f^p)\in\mathbf{HI}_k(\Delta,\Delta^p;n;s_1,\ldots,s_p)$ *with* $f(0)=\mathbf{0}$, *where* $p\geq 4$ *is an integer. Suppose that there is a point* $z_0\in\partial\Delta$ *such that* $\nu(R_{\sigma(j)}, f^{\sigma(j)}(z_0))\geq 2$ $(j = p-1, p)$ *and* $\nu(R_{\sigma(\mu)}, f^{\sigma(\mu)}(z_0)) = 1$ $(\mu = 1,\ldots,p-2)$ *for some* $\sigma\in\Sigma_p$. *Then,* $s_{\sigma(p-1)} = s_{\sigma(p)}$ *is an even integer and there exists* $\psi\in\mathrm{Aut}(\mathbb{P}^1)$ *with* $\psi(0)=0$ *such that* $\psi\circ f_1^{\sigma(p-1)} = f_1^{\sigma(p)}$ *so that* $R_{\sigma(p)}\circ\psi = R_{\sigma(p-1)}$ *and* ψ *is of the form* $\psi(z) = u_3z/(-(\det\mathbf{U})z+u_1)$ *for some unitary matrix* $\mathbf{U} = \left(\begin{smallmatrix} u_1 & u_2\\ u_3 & u_4\end{smallmatrix}\right)$ *satisfying* $u_1,u_3\in\mathbb{C}\setminus\{0\}$. *In particular, we have*

$$(1-|f^{\sigma(p-1)}(z)|^2)(1-|f^{\sigma(p)}(z)|^2) = 1-|h(z)|^2$$

for some holomorphic function h *on* Δ *and thus*

$$(f^{\sigma(1)},\ldots,f^{\sigma(p-2)},h):(\Delta,k\,ds^2_\Delta)\to(\Delta^{p-1},ds^2_{\Delta^{p-1}})$$

is a holomorphic isometric embedding.

REMARK. The assumption made in the theorem may be replaced by the existence of a certain branch of f which is of the form $\left(f_1^1, \dots, f_1^{p-2}, f_{l_{p-1}}^{p-1}, f_{l_p}^p\right)$ up to a permutation of the component functions of f, where $l_j \neq 1$ for $j = p-1, p$. Denote by B_π the branching locus of the finite branched covering π as a subset of $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Then, the assumption may be replaced by that of the existence of a continuous path $\gamma : [0,1] \to \mathbb{P}^1 \setminus B_\pi$ such that $\gamma(0) = \gamma(1) = 0$ and perform (multivalued) analytic continuation of $f = (f_1^1, \dots, f_1^p)$ along γ would come up with a branch of f which is of the form $(g_1, \dots, g_p)$, where $g_{\sigma(j)} := f_1^{\sigma(j)}$ for $1 \le j \le p-2$ and $g_{\sigma(\mu)} := f_{l_{\sigma(\mu)}}^{\sigma(\mu)}$ with $l_{\sigma(\mu)} \neq 1$ for $\mu = p-1, p$, and for some $\sigma \in \Sigma_p$.

Proof of Theorem 2.2. We may assume without loss of generality that $\sigma = \mathrm{Id}$ is the identity permutation. Starting with the branch $f = (f_1^1, \dots, f_1^p)$ at 0, we perform (multivalued) analytic continuation along some simple closed loop around z_0 once to obtain $(f_1^1, \dots, f_1^{p-2}, f_2^{p-1}, f_2^p)$. (Noting that we may relabel the branches of each f^j so that we can obtain f_2^j by performing (multivalued) analytic continuation of f_1^j along some simple closed loop around z_0 once for $j = p-1, p$.) By the polarized functional equation, we have

$$\left(1 - f_1^{p-1}(z)\overline{f_2^{p-1}(0)}\right)\left(1 - f_1^p(z)\overline{f_2^p(0)}\right) = 1$$

for $z \in \Delta$ so that $f_1^p(z) = \psi(f_1^{p-1}(z))$, where $\psi(w) := \left(1/\overline{f_2^p(0)}\right)\left(w/(w - 1/\overline{f_2^{p-1}(0)})\right)$. Note that $f_2^j(0) \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ for $j = p-1, p$, thus $\psi \in \mathrm{Aut}(\mathbb{P}^1)$ because

$$\det\begin{pmatrix} 1/\overline{f_2^p(0)} & 0 \\ 1 & -1/\overline{f_2^{p-1}(0)} \end{pmatrix} = -\frac{1}{\overline{f_2^p(0)}\,\overline{f_2^{p-1}(0)}} \neq 0.$$

In particular, $s_{p-1} = s_p$ and $R_p \circ \psi = R_{p-1}$. From the polarized functional equation, we also have

$$\left(1 - f_2^{p-1}(z)\overline{f_2^{p-1}(0)}\right)\left(1 - f_2^p(z)\overline{f_2^p(0)}\right) = 1$$

so that $\psi(f_2^{p-1}(z)) = f_2^p(z)$ for $z \in \Delta$. Now, we have $f_2^p(0) = \psi(f_2^{p-1}(0)) = |f_2^{p-1}(0)|^2/(\overline{f_2^p(0)} \cdot (|f_2^{p-1}(0)|^2 - 1))$ so that

$$\frac{1}{|f_2^p(0)|^2} + \frac{1}{|f_2^{p-1}(0)|^2} = 1.$$

Therefore, we have $|f_2^j(0)|^2 > 1$ for $j = p-1, p$. Then, one can verify that $\psi(z) = u_3 z/(-(\det \mathbf{U})z + u_1)$, where

$$\mathbf{U} = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} := \begin{pmatrix} \dfrac{-\lambda\overline{f_2^p(0)}}{\lambda f_2^{p-1}(0)} & \dfrac{1/f_2^p(0)}{f_2^{p-1}(0)\left(1 - 1/|f_2^p(0)|^2\right)} \end{pmatrix}$$

is a unitary matrix with $\lambda = \sqrt{\left(1 - 1/|f_2^p(0)|^2\right)\left(1/|f_2^p(0)|^2\right)}e^{i\theta_0}$ for some $\theta_0 \in [0, 2\pi)$. By Lemma 2.1, the holomorphic function h on Δ defined by

$$h(z) := \frac{f^{p-1}(z) - u_4(f^{p-1}(z))^2}{u_1 - (\det \mathbf{U}) f^{p-1}(z)}$$

satisfies

$$(1 - |f^{p-1}(z)|^2)(1 - |f^p(z)|^2) = 1 - |h(z)|^2.$$

Then, $(f^1, \ldots, f^{p-2}, h) : (\Delta, k\, ds_\Delta^2) \to (\Delta^{p-1}, ds^2_{\Delta^{p-1}})$ is clearly a holomorphic isometric embedding. Hence, there is a rational function R_h such that $R_h(h(z)) = z$, and we have $2 \cdot \deg R_h = \deg R_{p-1} = s_{p-1} = s_p$ so that $s_p = s_{p-1}$ is an even integer. □

2.2. *Special Sheeting Numbers of Holomorphic Isometries*

In the study of the structure of $\mathbf{HI}_1(\Delta, \Delta^p; n; s_1, \ldots, s_p)$ in [Ng10], if $s_j = 2$ for some j, then the study of holomorphic isometries $f = (f^1, \ldots, f^p) : (\Delta, ds_\Delta^2) \to (\Delta^p, ds^2_{\Delta^p})$ can be reduced to the study of holomorphic isometries from (Δ, ds_Δ^2) to $(\Delta^{p-1}, ds^2_{\Delta^{p-1}})$. For example, in the proof of Theorem 6.8 in [Ng10, pp. 2918–2919], Ng has reduced the study of certain $f \in \mathbf{HI}(\Delta, \Delta^p)$ to the understanding of the space $\mathbf{HI}(\Delta, \Delta^{p-1})$ and so on. For the study of the space $\mathbf{HI}_1(\Delta, \Delta^p; n; s_1, \ldots, s_p)$, one may ask whether $s_j = q$ for some prime number $q \geq 3$ and some j could lead to a similar phenomenon as in the case of $s_j = 2$ for some j. We do not have any general method to handle such a problem. However, for some small prime number $q \geq 3$, it may be possible for us to use the method in [Ch16a] to deal with the problem. In this section, we will show that when $q = 3$, a similar phenomenon occurs as in the case where $s_j = 2$ for some j.

LEMMA 2.3. *Suppose that h is a component function of a holomorphic isometric embedding $f : (\Delta, k\, ds_\Delta^2) \to (\Delta^p, ds^2_{\Delta^p})$ such that* $\deg R_h = 3$, *where $R_h : \mathbb{P}^1 \to \mathbb{P}^1$ is the rational function of degree* 3 *such that $R_h(h(z)) = z$, $R_h(\frac{1}{\overline{w}}) = 1/\overline{R_h(w)}$ and $R_h(\partial\Delta) \subset \partial\Delta$. Then, for any branch point $a \in \partial\Delta$ of R_h, we have $|w| = 1$ for all $w \in R_h^{-1}(a)$.*

Proof. We may assume without loss of generality that $f(0) = \mathbf{0}$. Let m be the number of distinct branch points of R_h, $\{a_1, \ldots, a_m\} \subset \partial\Delta$ be the set of all distinct branch points of R_h and the branching order of R_h at a_j is denoted by b_j for $1 \leq j \leq m$. Since $\deg R_h = 3$, we have $\sum_{i=1}^m b_i = 4$ so that $2 \leq m \leq 4$. After reordering the branch points of h if necessary, we may assume without loss of generality that $b_1 \leq \cdots \leq b_m$. Then, we have the following possibilities:

(1) $m = 2$ and $(b_1, b_2) = (2, 2)$;
(2) $m = 3$ and $(b_1, b_2, b_3) = (1, 1, 2)$;
(3) $m = 4$ and $(b_1, b_2, b_3, b_4) = (1, 1, 1, 1)$.

If $b_i = 1$ for some i, then $|R_h^{-1}(a_i)| = 2$ and thus $R_h^{-1}(a_i) = \{w_1, w_2\}$ such that the ramification index of R_h at w_1 (resp. w_2) equals 1 (resp. 2) for some distinct

$w_1, w_2 \in \mathbb{P}^1$. We have either $|w_1| = |w_2| = 1$ or $w_1 = 1/\overline{w_2}$ by [Ng10, Corollary 4.7]. If $w_1 = 1/\overline{w_2}$, then the ramification order of R_h at w_1 would be the same as that of R_h at w_2, which contradicts the assumption that $b_i = 1$. Thus, we have $|w_1| = |w_2| = 1$.

If $b_i = 2$, then clearly $|R_h^{-1}(a_i)| = 1$ and $w \in R_h^{-1}(a_i)$ would satisfy $|w| = 1$ because $(a_i, w) \in V_h$ if and only if $(a_i, \frac{1}{\overline{w}}) \in V_h$, where V_h is the projective-algebraic curve in $\mathbb{P}^1 \times \mathbb{P}^1$ containing the graph of h (cf. [Ng10, p. 2912]). Thus, we have verified that if h is a component function of a holomorphic isometric embedding from $(\Delta, k\, ds_\Delta^2)$ to $(\Delta^p, ds_{\Delta^p}^2)$ with $\deg R_h = 3$, then we have $|w| = 1$ for all $w \in R_h^{-1}(a_i)$ and for $i = 1, \ldots, m$. On the other hand, we have shown that for an arbitrary branch h_l of h, we have $|h_l(a_i)| = 1$ for $i = 1, \ldots, m$. □

Note that Lemma 6.7 in [Ng10, p. 2917] asserts that if the sheeting number of some component function g of a holomorphic isometry from (Δ, ds_Δ^2) to $(\Delta^p, ds_{\Delta^p}^2)$ is equal to 2, then there exists a holomorphic function $h : \Delta \to \Delta$ such that $(g, h) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$. The following proposition provides a similar result in the case where two component functions of a holomorphic isometry from (Δ, ds_Δ^2) to $(\Delta^p, ds_{\Delta^p}^2)$ have the sheeting numbers equal to 3.

PROPOSITION 2.4. *Let $p \geq 3$ be an integer. If $h^1, h^2 : \Delta \to \Delta$ are two distinct component functions of a holomorphic isometric embedding $f = (f^1, \ldots, f^p) : (\Delta, ds_\Delta^2) \to (\Delta^p, ds_{\Delta^p}^2)$ such that the sheeting numbers of h^2 and h^3 are equal to* 3, *then there is a holomorphic function $h^3 : \Delta \to \Delta$ such that $(h^1, h^2, h^3) : \Delta \to \Delta^3$ is the cube-root embedding up to reparametrizations*, *i.e.*, $(h^1, h^2, h^3) \in \mathbf{HI}_1(\Delta, \Delta^3; 3)$.

Proof. We may assume without loss of generality that $f^1 = h^1$, $f^2 = h^2$ and $f(0) = \mathbf{0}$. Let $\{a_1, \ldots, a_m\} \subset \partial\Delta$ be the set of all distinct branch points of f^1. Suppose that $m \geq 3$. Then, there is a branch point $a = a_i \in \partial\Delta$ such that $b_i = 1$. Therefore, there is a branch f_l^1 of f^1 such that the ramification index of π_1 at $(a, f_l^1(a))$ is equal to 1 and $|f_l^1(a)| = 1$. Then, we have a branch $\left(f_l^1, f_{l_2}^2, f_{l_3}^3, \ldots, f_{l_p}^p\right)$ of f for some l_j. Consider the functional equation

$$\left(1 - f_l^1(z)\overline{f_l^1(a)}\right) \cdot \prod_{j=2}^{p} \left(1 - f_{l_j}^j(z)\overline{f_{l_j}^j(a)}\right) = 1 - z\overline{a}. \tag{2.1}$$

By comparing the vanishing orders of both sides of Equation (2.1) at a, we see that $|f_{l_j}^j(a)| \neq 1$ for $2 \leq j \leq p$. Thus, a is not a branch point of π_2; otherwise we would have $|f_{l_j}^2(a)| = 1$ by Lemma 2.3 because the sheeting number of f^2 equals 3.

Since $\pi_2 : V_2 \to \mathbb{P}^1$ is not branched over $a \in \partial\Delta$, we have $|(\pi_2)^{-1}(a)| = 3$ and the set $(R_2)^{-1}(a)$ contains at least one unimodular value because $(z, w) \in V_2$ if and only if $(\frac{1}{\overline{z}}, \frac{1}{\overline{w}}) \in V_2$. Then, we may choose l' such that $|f_{l'}^2(a)| = 1$ and we have a branch $\left(f_{l_1'}^1, f_{l'}^2, f_{l_3'}^3, \ldots, f_{l_p'}^p\right)$ of f for some l_j'. Consider the functional equation

$$\left(1 - f_{l'}^2(z)\overline{f_{l'}^2(a)}\right) \prod_{1\le j\le p,\, j\ne 2} \left(1 - f_{l'_j}^j(z)\overline{f_{l'_j}^j(a)}\right) = 1 - z\overline{a}.$$

Since $a \in \partial\Delta$ is a branch point of π_1 and the sheeting number of f^1 equals 3, we have $|f_{l'_1}^1(a)| = 1$ by Lemma 2.3. Now, we have $|f_{l'_1}^1(a)| = |f_{l'}^2(a)| = 1$. Note that we have the Puiseux series $f_{l'_1}^1(z) = \varphi_{l'_1}^1((z-a)^{1/v})$ for $z \in B^1(a,\varepsilon)$, where $\varepsilon > 0$ such that $B^1(a,\varepsilon)\setminus\{a\}$ does not contain any branch point of any component function of f and $\varphi_{l'_1}^1$ is some holomorphic function on $B^1(0,\varepsilon^{1/v})$. Here $v = 1$ or $v = 2$. Then, we have

$$\left(1 - \varphi_{l'_1}^1(\xi)\overline{\varphi_{l'_1}^1(0)}\right)\left(1 - f_{l'}^2(\xi^v + a)\overline{f_{l'}^2(a)}\right)\psi(\xi) = -\overline{a}\xi^v, \qquad (2.2)$$

where $\psi(\xi) := \prod_{j=3}^p \left(1 - f_{l'_j}^j(\xi^v + a)\overline{f_{l'_j}^j(a)}\right)$. Note that $1 - \varphi_{l'_1}^1(\xi)\overline{\varphi_{l'_1}^1(0)}$ has a zero of order 1 at $\xi = 0$ and that $1 - f_{l'}^2(\xi^v + a)\overline{f_{l'}^2(a)}$ has a zero of order v at $\xi = 0$ since a is not a branch point of π_2. Thus, the left hand side of Equation (2.2) has a zero of order at least $v+1$ at $\xi = 0$. However, the right hand side of Equation (2.2) has a zero of order v at $\xi = 0$, which is a contradiction. Thus, $b_i \ne 1$ for all i, $1 \le i \le m$. Hence, we have $m = 2$, i.e., f^1 has precisely two distinct branch points. Similarly, f^2 can only have two distinct branch points. Then, f^1 and f^2 are component functions of the cube-root embedding up to reparametrizations by [Ng10, Lemma 4.9].

We claim that f^1 and f^2 have the same set of branch points, say $a_1, a_2 \in \partial\Delta$. Assume the contrary that $a = a_j$ for some j such that a is a branch point of R_1 but not a branch point of R_2. Then, $|f_l^1(a)| = 1$ for $l = 1, 2, 3$ by Lemma 2.3. But then there exists $l' \in \{1, 2, 3\}$ such that $|f_{l'}^2(a)| = 1$ since $|(R_2)^{-1}(a)| = 3$ and $(z, w) \in V_2$ if and only if $(\frac{1}{\overline{z}}, \frac{1}{\overline{w}}) \in V_2$ (cf. [Ng10, p. 2912]). Thus, we obtain a contradiction by considering the polarized functional equation as before. Therefore, if a is a branch point of f^1, then a is a branch point of f^2. Similarly, if a is a branch point of f^2, then a is a branch point of f^1. Hence, the branching loci of R_1 and R_2 are the same.

From [Ng10, Lemma 4.9] and the proof of Theorem 6.5 in [Ng10], there is a single reparmetrization such that f^1, f^2 would become one of the component functions of the cube-root embedding. Then, $f^1 \ne f^2$ since for each branch of $f = (f^1, \ldots, f^p)$, there is only one infinite value as $z \to \infty$ (cf. [Ng10, p. 2917]). Thus, f^1 and f^2 are precisely two distinct component functions of the cube-root embedding. Recall that $h^j = f^j$ for $j = 1, 2$. Therefore, there is a holomorphic function $h^3 : \Delta \to \Delta$ such that $h^3(0) = 0$ and $(h^1, h^2, h^3) : \Delta \to \Delta^3$ is the cube-root embedding up to reparametrizations, i.e., $(h^1, h^2, h^3) \in \mathbf{HI}_1(\Delta, \Delta^3; 3)$. □

REMARK. This proposition can be used for classifying all holomorphic isometric embeddings $f : (\Delta, ds_\Delta^2) \to (\Delta^p, ds_{\Delta^p}^2)$ with some special sheeting numbers

$s_1, \ldots, s_p$. For example, the structure of the space

$$\mathbf{HI}_1(\Delta, \Delta^{2q+1}; n; 3, 3, 3^2, 3^2, \ldots, 3^{q-1}, 3^{q-1}, 3^q, 3^q, 3^q) \tag{2.3}$$

can be completely described by induction as that in [Ng10, Theorem 6.8], where $q \geq 2$ and n satisfying $3^q \mid n$, $2q + 1 < n \leq 2^{2q}$. Actually, the space in Equation (2.3) is constructed by compositions of q holomorphic isometries in $\mathbf{HI}_1(\Delta, \Delta^3; 3)$. Similarly, the structure of the space

$$\mathbf{HI}_1(\Delta, \Delta^{2q'+2}; n'; 3, 3, 3^2, 3^2, \ldots, 3^{q'}, 3^{q'}, 2 \cdot 3^{q'}, 2 \cdot 3^{q'}) \tag{2.4}$$

can be completely described by induction, where $q' \geq 1$ and n' satisfying $(2 \cdot 3^{q'}) \mid n'$, $2q' + 2 < n' \leq 2^{2q'+1}$. Actually, the space in Equation (2.4) is constructed by compositions of q' holomorphic isometries in $\mathbf{HI}_1(\Delta, \Delta^3; 3)$ and a holomorphic isometry in $\mathbf{HI}_1(\Delta, \Delta^2)$. The author has written down the details in his Ph.D. thesis [Ch16b].

3. Proof of Theorem 1.1

From [Ng10, pp. 2914–2915], if $f \in \mathbf{HI}_k(\Delta, \Delta^4)$ is a holomorphic isometric embedding such that all component functions of f are non-constant, then we have $f \in \mathbf{HI}_k(\Delta, \Delta^4; n; s_1, s_2, s_3, s_4)$ for some positive integers n, s_1, s_2, s_3, s_4 satisfying $\frac{4}{k} \leq n \leq 8$, $\sum_{l=1}^4 (1/s_l) = k$ and $s_j \mid n$ for $j = 1, 2, 3, 4$. Recall that k is a positive integer satisfying $1 \leq k \leq 4$ by [Ng10, p. 2909]. It turns out that given some positive integers n, s_1, s_2, s_3, s_4 satisfying $\frac{4}{k} \leq n \leq 8$, $\sum_{l=1}^4 (1/s_l) = k$ and $s_j \mid n$ for $j = 1, 2, 3, 4$, it is possible that the space $\mathbf{HI}_k(\Delta, \Delta^4; n; s_1, s_2, s_3, s_4)$ is empty due to the structure of the irreducible projective-algebraic curve V and the branching behaviour of each component function of f.

3.1. *Classification of Holomorphic Isometries in* $\mathbf{HI}_1(\Delta, \Delta^4)$

LEMMA 3.1. *Let $p \geq 2$ be an integer and n be a prime number satisfying $p < n \leq 2^{p-1}$. Then, the space $\mathbf{HI}_1(\Delta, \Delta^p; n)$ is empty.*

REMARK. Note that such a prime n does not exist when $p = 2, 3$, thus the condition $p \geq 2$ could be replaced by $p \geq 4$.

Proof of Lemma 3.1. Assume the contrary that the space $\mathbf{HI}_1(\Delta, \Delta^p; n)$ is non-empty. Then, there is a holomorphic isometric embedding $f = (f^1, \ldots, f^p) : (\Delta, ds_\Delta^2) \to (\Delta^p, ds_{\Delta^p}^2)$ such that the sheeting number of f^j equals s_j, $s_j \mid n$ for $1 \leq j \leq p$ and $\sum_{j=1}^p (1/s_j) = 1$ (cf. [Ng10, pp. 2914–2915]). In particular, we have $s_j = n$ for $1 \leq j \leq p$ because $\sum_{j=1}^p (1/s_j) = 1$ so that $s_j \neq 1$ for any j. This would imply that $1 = \sum_{j=1}^p (1/s_j) = \frac{p}{n}$ so that $n = p$, which contradicts $n > p$. Hence, we have $\mathbf{HI}_1(\Delta, \Delta^p; n) = \emptyset$. □

By Lemma 3.1, we have $\mathbf{HI}_1(\Delta, \Delta^4; n) = \emptyset$ for $n = 5, 7$. Thus, we only need to consider the case where $n = 4$, 6 or 8. The following are all possibilities of the global sheeting number n and the sheeting numbers $s_1, \ldots, s_4$:

(1) $(n, s_1, s_2, s_3, s_4) = (4, 4, 4, 4, 4)$.
(2) $(n, s_1, s_2, s_3, s_4) = (6, 3, 6, 6, 3)$ or $(n, s_1, s_2, s_3, s_4) = (6, 2, 6, 6, 6)$.
(3) $(n, s_1, s_2, s_3, s_4) = (8, 4, 4, 4, 4)$ or $(n, s_1, s_2, s_3, s_4) = (8, 2, 4, 8, 8)$.

In the case where $(n, s_1, s_2, s_3, s_4) = (4, 4, 4, 4, 4)$, we can apply the global rigidity of the pth root embedding for $p \geq 2$ (cf. [Ch16a]). More precisely, any $f \in \mathbf{HI}_1(\Delta, \Delta^4; 4)$ is the 4th root embedding up to reparametrizations as we have mentioned at the beginning of the present paper.

PROPOSITION 3.2 (cf. Theorem 6.8, [Ng10]). *If* $f \in \mathbf{HI}_1(\Delta, \Delta^4; 8; 2, 4, 8, 8)$, *then*

$$f = (\alpha_1, \alpha_2 \circ \beta_1, \alpha_3 \circ (\beta_2 \circ \beta_1), \beta_3 \circ (\beta_2 \circ \beta_1))$$

up to reparametrizations, where $(\alpha_j, \beta_j) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$ *for* $j = 1, 2, 3$.

PROPOSITION 3.3. *If* $f \in \mathbf{HI}_1(\Delta, \Delta^4; 6; 2, 6, 6, 6)$, *then*

$$f = (\alpha_1, h^2 \circ \alpha_2, h^3 \circ \alpha_2, h^4 \circ \alpha_2)$$

up to reparametrizations, where $(\alpha_1, \alpha_2) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$ *and* $(h^2, h^3, h^4) \in \mathbf{HI}_1(\Delta, \Delta^3; 3)$.

Proof. We may suppose that $f(0) = \mathbf{0}$. From [Ng10, Lemma 6.7], we have $f^1 = \alpha_1$ for some holomorphic isometric embedding $(\alpha_1, \alpha_2) : \Delta \to \Delta^2$ with the isometric constant 1 and $\alpha_1(0) = \alpha_2(0) = 0$. Then, we have

$$(1 - |f^2(z)|^2)(1 - |f^3(z)|^2)(1 - |f^4(z)|^2) = 1 - |\alpha_2(z)|^2$$

because $(1 - |\alpha_1(z)|^2)(1 - |\alpha_2(z)|^2) = 1 - |z|^2$. Since 0 is not a branch point, locally there is an inverse $\alpha_2^{-1} : U \subset \Delta \to \Delta$ of α_2. Thus,

$$(1 - |f^2(\alpha_2^{-1}(z))|^2)(1 - |f^3(\alpha_2^{-1}(z))|^2)(1 - |f^4(\alpha_2^{-1}(z))|^2) = 1 - |z|^2,$$

i.e., $(f^2 \circ \alpha_2^{-1}, f^3 \circ \alpha_2^{-1}, f^4 \circ \alpha_2^{-1}) : U \to \Delta^3$ is a holomorphic isometric embedding with the isometric constant 1. From [Mok12, Theorem 1.3.1], we know that $(f^2 \circ \alpha_2^{-1}, f^3 \circ \alpha_2^{-1}, f^4 \circ \alpha_2^{-1})$ can be extended to the whole Δ, and we let $(h^2, h^3, h^4) : \Delta \to \Delta^3$ be the extension. Then, $f^j \circ \alpha_2^{-1} = h^j$ for $j = 2, 3, 4$ and thus $f^j = h^j \circ \alpha_2$ on some open subset. Now, we have a local inverse $(f^j)^{-1} = \alpha_2^{-1} \circ (h^j)^{-1}$. Since the degree of $(f^j)^{-1}$ equals 6 while the degree of α_2^{-1} equals 2, the degree of $(h^j)^{-1}$ should be equal to 3. Thus $(h^2, h^3, h^4) : \Delta \to \Delta^3$ is the cube-root embedding up to reparametrizations by [Ng10, Theorem 8.1]. Hence, $f = (f^1, f^2, f^3, f^4) = (\alpha_1, h^2 \circ \alpha_2, h^3 \circ \alpha_2, h^4 \circ \alpha_2)$ up to reparametrizations. □

PROPOSITION 3.4. *If* $f \in \mathbf{HI}_1(\Delta, \Delta^4; 6; 3, 6, 6, 3)$, *then*

$$f = (\beta_1, \alpha_1 \circ \beta_2, \alpha_2 \circ \beta_2, \beta_3)$$

up to reparametrizations, where $(\beta_1, \beta_2, \beta_3) \in \mathbf{HI}_1(\Delta, \Delta^3; 3)$ *and* $(\alpha_1, \alpha_2) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$.

Proof. We may assume without loss of generality that $f = (f^1, f^2, f^3, f^4)$ satisfying $f(0) = \mathbf{0}$. Then, there is a holomorphic function $g : \Delta \to \Delta$ with $g(0) = 0$ such that $(f^1, f^4, g) \in \mathbf{HI}_1(\Delta, \Delta^3; 3)$ by Proposition 2.4. From the functional equation, we have

$$(1 - |f^2(z)|^2)(1 - |f^3(z)|^2) = 1 - |g(z)|^2.$$

Since g is a component function of some holomorphic isometry in $\mathbf{HI}_1(\Delta, \Delta^3; 3)$, there is a local inverse g^{-1} of g around $0 \in \Delta$ so that

$$(1 - |f^2 \circ g^{-1}(z)|^2)(1 - |f^3 \circ g^{-1}(z)|^2) = 1 - |z|^2$$

on some open neighborhood of 0 in Δ (cf. [Ng10, p. 2918]). Thus $(f^2 \circ g^{-1}, f^3 \circ g^{-1}) : \Delta \to \Delta^2$ is a germ of holomorphic isometric embedding with the isometric constant 1. In particular, $(f^2 \circ g^{-1}, f^3 \circ g^{-1})$ is a germ of the square-root embedding at 0 up to reparametrizations. From [Mok12, Theorem 1.3.1], such a germ of holomorphic isometric embedding can be extended to a holomorphic isometric embedding from (Δ, ds^2_Δ) to $(\Delta^2, ds^2_{\Delta^2})$. Therefore, we have $f^2 \circ g^{-1} = \alpha_1|_U$, $f^3 \circ g^{-1} = \alpha_2|_U$ for some neighborhood U of 0 in Δ, where $(\alpha_1, \alpha_2) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$. Then, $f^2 = \alpha_1 \circ g$ and $f^3 = \alpha_2 \circ g$ on Δ. Hence, we have $f = (\beta_1, \alpha_1 \circ \beta_2, \alpha_2 \circ \beta_2, \beta_3)$, where $(\beta_1, \beta_2, \beta_3) \in \mathbf{HI}_1(\Delta, \Delta^3; 3)$ and $(\alpha_1, \alpha_2) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$. □

Let $f = (f^1, f^2, f^3, f^4) \in \mathbf{HI}_1(\Delta, \Delta^4; 8; 4, 4, 4, 4)$ and $\nu : X \to V$ be the normalization, where X is a compact Riemann surface of genus $g(X)$. Without loss of generality, we may assume that $f(0) = \mathbf{0}$. The universal cover of X is either $\mathbb{P}^1$, $\mathbb{C}$ or Δ by the Uniformization Theorem. In any case, we may use the global holomorphic coordinate ζ on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, $\mathbb{C}$ or Δ to represent a point in X. Given a non-constant meromorphic function $\hat{S}$ on X, denote by $\mathrm{Zeros}(\hat{S}(\zeta))$ (resp. $\mathrm{Poles}(\hat{S}(\zeta))$) the set of all zeros (resp. poles) of $\hat{S}$ not counting multiplicities.

Recall that $\pi : V \to \mathbb{P}^1$ is the finite branched covering defined by $(z, w_1, w_2, w_3, w_4) \mapsto z$. Then, $\pi \circ \nu(\zeta) = R(\zeta)$ is a non-constant meromorphic function on X with precisely 8 distinct poles and 8 distinct zeros. Let $S_j(\zeta) := (\mathrm{Pr}_2 \circ (P_j \circ \nu))(\zeta)$ for $1 \le j \le 4$, where $\mathrm{Pr}_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is the projection onto the second factor, $P_j : V \to \mathbb{P}^1 \times \mathbb{P}^1$ is defined by $(z, w_1, w_2, w_3, w_4) \mapsto (z, w_j)$ and $V_j = P_j(V)$ for $1 \le j \le 4$. Then, S_j is a non-constant meromorphic function on X with precisely two distinct poles and two distinct zeros. Moreover, we have $R(\zeta) = R_j(S_j(\zeta))$ for $1 \le j \le 4$.

Let $(f^1_{l_1}, f^2_{l_2}, f^3_{l_3}, f^4_{l_4})$ be a branch of f over Δ for some $l_j \in \{1, 2, 3, 4\}$. For $\zeta \in U' := \nu^{-1}(\mathrm{Graph}(f))$, we have $f^j(R(\zeta)) = S_j(\zeta)$ for $1 \le j \le 4$. Note that for any branch f^j_l of f^j, $1 \le l, j \le 4$, there are precisely two distinct branches of f over Δ with the jth-component function being equal to f^j_l because $S_j : X \to \mathbb{P}^1$ is a degree 2 branched covering and the graph of each branch of f over Δ (resp. $\mathbb{P}^1 \setminus \overline{\Delta}$) lies in the regular part of the variety V. The following consideration comes from [Mok]. From the polarized functional equation, for $\zeta \in$

$U' := \nu^{-1}(\mathrm{Graph}(f))$ and $w \in \Delta$, we have

$$\prod_{j=1}^{4}\left(1 - S_j(\zeta)\overline{f^j_{l_j}(w)}\right) = 1 - R(\zeta)\overline{w}. \tag{3.1}$$

Fixing $w \in \Delta$, both sides of Equation (3.1) are meromorphic functions on X. Thus, by the Identity Theorem of meromorphic functions on compact Riemann surfaces, the above equality holds true for $\zeta \in X$ and $w \in \Delta$. Putting $w = 0$ in Equation (3.1), we have

$$\prod_{j=1}^{4}\left(1 - S_j(\zeta)\overline{f^j_{l_j}(0)}\right) = 1 \quad \forall \zeta \in X.$$

LEMMA 3.5. *If $f = (f^1, f^2, f^3, f^4) \in \mathbf{HI}_1(\Delta, \Delta^4; 8; 4, 4, 4, 4)$, then there is a branch of f over Δ which is of the form $(g_1, \ldots, g_4)$, where $g_{\sigma(j)} := f_1^{\sigma(j)}$ $(j = 1, 2)$ and $g_{\sigma(\mu)} := f^{\sigma(\mu)}_{l_{\sigma(\mu)}}$ with $l_{\sigma(\mu)} \neq 1$ $(\mu = 3, 4)$ for some $\sigma \in \Sigma_4$.*

Proof. We assume without loss of generality that $f(0) = \mathbf{0}$. Let $\nu : X \to V$ be the normalization. Assume the contrary that f does not have a branch of the desired form. From the functional equation, it is known that f cannot have a branch of the form $\left(f^{\sigma(1)}, f^{\sigma(2)}, f^{\sigma(3)}, f^{\sigma(4)}_{j_{\sigma(4)}}\right)$ over Δ up to a permutation of component functions of f, where $\sigma \in \Sigma_4$ and $j_{\sigma(4)} \neq 1$. Otherwise, we would have $|f^{\sigma(4)}_{j_{\sigma(4)}}(z)|^2 = |f^{\sigma(4)}(z)|^2$ so that $f^{\sigma(4)}_{j_{\sigma(4)}}(0) = f^{\sigma(4)}(0) = 0$, which contradicts the fact that $f^{\sigma(4)}_{j_{\sigma(4)}}$ and $f^{\sigma(4)}$ are distinct branches and 0 is not a branch point of $R_{\sigma(4)}$. Then, we have some branches of f over Δ which are of the forms

$$\begin{aligned} &\left(f^1, f^2_{l_2^{(1)}}, f^3_{l_3^{(1)}}, f^4_{l_4^{(1)}}\right), \qquad \left(f^1_{l_1^{(2)}}, f^2, f^3_{l_3^{(2)}}, f^4_{l_4^{(2)}}\right),\\ &\left(f^1_{l_1^{(3)}}, f^2_{l_2^{(3)}}, f^3, f^4_{l_4^{(3)}}\right), \qquad \left(f^1_{l_1^{(4)}}, f^2_{l_2^{(4)}}, f^3_{l_3^{(4)}}, f^4\right), \end{aligned} \tag{3.2}$$

where $l_j^{(k)} \neq 1$ for each j, k. Note that performing (multivalued) analytic continuation of (f^1, f^2, f^3, f^4) along some simple closed loop around each branch point of R_j in $\mathbb{C}$, $1 \leq j \leq 4$, would produce all branches of f over Δ because $\mathrm{Reg}(V)$ is connected (cf. Proposition 1 in [MN10, pp. 2634–2635] for the structure of V and properties of the branches of f). From the polarized functional equation, we have

$$\prod_{j=1}^{3}\left(1 - S_{\sigma(j)}(\zeta)\overline{\beta^{(\sigma(4))}_{\sigma(j)}}\right) = 1$$

for each $\sigma \in \Sigma_4$, where for each $k \in \{1, 2, 3, 4\}$, $\beta_j^{(k)} := f^j_{l_j^{(k)}}(0) \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ for $j \in \{1, 2, 3, 4\} \setminus \{k\}$. Note that the poles of $1 - S_j(\zeta)\overline{\beta_j^{(l)}}$ are precisely the poles of $S_j(\zeta)$ for $j \in \{1,2,3,4\} \setminus \{l\}$ and $l = 1,2,3,4$. Moreover, $1 - S_j(\zeta)\overline{\beta_j^{(l)}}$ has precisely two distinct zeros and two distinct poles for $j \in \{1,2,3,4\} \setminus \{l\}$ and $l = 1,2,3,4$.

Consider the branch $\big(f^1_{l_1^{(4)}}, f^2_{l_2^{(4)}}, f^3_{l_3^{(4)}}, f^4\big)$. Then, there is a unique branch of f over Δ which is of the form $\big(f^1_{k_1}, f^2_{k_2}, f^3_{l_3^{(4)}}, f^4_{k_4}\big)$ with $k_4 \neq 1$ because we already have the branch (f^1, f^2, f^3, f^4) of f, S_j is a degree 2 branched covering and all points in $\nu^{-1}(\pi^{-1}(\infty))$ are not ramification points of S_l for $1 \le l \le 4$. We claim that $k_j \neq l_j^{(4)}$ for $j = 1,2$.

If $k_j = l_j^{(4)}$ for $j = 1,2$, then we would have $|f^4(z)|^2 = |f^4_{k_4}(z)|^2$ for $z \in \Delta$, which leads to a contradiction by the arguments above. If $k_1 = l_1^{(4)}$ and $k_2 \neq l_2^{(4)}$, then we have

$$1 - S_2(\zeta)\overline{\beta_2^{(4)}} = \Big(1 - S_2(\zeta)\overline{f^2_{k_2}(0)}\Big)\Big(1 - S_4(\zeta)\overline{f^4_{k_4}(0)}\Big)$$

from the functional equation so that

$$S_4(\zeta) = \frac{1}{\overline{f^4_{k_4}(0)}} \frac{\Big(\overline{\beta_2^{(4)}} - \overline{f^2_{k_2}(0)}\Big) \cdot S_2(\zeta)}{1 - S_2(\zeta)\overline{f^2_{k_2}(0)}}.$$

Thus, $S_4 = \varphi \circ S_2$ for some $\varphi \in \mathrm{Aut}(\mathbb{P}^1)$. But then this implies that all branches of f are of the form $(f^1_{l_1}, f^2_l, f^3_{l_3}, f^4_l)$ for some $l_1, l_3, l \in \{1,2,3,4\}$ by performing (multivalued) analytic continuation, which contradicts the existence of the branch $\big(f^1_{l_1^{(4)}}, f^2_{l_2^{(4)}}, f^3_{l_3^{(4)}}, f^4\big)$. Similarly, if $k_2 = l_2^{(4)}$ and $k_1 \neq l_1^{(4)}$, then this also leads to a contradiction. Hence, $k_j \neq l_j^{(4)}$ for $j = 1,2$.

From the functional equation, we have

$$1 - S_4(\zeta)\overline{f^4_{k_4}(0)} = \frac{1 - S_1(\zeta)\overline{\beta_1^{(4)}}}{1 - S_1(\zeta)\overline{f^1_{k_1}(0)}} \frac{1 - S_2(\zeta)\overline{\beta_2^{(4)}}}{1 - S_2(\zeta)\overline{f^2_{k_2}(0)}}$$

and $\prod_{j=1}^3 \big(1 - S_j(\zeta)\overline{\beta_j^{(4)}}\big) = 1$. Thus, we have

$$\begin{aligned}\mathrm{Zeros}\Big(1 - S_4(\zeta)\overline{f^4_{k_4}(0)}\Big) &\subseteq \mathrm{Zeros}\Big(\Big(1 - S_1(\zeta)\overline{\beta_1^{(4)}}\Big)\Big(1 - S_2(\zeta)\overline{\beta_2^{(4)}}\Big)\Big) \\ &= \mathrm{Zeros}\bigg(\frac{1}{1 - S_3(\zeta)\overline{\beta_3^{(4)}}}\bigg) = \mathrm{Poles}(S_3(\zeta))\end{aligned}$$

Since S_3 has two distinct simple poles and $1 - S_4(\zeta)\overline{f^4_{k_4}(0)}$ has two distinct simple zeros, we have $\mathrm{Zeros}\big(1 - S_4(\zeta)\overline{f^4_{k_4}(0)}\big) = \mathrm{Poles}(S_3(\zeta))$. Therefore, there are two distinct points $y_1, y_2 \in V$ (resp. $x_1, x_2 \in X$) such that $\nu(x_j) = y_j = \big(\infty, \alpha_1^j, \alpha_2^j, \infty, 1/\overline{f^4_{k_4}(0)}\big)$ for $j = 1,2$, and $\{x_1, x_2\} = \mathrm{Zeros}\big(1 - S_4(\zeta)\overline{f^4_{k_4}(0)}\big) = \mathrm{Poles}(S_3(\zeta))$, where $\alpha_1^j, \alpha_2^j \in \mathbb{C}^*$, $j = 1,2$. Note that $x_1, x_2 \in X$ are two distinct unramified points of $\pi \circ \nu : X \to \mathbb{P}^1$ and $y_1, y_2 \in V$ are smooth points on V.

Then, we have two distinct branches of f over $\mathbb{P}^1 \setminus \overline{\Delta}$ which are of the forms $(f^1_{l_1,-}, f^2_{l_2,-}, f^3_{l_3,-}, f^4_{l_4,-})$, $(f^1_{n_1,-}, f^2_{n_2,-}, f^3_{l_3,-}, f^4_{l_4,-})$ such that

$$y_1 = \left(\infty, f^1_{l_1,-}(\infty), f^2_{l_2,-}(\infty), f^3_{l_3,-}(\infty), f^4_{l_4,-}(\infty)\right),$$
$$y_2 = \left(\infty, f^1_{n_1,-}(\infty), f^2_{n_2,-}(\infty), f^3_{l_3,-}(\infty), f^4_{l_4,-}(\infty)\right).$$

If $n_j = l_j$ and $n_i \neq l_i$ for distinct $i, j \in \{1, 2\}$, then we have

$$1 - f^i_{l_i,-}(z)\overline{f^i_{l_i,-}(w)} = 1 - f^i_{n_i,-}(z)\overline{f^i_{l_i,-}(w)}$$

for $z, w \in \mathbb{P}^1 \setminus \overline{\Delta}$ from the functional equation, which implies that $f^i_{l_i,-} = f^i_{n_i,-}$ so that $l_i = n_i$, a plain contradiction. Thus, $n_j \neq l_j$ for $j = 1, 2$. Now, we have $\alpha^1_l \neq \alpha^2_l$ for $l = 1, 2$. From the functional equation, we have

$$\begin{aligned}&\left(1 - f^1_{l_1,-}(z)\overline{f^1_{n_1,-}(w)}\right)\left(1 - f^2_{l_2,-}(z)\overline{f^2_{n_2,-}(w)}\right)\\ =&\left(1 - f^1_{l_1,-}(z)\overline{f^1_{l_1,-}(w)}\right)\left(1 - f^2_{l_2,-}(z)\overline{f^2_{l_2,-}(w)}\right)\end{aligned}$$

so that

$$\frac{1 - f^1_{l_1,-}(z)\overline{\alpha^2_1}}{1 - f^1_{l_1,-}(z)\overline{\alpha^1_1}} = \frac{1 - f^2_{l_2,-}(z)\overline{\alpha^1_2}}{1 - f^2_{l_2,-}(z)\overline{\alpha^2_2}},$$

which implies that $f^1_{l_1,-}(z) = \varphi(f^2_{l_2,-}(z))$ for some $\varphi \in \mathrm{Aut}(\mathbb{P}^1)$ satisfying $\varphi(0) = 0$. Denote by $\mathcal{O} := \mathbb{P}^1 \setminus \overline{\Delta}$. Thus, $R_1 \circ \varphi|_{f^2_{l_2,-}(\mathcal{O})} = R_2|_{f^2_{l_2,-}(\mathcal{O})}$. Since $f^2_{l_2,-}(\mathcal{O}) \subset \mathbb{P}^1$ is open, we have $R_1 \circ \varphi = R_2$ by the Identity Theorem for meromorphic functions on irreducible holomorphic varieties [Gun90, p. 177]. We claim that $R_j(h(z)) = z$ for some holomorphic function h on Δ implies $h = f^j_l$ for some l and $h(0) = f^j_l(0)$. Actually, there is an open neighborhood B_0 of 0 in Δ such that $R_j|_{U_l} : U_l \to B_0$ is biholomorphic and $h(0) = f^j_l(0)$ for some l since 0 is not a branch point of R_j, where U_l is some open neighborhood of $f^j_l(0)$ in $\mathbb{P}^1$. Then, $(R_j|_{U_l})^{-1}|_{B_0} = h|_{B_0} = f^j_l|_{B_0}$ and thus $h = f^j_l$ by the Identity Theorem. Therefore, this implies that $\varphi \circ f^2$ is one of the branches of f^1 over Δ. Since $(\varphi \circ f^2)(0) = 0$, we have $\varphi \circ f^2 = f^1$ because 0 is not a branch point of any R_j, $1 \leq j \leq 4$. But then performing (multivalued) analytic continuation of (f^1, f^2, f^3, f^4) could only produce branches of f over Δ of the form $(f^1_l, f^2_l, f^3_{l_3}, f^4_{l_4})$ for some $l, l_3, l_4 \in \{1, 2, 3, 4\}$, which contradicts Equation (3.2). Hence, there is a branch of f over Δ which is of the desired form. □

PROPOSITION 3.6. *If* $f \in \mathbf{HI}_1(\Delta, \Delta^4; 8; 4, 4, 4, 4)$, *then*

$$f = (\alpha_1 \circ \alpha_2, \beta_1 \circ \alpha_2, \alpha_3 \circ \beta_2, \beta_3 \circ \beta_2)$$

up to reparametrizations, where $(\alpha_j, \beta_j) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$, $j = 1, 2, 3$.

Proof. We may assume without loss of generality that $f(0) = \mathbf{0}$. By Lemma 3.5, there is a branch of f over Δ which is of the form $(g_1, \ldots, g_4)$, where $g_{\sigma(j)} :=$

$f_1^{\sigma(j)}$ for $1 \le j \le 2$ and $g_{\sigma(\mu)} := f_{l_{\sigma(\mu)}}^{\sigma(\mu)}$ with $l_{\sigma(\mu)} \neq 1$ for $\mu = 3, 4$, and for some $\sigma \in \Sigma_4$. Then, it follows from Theorem 2.2 that

$$(1 - |f^{\sigma(3)}(z)|^2)(1 - |f^{\sigma(4)}(z)|^2) = 1 - |h(z)|^2$$

for some holomorphic function $h : \Delta \to \mathbb{C}$. Thus, it follows from the functional equation that $(f^{\sigma(1)}, f^{\sigma(2)}, h) \in \mathbf{HI}_1(\Delta, \Delta^3)$. Since the sheeting numbers of both $f^{\sigma(1)}$ and $f^{\sigma(2)}$ are equal to 4, the sheeting number of h equals 2 and h is a component function of some isometry in $\mathbf{HI}_1(\Delta, \Delta^2; 2)$ (cf. [Ng10, Theorem 8.1]). This shows that $(f^{\sigma(1)}, f^{\sigma(2)}, h) \in \mathbf{HI}_1(\Delta, \Delta^3; 4; 4, 4, 2)$. From [Ng10, Theorem 8.1], we have $(f^{\sigma(1)}, f^{\sigma(2)}, h) = (\alpha_5 \circ g, \beta_5 \circ g, h)$ up to reparametrizations, where $(\alpha_5, \beta_5) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$ and $(g, h) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$ for some holomorphic function $g : \Delta \to \Delta$. Moreover, $(1 - |f^{\sigma(3)}(h^{-1}(z))|^2)(1 - |f^{\sigma(4)}(h^{-1}(z))|^2) = 1 - |z|^2$ for $z \in B^1(0, \varepsilon) \subset \Delta$, where $\varepsilon > 0$ is some real constant. Thus, $(f^{\sigma(3)} \circ h^{-1}, f^{\sigma(4)} \circ h^{-1}) : B^1(0, \varepsilon) \to \Delta^2$ is a local holomorphic isometric embedding which can be extended to the whole unit disk Δ (cf. [Mok12, Theorem 1.3.1]), where the isometric constant equals 1. Therefore, we have $f^{\sigma(3)} = \alpha_4 \circ h$ and $f^{\sigma(4)} = \beta_4 \circ h$ for some $(\alpha_4, \beta_4) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$. Hence, $(f^{\sigma(1)}, f^{\sigma(2)}, f^{\sigma(3)}, f^{\sigma(4)}) = (\alpha_5 \circ g, \beta_5 \circ g, \alpha_4 \circ h, \beta_4 \circ h)$ up to reparametrizations so that $f = (\alpha_1 \circ \alpha_2, \beta_1 \circ \alpha_2, \alpha_3 \circ \beta_2, \beta_3 \circ \beta_2)$ up to reparametrizations, where $(\alpha_j, \beta_j) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$ for $j = 1, 2, 3$. □

Combining the above results, part (1) of the Theorem 1.1 is proved.

3.2. *Classification of Holomorphic Isometries in* $\mathbf{HI}_k(\Delta, \Delta^4)$ *for* $k \ge 2$

In this section, we consider the case where $k = 2, 3$ or 4. The following is part (2) of Theorem 1.1.

PROPOSITION 3.7. *Let* $f : (\Delta, 2ds_\Delta^2) \to (\Delta^4, ds_{\Delta^4}^2)$ *be a holomorphic isometric embedding. Then,* $f(z)$ *is of one of the following forms up to reparametrizations*:

(1) $(\alpha_1(z), \beta_1(z), \alpha_2(z), \beta_2(z))$, *where* $(\alpha_j, \beta_j) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$ *for* $j = 1, 2$.
(2) $(z, \alpha_1(z), (\alpha_2 \circ \beta_1)(z), (\beta_2 \circ \beta_1)(z))$, *where* $(\alpha_j, \beta_j) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$ *for* $j = 1, 2$.
(3) $(z, \alpha_1(z), \alpha_2(z), \alpha_3(z))$, *where* $(\alpha_1, \alpha_2, \alpha_3) \in \mathbf{HI}_1(\Delta, \Delta^3; 3)$.

Moreover, the space $\mathbf{HI}_2(\Delta, \Delta^4; n; 2, 2, 2, 2)$ *is non-empty only if* $n = 2$ *or* $n = 4$.

Proof. We may assume without loss of generality that $f(0) = \mathbf{0}$. Let s_j be the sheeting number of f^j and n be the global sheeting number (cf. [Ng10, p. 2911]). In the case where $k = 2$, we have $2 \le n \le 8$. If $n = 5$, then we have $\sum_{j=1}^4 (1/s_j) = 2$ with $s_j \mid 5$ for $1 \le j \le 4$. Thus, $l + \frac{4-l}{5} = 2$ for some integer $l \ge 0$, but this would imply that $4l = 6$, which is a contradiction. If $n = 7$, then we have $\sum_{j=1}^4 (1/s_j) = 2$ with $s_j \mid 7$ for $1 \le j \le 4$. Therefore, $l + \frac{4-l}{7} = 2$ for some integer $l \ge 0$, but this would imply that $6l = 10$, which is again a contradiction. Then, we have $n \notin \{5, 7\}$ so that $n = 2, 3, 4, 6$ or 8.

In a priori for $n = 6$ or $n = 8$, it is possible that $(n, s_1, s_2, s_3, s_4) = (6, 2, 2, 2, 2), (6, 1, 3, 3, 3), (6, 1, 2, 3, 6), (8, 2, 2, 2, 2)$ or $(8, 1, 2, 4, 4)$.

If $s_1 = 1$, then $f^1(z) = z$ up to reparametrizations so that the problem reduces to the study of $\mathbf{HI}_1(\Delta, \Delta^3)$, which is completely described by [Ng10, Theorem 8.1]. If $(n, s_1, s_2, s_3, s_4) = (6, 1, 3, 3, 3)$, then (f^2, f^3, f^4) is the cube-root embedding up to reparametrizations by [Ng10, Theorem 8.1] and this implies that $n = 3$, which is a contradiction. If $(n, s_1, s_2, s_3, s_4) = (6, 1, 2, 3, 6)$, then we would have a holomorphic isometry in $\mathbf{HI}_1(\Delta, \Delta^3; n'; 2, 3, 6)$ so that $n' \geq 6$, which contradicts $n' \leq 4$ (cf. [Ng10, Proposition 5.2]). If $(n, s_1, s_2, s_3, s_4) = (8, 1, 2, 4, 4)$, then (f^2, f^3, f^4) is of the form $(\alpha_1, \alpha_2 \circ \beta_1, \beta_2 \circ \beta_1)$ for $(\alpha_j, \beta_j) \in \mathbf{HI}_1(\Delta, \Delta^2)$, $j = 1, 2$, by [Ng10, Theorem 8.1] and thus $n = 4$, which is a contradiction. This rules out the cases where $(n, s_1, s_2, s_3, s_4) = (6, 1, 3, 3, 3)$, $(n, s_1, s_2, s_3, s_4) = (6, 1, 2, 3, 6)$ or $(n, s_1, s_2, s_3, s_4) = (8, 1, 2, 4, 4)$. Therefore, the only possible global sheeting numbers n and sheeting numbers $s_1, \ldots, s_4$ are the following:

(1) $(n, s_1, s_2, s_3, s_4) = (n, 2, 2, 2, 2)$, $n = 2, 4, 6$ or 8,
(2) $(n, s_1, s_2, s_3, s_4) = (4, 1, 2, 4, 4)$,
(3) $(n, s_1, s_2, s_3, s_4) = (3, 1, 3, 3, 3)$.

Now, we deal with these cases:

(1) Let $f = (f^1, f^2, f^3, f^4) \in \mathbf{HI}_2(\Delta, \Delta^4; n; 2, 2, 2, 2)$. Then, each f^j becomes one of the component functions of the square-root embedding from [Ng10, Lemma 6.7]. From [Ng10, Colloary 4.7], for each branch point $a \in \partial\Delta$ of some component function f^j of f, we have $|f^j(a)|^2 = 1$. From the use of the Puiseux series of each component function f^j of f around a branch point $a \in \partial\Delta$ of f^j, we see that either a is a branch point of all component functions of f or a is a branch point of another component f^l of f $(l \neq j)$ and a is not a branch point of other component functions f^μ of f $(\mu \notin \{l, j\})$.

Then, either (i) the branching loci of all component functions of f are the same or (ii) for any branch point $a \in \partial\Delta$ of each component function f^j of f, a is only a branch point of f^l for some $l \neq j$ and not a branch point of f^μ for $\mu \notin \{l, j\}$.

(i) If the branching loci of all component functions of f are the same, then there is a single reparametrization of f so that each f^j is one of the α_1, β_1, where $(\alpha_1, \beta_1) \in \mathbf{HI}_1(\Delta, \Delta^2)$ is the square-root embedding. From the proof of Theorem 6.5 in [Ng10], since for every branch of f there are precisely two component functions of f which take the value ∞ at ∞, only two of the f^j's are α_1 and the remaining two component functions of f are β_1 up to reparametrizations. In particular, f is $(\alpha_1, \beta_1, \alpha_1, \beta_1)$ up to reparametrizations for some $(\alpha_1, \beta_1) \in \mathbf{HI}_1(\Delta, \Delta^2)$.

(ii) Suppose that for any branch point $a \in \partial\Delta$ of each component function f^j of f, a is only a branch point of f^l for some $l \neq j$ and not a branch point of f^μ for $\mu \notin \{l, j\}$. We may assume that f^1 and f^2 have a common branch point $a \in \partial\Delta$ and a is not a branch point of f^3, f^4. Then, after performing (multivalued) analytic continuation along a simple continuous closed loop around $a \in \partial\Delta$ once, we obtain another branch (f_l^1, f_l^2, f^3, f^4) of f for some $l \neq 1$. From the proof of Theorem 2.2, we have

$$(1-|f^1(z)|^2)(1-|f^2(z)|^2) = 1-|h(z)|^2$$

for some holomorphic function $h : \Delta \to \Delta$. Thus, $(h, f^3, f^4) \in \mathbf{HI}_2(\Delta, \Delta^3)$. Since both f^3 and f^4 have sheeting numbers equal to 2, it follows from [Ng10] that the sheeting number of h is equal to 1, i.e., $h(z) = z$ up to reparametrizations. In particular, $(f^1, f^2) \in \mathbf{HI}_1(\Delta, \Delta^2)$ and thus $(f^3, f^4) \in \mathbf{HI}_1(\Delta, \Delta^2)$. Hence, f is $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ up to reparametrizations for some $(\alpha_j, \beta_j) \in \mathbf{HI}_1(\Delta, \Delta^2)$, $j = 1, 2$.

In any case, it follows that any $f \in \mathbf{HI}_2(\Delta, \Delta^4; n; 2, 2, 2, 2)$ is $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ up to reparametrizations for some $(\alpha_j, \beta_j) \in \mathbf{HI}_1(\Delta, \Delta^2)$, $j = 1, 2$. Note that the branching loci of α_j and β_j are the same for each j, where $j = 1, 2$. By performing (multivalued) analytic continuation of the given isometry $f \in \mathbf{HI}_2(\Delta, \Delta^4; n; 2, 2, 2, 2)$, the global sheeting number n is at most 4, i.e., either $n = 2$ or $n = 4$. This rules out the possibility of n being equal to 6 or 8.

If $f = (f^1, f^2, f^3, f^4) \in \mathbf{HI}_2(\Delta, \Delta^4; 2; 2, 2, 2, 2)$, then the branching loci of all f^j are the same so that there is a single parametrization of f to make f^j to be either α_1 or β_1, where $(\alpha_1, \beta_1) : \Delta \to \Delta^2$ is the square-root embedding. Moreover, since for each branch of f, there are only two component functions take the value ∞ at ∞, so $f = (\alpha_1, \beta_1, \alpha_1, \beta_1)$ up to reparametrizations.

If $f \in \mathbf{HI}_2(\Delta, \Delta^4; 4; 2, 2, 2, 2)$, then $f = (\alpha_1, \beta_1, \alpha_2, \beta_2)$ up to reparametrizations, where $(\alpha_j, \beta_j) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$ for $j = 1, 2$ such that the branching loci of (α_1, β_1) is different from that of (α_2, β_2).

(2) Let $f = (f^1, f^2, f^3, f^4) \in \mathbf{HI}_2(\Delta, \Delta^4; 4; 1, 2, 4, 4)$. Then, $f^1(z) = z$ up to reparametrizations so that $(f^2, f^3, f^4) \in \mathbf{HI}_1(\Delta, \Delta^3; 4; 2, 4, 4)$. From [Ng10], we have $(f^2, f^3, f^4) = (\alpha_1, \alpha_2 \circ \beta_1, \beta_2 \circ \beta_1)$ up to reparametrizations, where $(\alpha_j, \beta_j) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$ for $j = 1, 2$.

(3) Now, we consider the case where $n = 3$. The only possibility is that $(s_1, s_2, s_3, s_4) = (1, 3, 3, 3)$. Then, we have $f^1(z) = z$ up to reparametrizations so that

$$(1-|f^2(z)|^2)(1-|f^3(z)|^2)(1-|f^4(z)|^2) = 1-|z|^2$$

and thus $(f^2, f^3, f^4) : \Delta \to \Delta^3$ is a holomorphic isometric embedding with the isometric constant equal to 1. From [Ng10, Theorem 8.1], (f^2, f^3, f^4) has to be the cube-root embedding up to reparametrizations. Thus $f(z) = (z, \alpha_1(z), \alpha_2(z), \alpha_3(z))$ up to reparametrizations, where $(\alpha_1, \alpha_2, \alpha_3) : \Delta \to \Delta^3$ is the cube-root embedding. □

The following is part (3) of Theorem 1.1.

PROPOSITION 3.8. *Let $f : (\Delta, 3\,ds_\Delta^2) \to (\Delta^4, ds_{\Delta^4}^2)$ be a holomorphic isometric embedding. Then, $f(z) = (z, z, \alpha(z), \beta(z))$ up to reparametrizations, where $(\alpha, \beta) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$.*

Proof. We may assume without loss of generality that $f(0)=\mathbf{0}$. Note that $\sum_{j=1}^{4}(1/s_j)=3$, so there exists j such that $1/s_j\geq\frac{3}{4}$. But then $s_j\leq\frac{4}{3}<2$ implies $s_j=1$ so that $f^j(z)=z$ up to reparametrizations. We may assume without loss of generality that $f^1(z)=z$. Then, $(1-|f^2(z)|^2)(1-|f^3(z)|^2)(1-|f^4(z)|^2)=(1-|z|^2)^2$ so that $(f^2,f^3,f^4)\in\mathbf{HI}_2(\Delta,\Delta^3)$. It follows from Theorem 8.2 in [Ng10] that $(f^2(z),f^3(z),f^4(z))=(z,\alpha(z),\beta(z))$ up to reparametrizations, where $(\alpha,\beta)\in\mathbf{HI}_1(\Delta,\Delta^2;2)$. The result follows. □

Combining the results, Theorem 1.1 is proved when $k=1,2,3$. For the case where the isometric constant k equals 4, it is known from [Ng10, p. 2909] that $f(z)=(z,z,z,z)$ is the diagonal embedding up to reparametrizations. Hence, Theorem 1.1 is proved completely.

4. Generalizations of the Global Rigidity of the pth Root Embedding

In [Ch16a], the author has proven that any holomorphic isometric embedding in $\mathbf{HI}_1(\Delta,\Delta^p;p)$ is the pth root embedding F_p up to reparametrizations, which means that F_p is globally rigid in $\mathbf{HI}_1(\Delta,\Delta^p;p)$ in the sense of [Mok11]. This kind of phenomenon also occurs for the space $\mathbf{HI}_k\big(\Delta,\Delta^p;\frac{p}{k}\big)$, where k, p are positive integers satisfying $p\geq 2$, $k\mid p$ and $\frac{p}{k}\geq 2$. Note that the case of $\mathbf{HI}_k\big(\Delta,\Delta^p;\frac{p}{k}\big)$ is precisely the minimal case of $\mathbf{HI}_k(\Delta,\Delta^p)$ in terms of the global sheeting number. More precisely, we will show that all holomorphic isometries in $\mathbf{HI}_k(\Delta,\Delta^{qk};q)$ are globally rigid for positive integers q, k satisfying $q\geq 2$ and $k\geq 1$. The following may be regarded as an analogue of [Ch16a, Theorem 1.1] because the technique of proving [Ch16a, Theorem 1.1] is still valid for a more general situation with slight modifications.

PROPOSITION 4.1. *Let p and k be integers satisfying $p\geq 2$, $1\leq k\leq p$, $\frac{p}{k}\in\mathbb{Z}$ and $\frac{p}{k}\geq 2$. Let $f=(f^1,\ldots,f^p):(\Delta,k\,ds^2_\Delta)\to(\Delta^p,ds^2_{\Delta^p})$ be a holomorphic isometric embedding with the global sheeting number $q:=\frac{p}{k}$ and the isometric constant k. Then, $f=(g_1,\ldots,g_k)$ up to reparametrizations, where $g_j=F_q$ up to reparametrizations for $1\leq j\leq k$ such that the branching loci of all g_j's are the same and $F_q=(F_q^1,\ldots,F_q^q):\Delta\to\Delta^q$ is the qth root embedding.*

LEMMA 4.2 (Analogue of Lemma 4.9 in [Ch16a]). *Under the same assumptions as in Proposition 4.1, suppose that $q\geq 4$ is an even integer and π has 3 distinct branch points $a_1,a_2,a_3\in\partial\Delta$. Then, there is a component function f^j of f such that $\widetilde{f^j}(\overline{\Delta})\subset\Delta$, where $\widetilde{f}=(\widetilde{f^1},\ldots,\widetilde{f^{qk}}):\overline{\Delta}\to\overline{\Delta^{qk}}$ is the continuous mapping such that $\widetilde{f}|_\Delta=f$.*

Proof. From the proof of [Ch16a, Proposition 4.4], we see that the ramification index $v(\pi,x)$ is independent of the choice of $x\in\pi^{-1}(a_j)$ for each j. Moreover, we will see in the proof of Proposition 4.1 that the branching loci of all component functions of f are the same and coincide with the branching locus of π. Let the ramification index of π at $x\in\pi^{-1}(a_j)$ be v_j for $j=1,2,3$. Then, from [Ch16a, Remark 4.5] we also have the Riemann-Hurwitz formula $2q-2=\sum_{j=1}^{3} q\big(1-\frac{1}{v_j}\big)$ and all possible (v_1,v_2,v_3) are listed on Table 1 in [Ch16a, p. 355]. We may write $a_j=e^{i\theta_j}$ for $j=1,2,3$ and assume that $0\le\theta_1<\theta_2<\theta_3<2\pi$ without loss of generality. Let $A_{3,1}=\{e^{i\theta}\in\partial\Delta\mid\theta\in(\theta_3,\theta_1+2\pi)\}$, $A_{1,2}=\{e^{i\theta}\in\partial\Delta\mid\theta\in(\theta_1,\theta_2)\}$ and $A_{2,3}=\{e^{i\theta}\in\partial\Delta\mid\theta\in(\theta_2,\theta_3)\}$. Since $m=3$, each component function of f can only map precisely one connected component $A\subset\partial\Delta\setminus\{a_1,a_1,a_3\}$ into $\partial\Delta$. Then, by properness of the holomorphic isometric embedding f (cf. [Mok12]), we may suppose that $\widetilde{f^{\mu}}(A_{3,1})\subset\partial\Delta$ for $1\le\mu\le k$ and $\widetilde{f^{j}}(A_{3,1})\not\subset\partial\Delta$ for $k+1\le j\le qk$; $\widetilde{f^{\mu}}(A_{1,2})\subset\partial\Delta$ for $k+1\le\mu\le 2k$ and $\widetilde{f^{j}}(A_{1,2})\not\subset\partial\Delta$ for $1\le j\le k$ or $2k+1\le j\le qk$; $\widetilde{f^{\mu}}(A_{2,3})\subset\partial\Delta$ for $2k+1\le\mu\le 3k$ and $\widetilde{f^{j}}(A_{2,3})\not\subset\partial\Delta$ for $1\le j\le 2k$ or $3k+1\le j\le qk$.

For all cases listed on Table 1 in [Ch16a, p. 355], we have $v_3=2$. In order to be consistent to the above setting, by continuity of the map $\widetilde{f}$, we would have $|\widetilde{f^{\mu}}(a_3)|=1$ for $1\le\mu\le k$ or $2k+1\le\mu\le 3k$, $|\widetilde{f^{j}}(a_3)|<1$ for $k+1\le j\le 2k$ or $3k+1\le j\le qk$ by arguments in the proof of Lemma 4.3 in [Ch16a]; $|\widetilde{f^{\mu'}}(a_2)|=1$ for $2k+1\le\mu'\le 3k$ or $k+1\le\mu'\le 2k$ and $|\widetilde{f^{\mu''}}(a_1)|=1$ for $k+1\le\mu''\le 2k$ or $1\le\mu''\le k$. Actually, arguments in the proof of Lemma 4.3 in [Ch16a] would imply that if the ramification index of π at $(a_i,f_l^1(a_i),\dots,f_l^{qk}(a_i))$ equals s, then there exist distinct $j_1,\dots,j_{sk}\in\{1,\dots,qk\}$ such that $|f_l^{j_\mu}(a_i)|=1$ for $1\le\mu\le sk$. If $2\le s<q$, then $|f_l^{j}(a_i)|\ne 1$ for $j\notin\{j_1,\dots,j_{sk}\}$. The only difference is that in the proof of Lemma 4.3 in [Ch16a, p. 352], we replace the term $1-|z|^2$ by $(1-|z|^2)^k$ in the functional equation, replace the term $-\overline{a_i}\xi^s$ by $(-\overline{a_i})^k\xi^{ks}$ in the polarized functional equation and also replace p by q. The argument of comparing the vanishing orders of holomorphic functions at $\xi=0$ is still valid. Now, we assume the contrary that

$$\nexists j\in\{1,\dots,kq\}\ \text{such that}\quad \widetilde{f^{j}}(\overline{\Delta})\subset\Delta. \tag{4.1}$$

Then, for $3k+1\le\mu\le qk$, we should have $|\widetilde{f^{\mu}}(a_2)|=1$ or $|\widetilde{f^{\mu}}(a_1)|=1$.

In any case listed on Table 1 in [Ch16a, p. 355], the number of elements in the set

$$I_2:=\{\mu\in\mathbb{Z}\mid 3k+1\le\mu\le qk,\ |\widetilde{f^{\mu}}(a_2)|=1\text{ or }|\widetilde{f^{\mu}}(a_1)|=1\}$$

is at most $2(\frac{q}{2}\cdot k-2k)=(q-4)k$ because we already have $|\widetilde{f^{\mu'}}(a_2)|=1$ for $2k+1\le\mu'\le 3k$ or $k+1\le\mu'\le 2k$, $|\widetilde{f^{\mu''}}(a_1)|=1$ for $k+1\le\mu''\le 2k$ or $1\le\mu''\le k$ and $v_1,v_2\le\frac{q}{2}$. Note that $|\widetilde{f^{j}}(a_3)|<1$ for $k+1\le j\le 2k$ or $3k+1\le j\le qk$, by the assumption made in Equation (4.1), the set I_2 would have precisely $(q-3)k$ elements. This leads to a contradiction. Hence, we conclude that there exists $j\in\{1,\dots,qk\}$ such that $\widetilde{f^{j}}(\overline{\Delta})\subset\Delta$. □

Proof of Proposition 4.1. Assume without loss of generality that $f(0)=\mathbf{0}$. Note that $\sum_{j=1}^{kq}(1/s_j)=k$ and $s_j\mid q$ so that $s_j\le q$. Then, $k=\sum_{j=1}^{kq}\frac{1}{q}\le\sum_{j=1}^{kq}(1/s_j)=k$ implies that $s_j=q$ for $1\le j\le p$. The method used in the proof of the global rigidity of the pth root embedding can be applied to the study of $\mathbf{HI}_k(\Delta,\Delta^{kq};q)$ since $s_j=q$ for $1\le j\le kq$, so that all rational functions R_j

are equivalent, i.e., $R_i = R_j \circ \varphi_{ji}$ for some $\varphi_{ji} \in \mathrm{Aut}(\mathbb{P}^1)$. From the arguments in the study of the minimal case in [Ng10], the branching loci of all component functions of f are the same, and for each point $(z, w_1, \ldots, w_p) \in V$, the ramification index of π_j at (z, w_j) equals the ramification index of π_i at (z, w_i) for distinct i, j, $1 \le i, j \le p$. Let $\{a_1, \ldots, a_m\} \subset \partial\Delta$ be the set of all distinct branch points of $\pi : V \to \mathbb{P}^1$. Then, for each connected component $A \subset \partial\Delta \setminus \{a_1, \ldots, a_m\}$, there are precisely k component functions of f that map A into $\partial\Delta$. From the arguments in the proof of Proposition 4.4 in [Ch16a], we have $2 \le m \le 3$, and Table 1 in [Ch16a, p. 355] still provides all possible cases when $q \ge 4$ is even and $m = 3$. In fact, we only need to modify the arguments in the proof of Proposition 4.4 in [Ch16a], namely replacing the term $1 - |z|^2$ (resp. $-\overline{a_i}\xi^s$) by $(1 - |z|^2)^k$ (resp. $(-\overline{a_i})^k \xi^{ks}$) in the functional equation (resp. polarized functional equation) and also replacing p by q. The argument of comparing the vanishing orders of holomorphic functions at $\xi = 0$ is still valid.

If $q = 2$ or $q \ge 3$ is odd, then it follows from the arguments in the proof of both Proposition 4.4 and Corollary 4.6 in [Ch16a] that f has precisely two distinct branch points. If $q \ge 4$ is an even integer and $m = 3$, then it follows from Lemma 4.2 that $\widetilde{f^j}(\overline{\Delta}) \subset \Delta$ for some j, which contradicts the maximum principle as in the proof of Proposition 4.8 in [Ch16a]. Thus $m \ne 3$, so that $m = 2$. Therefore, all component functions of f are some component functions of the qth root embedding up to reparameterizations (see Lemma 4.9 in [Ng10, p. 2913]). Note that $\pi : V \to \mathbb{P}^1$ is also q-sheeted. By the proof of Theorem 6.5 in [Ng10] and the polarized functional equation

$$\prod_{j=1}^{qk} \left(1 - f^j(z)\overline{f^j(w)}\right) = (1 - z\overline{w})^k$$

for fixed $w \in \Delta \setminus \{0\}$, each branch of f has precisely k distinct component functions that take the value ∞ at ∞. Thus, these k component functions of f are the same component function of the qth root embedding up to reparameterizations. We may suppose without loss of generality that $f^{\mu k+1}, \ldots, f^{\mu k+k}$ are the same component function of F_q up to reparameterizations for each $\mu = 0, \ldots, q-1$ and that for $1 \le j, i \le k$, $f^{\mu k+j}$ and $f^{\mu' k+i}$ are not congruent to the same component function of F_q, provided that $\mu \ne \mu'$. In addition, $(f^j, f^{j+k}, \ldots, f^{j+(q-1)k})$ is the qth root embedding F_q up to reparameterizations for $1 \le j \le k$. The result follows. □

Remark. As an application of Theorem 1.1, we can solve the classification problem for the space $\mathbf{HI}_{p-l}(\Delta, \Delta^p)$ when $l = 1, 2$. In fact, given any $f \in \mathbf{HI}_{p-l}(\Delta, \Delta^p)$ for $p \ge 5$ and $l = 1$ (resp. $l = 2$), it follows from direct computation via Ng's identity $\sum_{j=1}^{p}(1/s_j) = p - l$ (see [Ng10]) that there are $p - 2$ (resp. at least $p - 4$) component functions f^j of f with sheeting numbers equal to 1, so that $f^j(z) = z$ up to reparameterizations. This shows that such a holomorphic isometry f is given by $f(z) = (g_1(z), g_2(z))$ up to reparameterizations for some $g_1 \in \mathbf{HI}_\mu(\Delta, \Delta^\mu)$ and $g_2 \in \mathbf{HI}_{p-l-\mu}(\Delta, \Delta^{p-\mu})$, where μ is the number of component functions of f with the sheeting numbers equal to 1. Here we apply Theorem 1.1 precisely when $l = 2$ and $\mu = p - 4$. The details can be found in [Ch16b]. This gives a complete classification of all holomorphic isometries in $\mathbf{HI}_{p-l}(\Delta, \Delta^p)$ when $l = 1, 2$ and $p \geq 3$. (Noting that the cases where $p = 3, 4$ have been done by Ng [Ng10] and the author in Theorem 1.1, respectively.) Moreover, the result obtained for the case where $p = 4$ and $l = 1$ is precisely that in Proposition 3.8.

On the other hand, by applying both Proposition 4.1 and Theorem 1.1 we have solved the classification problem for the subspace $\mathbf{HI}_k(\Delta, \Delta^p; n)$ of $\mathbf{HI}_k(\Delta, \Delta^p)$ whenever the global sheeting number n is a prime number such that $\mathbf{HI}_k(\Delta, \Delta^p; n)$ is nonempty. More precisely, if $f \in \mathbf{HI}_k(\Delta, \Delta^p; n)$ for some prime number n, then f is parameterized by the nth root embedding, the diagonal embeddings, and automorphisms of Δ and Δ^p, respectively. This has been done in the Ph.D. thesis of the author [Ch16b].

ACKNOWLEDGMENTS. This work is part of the author's Ph.D. thesis at The University of Hong Kong, and he would like to express his gratitude to his supervisor Professor Ngaiming Mok for his guidance. The author would like to thank Professor Tuen Wai Ng for discussions on the research, which is related to the study on the classification of all holomorphic isometries in the space $\mathbf{HI}_1(\Delta, \Delta^4; 8; 4, 4, 4, 4)$. He would also like to thank the referee for reading the paper carefully and providing useful suggestions to improve its readability.

References

[Ch16a] S. T. Chan, *On global rigidity of the p-th root embedding,* Proc. Amer. Math. Soc. 144 (2016), 347–358.

[Ch16b] ———, *On holomorphic isometric embeddings of complex unit balls into bounded symmetric domains,* thesis, The University of Hong Kong, 2016.

[GH78] P. Griffiths and J. Harris, *Principles of algebraic geometry,* Pure Appl. Math., Wiley-Interscience [John Wiley & Sons], New York, 1978.

[Gun90] R. C. Gunning, *Introduction to holomorphic functions of several variables, volume II: local theory,* The Wadsworth & Brooks/Cole Mathematics Series, Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, California, 1990.

[Mok11] N. Mok, *Geometry of holomorphic isometries and related maps between bounded domains,* Geometry and Analysis, No. 2, Adv. Lectures Math., 18, pp. 225–270, Int. Press, Somerville, MA, 2011.

[Mok12] ———, *Extension of germs of holomorphic isometries up to normalizing constants with respect to the Bergman metric,* J. Eur. Math. Soc. (JEMS) 14 (2012), 1617–1656.

[Mok] ———, private communication.

[MN10] N. Mok and S.-C. Ng, *Second fundamental forms of holomorphic isometries of the Poincaré disk into bounded symmetric domains and their boundary behavior along the unit circle,* Sci. China Ser. A 52 (2009), 2628–2646.

[Ng08] S.-C. Ng, *On holomorphic isometric embeddings from the unit disk into polydisks and their generalizations,* thesis, The University of Hong Kong, 2008.

[Ng10] ______, *On holomorphic isometric embeddings of the unit disk into polydisks,* Proc. Amer. Math. Soc. 138 (2010), 2907–2922.

Department of Mathematics
Syracuse University
Syracuse, NY 13244-1150
USA

schan08@syr.edu