Michigan Math. J. 61 (2012), 475–491

Continuous Closure of Sheaves

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Definition 1. Let $I = (f_1, \dots, f_r) \subset \mathbb{C}[z_1, \dots, z_n]$ be an ideal. Following [Br], a polynomial $g(z_1, \dots, z_n)$ is in the *continuous closure* of I if and only if there are continuous functions ϕ_i such that $g = \phi_1 f_1 + \cdots + \phi_r f_r$. These polynomials form an ideal $I^C \supset I$. For example,

$$z_1^2 z_2^2 = \frac{\bar{z}_1 z_2^2}{|z_1|^2 + |z_2|^2} z_1^3 + \frac{\bar{z}_2 z_1^2}{|z_1|^2 + |z_2|^2} z_2^3$$

shows that $z_1^2 z_2^2 \in (z_1^3, z_2^3)^C \setminus (z_1^3, z_2^3)$.

Definition 1 is very natural, but it is not clear that it gives an algebraic notion (since $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ does not map continuous functions to continuous functions) or that it defines a sheaf in the Zariski topology (since a continuous function may grow faster than any polynomial).

This paper has three aims.

- We give a purely algebraic construction of the continuous closure of any torsion-free coherent sheaf (Definition 6). Although the construction makes sense for any reduced scheme, even in positive and mixed characteristic, it is not clear that it corresponds to a more intuitive version in general.
- In characteristic 0 we prove that one gets the same definition of I^C using various subclasses of continuous functions (Corollary 19).
- We show that taking continuous closure commutes with flat morphisms whose fibers are seminormal (Corollary 21), at least in characteristic 0. In particular, the continuous closure of a coherent ideal sheaf is again a coherent ideal sheaf (both in the Zariski and in the étale topologies) and it commutes with field extensions.

It should be noted that, although our definition of the continuous closure is purely algebraic and without any reference to continuity, the proof of these base change properties uses continuous functions in an essential way.

Instead of working with $\mathbb{C}$ or other algebraically closed fields, one can also define the continuous closure over any topological field. The most interesting is the real case, considered in [FK]. The answer turns out to be quite different; for instance, over $\mathbb{C}$ the continuous closure of $(x^2 + y^2)$ is itself but over $\mathbb{R}$ it is the

Received December 16, 2010. Revision received January 17, 2012.
Partial financial support was provided by NSF Grant no. DMS-0758275.

much larger ideal (x^2+y^2, x^3, y^3). The methods, however, are quite similar. The main difference is that the base change properties are not considered in [FK] and the key construction (Proposition 24) is more complicated over nonclosed fields.

The methods of this paper provide a way to compute the continuous closure in principle, but it is unlikely to be practical in its current form.

Descent Problems

Instead of working with ideals, I work with maps of locally free sheaves $f: E \to F$. Thus an ideal sheaf $I = (f_1, \dots, f_r) \subset \mathcal{O}_X$ corresponds to the map $(f_1, \dots, f_r)$: $\mathcal{O}_X^r \to \mathcal{O}_X$. For inductive purposes we need the case when E and F live on different schemes.

DEFINITION 2. Fix a base scheme S. A *descent problem* over S is a compound object

$$\mathbf{D} = (p: Y \to X,\ f: p^*E \to F) \tag{2.1}$$

consisting of a proper morphism $p: Y \to X$ of reduced S-schemes of finite type, a locally free sheaf E on X, a locally free sheaf F on Y, and a map of sheaves $f: p^*E \to F$.

The original setting corresponds to the cases

$$(p: X \cong X,\ f: \mathcal{O}_X^r \to \mathcal{O}_X) \quad \text{with } S = \operatorname{Spec} \mathbb{C}. \tag{2.2}$$

When X is normal or seminormal, the continuous closure is

$$H^0(Y, F) \cap \operatorname{im}[C^0(X(\mathbb{C}), E) \to C^0(Y(\mathbb{C}), F)], \tag{2.3}$$

where C^0 denotes the space of continuous sections.

Our claim is that the primary task should be to understand to continuous aspects of the problem, that is, the image of

$$f \circ p^*: C^0(X(\mathbb{C}), E) \to C^0(Y(\mathbb{C}), F). \tag{2.4}$$

Once that is done, the answers to the algebraic questions should follow.

A descent problem over $\mathbb{C}$ is called *finitely determined* if for every $\phi_Y \in C^0(Y(\mathbb{C}), F)$ the following are equivalent.

(2.5a) There is a $\phi_X \in C^0(X(\mathbb{C}), E)$ such that $\phi_Y = f \circ p^*(\phi_X)$.

(2.5b) For every finite subset $Z \subset Y(\mathbb{C})$ there is a $\phi_{X,Z} \in C^0(X(\mathbb{C}), E)$ such that $\phi_Y(z) = f \circ p^*(\phi_{X,Z})(z)$ for every $z \in Z$.

For finitely determined descent problems it is quite easy to pass between the continuous and the algebraic sides.

The original descent problems (2.2) are finitely determined only in the trivial case $I = \mathcal{O}_X$. A better example is given by the following construction. Given $I = (f_1, \dots, f_r)$, let $Y := B_I X$ denote the blow-up of I with projection $p: Y \to X$. The inverse image ideal sheaf $f^{-1}I \cdot \mathcal{O}_Y \subset \mathcal{O}_Y$ is locally free; denote it by $\mathcal{O}_Y(-E)$, where E is an exceptional divisor. We get a descent problem

$$(p: Y \to X,\ f: p^*\mathcal{O}_X^r \to \mathcal{O}_Y(-E)), \tag{2.6}$$

which is equivalent to the original one. The concept of finite determinacy for (2.6) is my variant of the axis closure condition of [Br]. The latter looks at morphisms

$$\phi\colon \operatorname{Spec} k[[x_1,\dots,x_n]]/(x_ix_j : i \neq j) \to X.$$

These maps do not lift to the blow-up, but, if ϕ does not map any generic point to $V(I)$, then ϕ lifts to a morphism of the normalization

$$\bar{\phi}\colon \coprod_i \operatorname{Spec} k[[x_i]] \to B_IX.$$

The image of the closed points is n points in E, and the primary obstruction to the axis closure condition is given by the possible failure of (2.5b). If (2.6) is finitely determined then the axis closure of I equals the continuous closure but the converse probably fails.

It turns out that (2.6) is finitely determined in many cases but not always. Such examples were discovered by [EH]; an especially nice one is $I = (x^2, y^2, xyz^2)$.

The definition of a descent problem and the proof grew out of first reducing (2.2) to (2.6) and then studying the latter by restriction to E and induction.

The key technical result (Theorem 17) shows that every descent problem is equivalent to a finitely determined descent problem. To achieve this, we need various ways of modifying descent problems. The following definition is chosen to consist of simple and computable steps yet be broad enough for the proofs to work. (It should become clear that several variants of the definition would also work; the present one is meant to supersede the choice in [K2].)

Definition 3 (Scions of descent problems). Let $\mathbf{D} = (p\colon Y \to X,\ f\colon p^*E \to F)$ and $\mathbf{D}_i = (p\colon Y \to X,\ f_i\colon p^*E \to F_i)$ be descent problems over S. The following three operations create new descent problems.

(3.1) (Proper pull-back) For a proper morphism $r\colon Y_1 \to Y$ set

$$r^*\mathbf{D} := (p \circ r\colon Y_1 \to X,\ r^*f\colon (p\circ r)^*E \to r^*F).$$

(3.2) (Direct sum) Using the natural diagonal map $\bigoplus_{i=1}^m f_i$, set

$$\bigoplus_{i=1}^m \mathbf{D}_i := \bigl(p\colon Y \to X,\ \bigoplus_{i=1}^m f_i\colon p^*E \to \bigoplus_{i=1}^m F_i\bigr).$$

(3.3) (Sheaf change) Assume that f factors as $p^*E \stackrel{q}{\to} F' \stackrel{j}{\hookrightarrow} F$, where F' is a locally free sheaf and $\operatorname{rank}_y j = \operatorname{rank}_y F'$ for all y in a dense open subscheme $Y^0 \subset Y$. Then set

$$\mathbf{D}' := (p\colon Y \to X,\ f' := q\colon p^*E \to F').$$

Informally speaking, a *scion* of $\mathbf{D}$ is a descent problem

$$\mathbf{D}_s = (p_s\colon Y_s \to X,\ f_s\colon p_s^*E \to F_s)$$

that is obtained from $\mathbf{D}$ using the constructions (3.1)–(3.3) and remembers its forebears. More precisely, a scion is a sequence of descent problems

$$\{\mathbf{D}_i = (p_i\colon Y_i \to X,\ f_i\colon p_i^*E \to F_i) : i = 0,\dots,s\},$$

where $\mathbf{D}_0 = \mathbf{D}$ and for each $\mathbf{D}_{i+1}$ we specify how it is constructed from the $\mathbf{D}_0, \dots, \mathbf{D}_i$. The actual process of construction is important in several of the basic definitions.

Each scion $\mathbf{D}_s = (p_s : Y_s \to X,\ f_s : p_s^* E \to F_s)$ comes equipped with a morphism $r_s : Y_s \to Y$, called the *structure map*. The class of all scions of $\mathbf{D}$ is denoted by $\mathrm{Sci}(\mathbf{D})$.

Simple examples of scions are given by *restrictions*. If $Y_1 \subset Y$ is a closed subscheme, we set

$$\mathbf{D}|_{Y_1} := (p|_{Y_1} : Y_1 \to X,\ f|_{Y_1} : (p|_{Y_1})^* E \to F|_{Y_1}).$$

If $X_1 \subset X$ is a subscheme and $Y_1 := \mathrm{red}\, p^{-1}(X_1)$, we set $\mathbf{D}|_{X_1} := \mathbf{D}|_{Y_1}$.

4 (Seminormalization). (For more details, see [K1, Sec. I.7.2].) A morphism $p : X' \to X$ is a *partial seminormalization* if X' is reduced, p is a finite homeomorphism, and $k(p^{-1}(x)) = k(x)$ for every point $x \in X$. Under mild conditions (for instance, if X is excellent) there is a unique largest partial seminormalization $\pi : X^{\mathrm{sn}} \to X$, called the *seminormalization* of X.

If $p : Y \to X$ is a proper surjection of reduced schemes, then composing by p identifies $\mathcal{O}_{X^{\mathrm{sn}}}$ with those sections of $\mathcal{O}_{Y^{\mathrm{sn}}}$ that are constant on the fibers of p.

Note that the seminormalization is dominated by the normalization; thus we can think of the seminormalization as a partial normalization. In some respects, seminormalizations behave better than the normalization. For instance, any morphism $g : Y \to X$ induces a morphism between the seminormalizations $g^{\mathrm{sn}} : Y^{\mathrm{sn}} \to X^{\mathrm{sn}}$. (For normalization this can fail if g is not dominant.)

A flat morphism is called seminormal if its geometric fibers are seminormal. If X and $g : X' \to X$ are both seminormal then so is X'. For normal fibers this is proved in [K1, I.7.2.6]. By localization, the general case is a consequence of the following claim.

Claim 4.1. *Let $f : (y \in Y) \to (0 \in X)$ be a flat morphism of finite type. Assume that X, $Y_0 := f^{-1}(0)$, and $Y \setminus \{y\}$ are seminormal. Then Y is seminormal.*

Proof. If $\dim Y_0 = 0$ then f is smooth and we are done by [K1, I.7.2.6]. Let $h \in \mathcal{O}_{Y^{\mathrm{sn}}}$ be a section. If $\dim Y_0 \geq 1$, we prove by induction on r that $h \in \mathcal{O}_Y + m_{0,X}^r \mathcal{O}_{Y^{\mathrm{sn}}}$ for every r. We can start with $r = 0$. In general, assume that we have $p_r \in \mathcal{O}_Y$ such that $h - p_r \in m_{0,X}^r \mathcal{O}_{Y^{\mathrm{sn}}}$. By restricting to Y_0 we see that $(h - p_r)|_{Y_0}$ is a section of

$$\mathcal{O}_{Y_0^{\mathrm{sn}}} \otimes (m_{0,X}^r / m_{0,X}^{r+1}) = \mathcal{O}_{Y_0} \otimes (m_{0,X}^r / m_{0,X}^{r+1}).$$

Thus there is a $q_{r+1} \in m_{0,X}^r \mathcal{O}_Y$ such that $h - p_r - q_{r+1}$ vanishes along Y_0 to order $r+1$. This shows that the completion of $\mathcal{O}_Y$ equals the completion of $\mathcal{O}_{Y^{\mathrm{sn}}}$, hence $\mathcal{O}_Y = \mathcal{O}_{Y^{\mathrm{sn}}}$. □

If F is a coherent sheaf on X, its pull-back to X^{sn} is denoted by F^{sn}. We frequently view F^{sn} as an $\mathcal{O}_X$-sheaf.

If X is a variety over $\mathbb{C}$, then $\mathcal{O}_{X^{\mathrm{sn}}}$ consists of those rational functions that are continuous. Thus it appears that the continuous closure is a concept that naturally lives on seminormal schemes.

It would be possible to consider descent problems only for seminormal schemes. This, however, would be inconvenient because various constructions do not yield seminormal schemes, and we would have to take seminormalizations all the time. Instead, next we build the seminormalizations into the definition of the global sections of $\mathrm{Sci}(\mathbf{D})$.

DEFINITION 5. Let $\mathbf{D} = (p\colon Y \to X,\ f\colon p^*E \to F)$ be a descent problem with scions

$$\mathrm{Sci}(\mathbf{D}) = \{(p_i\colon Y_i \to X,\ f_i\colon p_i^*E \to F_i) : i \in I\}.$$

An *algebraic global section* of F over $\mathrm{Sci}(\mathbf{D})$ is a collection of sections

$$\Phi := \{\phi_i \in H^0(Y_i^{\mathrm{sn}}, F_i^{\mathrm{sn}}) : i \in I\}$$

such that the ϕ_i commute with pull-backs for the operations (3.1) with direct sums for the operations (3.2) and with push-forwards for the operations (3.3). All sections form an $\mathcal{O}_S$-module

$$H^0(\mathrm{Sci}(\mathbf{D}), F).$$

We call ϕ_i the *restriction* of Φ to Y_i, denoted by $\Phi|_{Y_i}$. The most important of these restrictions is $\Phi|_Y$. Note that $\Phi|_Y$ uniquely determines Φ. Indeed, the constructions (3.1) and (3.2) automatically carry along ϕ and in (3.3) the natural map $H^0(Y^{\mathrm{sn}}, F') \to H^0(Y^{\mathrm{sn}}, F)$ is an injection.

We usually think of $H^0(\mathrm{Sci}(\mathbf{D}), F)$ as an $\mathcal{O}_S$-submodule of $H^0(Y^{\mathrm{sn}}, F^{\mathrm{sn}})$. Note also that every $\phi_X \in H^0(X, E)$ defines a global section of F over $\mathrm{Sci}(\mathbf{D})$ by setting $\phi_i := f_i(p_i^*\phi_X)$. Thus we have natural maps

$$H^0(X, E) \to H^0(\mathrm{Sci}(\mathbf{D}), F) \hookrightarrow H^0(Y^{\mathrm{sn}}, F^{\mathrm{sn}}). \tag{5.1}$$

We can now define a notion of continuous closure of sheaves. A justification of the definition will be given in Corollary 19.

DEFINITION 6 (Continuous closure of sheaves). Let X be a reduced, affine scheme over a field of characteristic 0 and let J be a torsion-free coherent sheaf on X. One can realize J as the image of a map between locally free sheaves $f\colon E \to F$. Let $\mathbf{D}_J = (p\colon Y \cong X,\ f\colon E \to F)$ be the corresponding descent problem. Define the *continuous closure* of J as

$$J^C := H^0(\mathrm{Sci}(\mathbf{D}_J), F) \subset H^0(X^{\mathrm{sn}}, F^{\mathrm{sn}}).$$

We shall see in Proposition 23 that J^C does not depend on the choice of $f\colon E \to F$.

Definition 6 is purely algebraic, but it does not connect with continuity in any obvious way. Actually, for base fields that are not naturally subfields of $\mathbb{C}$, it is not even clear what continuity should mean. This is the question we consider next.

Although the basic definitions make sense in general, all the examples that I know are in characteristic 0. Thus from now on we assume that we work over a fixed base field k of characteristic 0.

Classes of Continuous Functions

Here we describe various classes of functions where our proof works.

ASSUMPTION 7. Let k be a field and $K \supset k$ an algebraically closed field. For a reduced k-scheme Z of finite type, let $C^K(Z)$ denote the K-algebra of all functions $Z(K) \to K$. One can naturally view $\Gamma(Z, \mathcal{O}_Z)$ as a k-subalgebra of $C^K(Z)$.

We consider K-subalgebras $C^*(Z) \subset C^K(Z)$ that satisfy the following properties.

(7.1) (Sheaf) $Z \mapsto C^*(Z)$ is a sheaf in the Zariski topology. That is, if $Z = \bigcup_i U_i$ is an open cover of Z then $\phi \in C^*(Z)$ iff $\phi|_{U_i} \in C^*(U_i)$ for every i.
(7.2) (Contains $\mathcal{O}_Z$) $\Gamma(Z, \mathcal{O}_Z) \subset C^*(Z)$.
(7.3) (Pull-back) For every k-morphism $g\colon Z_1 \to Z_2$, composing with g maps $C^*(Z_2)$ to $C^*(Z_1)$.
(7.4) (Zariski dense is dense) Let $\phi \in C^*(Z)$ and let h be a rational function on Z such that ϕ equals h on a dense Zariski open subset. Then $\phi = h$ everywhere and h is a regular function on Z^{sn}. This implies that the Zariski closure of the support of every $\phi \in C^*(Z)$ is a union of irreducible components of Z.
(7.5) (Descent property) Let $g\colon Z_1 \to Z_2$ be a proper, dominant k-morphism, let $\phi \in C^K(Z_2)$, and assume that $\phi \circ g \in C^*(Z_1)$. Then $\phi \in C^*(Z_2)$. In particular, assume that Z is a union of its closed subvarieties Z_i and we have $\phi_i \in C^*(Z_i)$ such that $\phi_i|_{Z_i \cap Z_j} = \phi_j|_{Z_i \cap Z_j}$ for every i, j. The descent property for $\coprod_i Z_i \to Z$ shows that there is a $\phi \in C^*(Z)$ such that $\phi|_{Z_i} = \phi_i$ for every i.
(7.6) (Extension property) Let $Z_1 \subset Z_2$ be a closed subscheme. Then the restriction map $C^*(Z_2) \to C^*(Z_1)$ is surjective.
(7.7) (Cartan–Serre A and B) Every locally free sheaf is generated by finitely many C^*-sections and every surjection of locally free sheaves has a C^*-valued splitting. (For more details, see Section 9.)

We can unite (7.5) and (7.6) as follows.

(7.8) (Strong descent property) Let $g\colon Z_1 \to Z_2$ be a proper k-morphism and let $\psi \in C^*(Z_2)$. Then $\psi = \phi \circ g$ for some $\phi \in C^*(Z_2)$ iff ψ is constant on every fiber of g.

EXAMPLE 8. Here are some natural examples satisfying the assumptions (7.1)–(7.7). Let us start with the cases when $k \subset K = \mathbb{C}$.

(8.1) Let $C^0(Z)$ denote all continuous functions on $Z(\mathbb{C})$.
(8.2) Let $C^h(Z)$ denote all locally Hölder continuous functions on $Z(\mathbb{C})$.
(8.3) Let $S^0(Z)$ be the sheaf of $\mathbb{C}$-valued continuous semialgebraic functions on $Z(\mathbb{C})$, viewed as a real algebraic variety. (If $Z \subset \mathbb{C}^m$, we identify $\mathbb{C}^m$ with

$\mathbb{R}^{2m}$ and view $Z(\mathbb{C})$ as a real variety. A function on $\mathbb{R}^{2m}$ is semialgebraic iff its graph is semialgebraic, that is, a finite union of sets defined by polynomial inequalities of the form $f \geq 0$.) See [BCR, Chap. 2] for details and proofs of the properties (7.1)–(7.7). (Let me just note that (7.4) is more interesting than it sounds. For instance, on the Whitney umbrella $(x^2 = y^2 z) \subset \mathbb{R}^3$ not every Zariski dense open set is Euclidean dense.)

I do not know how to generalize the first two of these in case k is not embedded into $\mathbb{C}$, but the third variant can be extended to any characteristic-0 field.

(8.4) Let $\mathbf{R}$ be a real closed field, let $\mathbf{C} := \mathbf{R}(\sqrt{-1})$, and assume that $k \subset \mathbf{C}$. Let $S^0_{\mathbf{R}}(Z)$ be the sheaf of $\mathbf{C}$-valued continuous semialgebraic functions on $Z(\mathbf{C})$, viewed as an $\mathbf{R}$-variety. (See [BCR, Chap. 2] for details.)

9 (C^*-valued sections). Let F be a locally free sheaf on Z, and let $Z = \bigcup_i U_i$ be an open cover with trivializations $t_i \colon F|_{U_i} \cong \mathcal{O}^r_{U_i}$ for every i. Let $C^*(Z, F)$ denote the set of those sections such that $t_i(\phi|_{U_i}) \in C^*(U_i)^r$ for every i. If C^* satisfies properties (7.1) and (7.2), this is independent of the trivializations and the choice of the covering.

Assume next that (7.7) holds. We claim that if C^* satisfies properties (7.1)–(7.6) then their natural analogues also hold for $C^*(Z, F)$. This is clear for the properties (7.2)–(7.5).

In order to check the extension property (7.6), let $Z_1 \subset Z_2$ be a closed subvariety and F a locally free sheaf on Z_2. Write it as a quotient of a trivial bundle $\mathcal{O}^N_{Z_2}$. Every section $\phi_1 \in C^*(Z_1, F|_{Z_1})$ lifts to a section in $C^*(Z_1, \mathcal{O}^N_{Z_1})$, which in turn extends to a section in $C^*(Z_2, \mathcal{O}^N_{Z_2})$ by (7.6). The image of this lift in $C^*(Z_2, F|_{Z_2})$ gives the required lifting of ϕ_1.

Let $\mathbf{D}$ be a descent problem with scions Sci($\mathbf{D}$). If $\phi \in C^*(Y, F)$ then $r^*\phi \in C^*(Y_1, r^*F)$ and $\bigoplus_{i=1}^m r_i^*\phi \in C^*\big(Y_w, \bigoplus_{i=1}^m r_i^*F\big)$ are well-defined. In (3.3), $j \colon C^*(Y, F') \to C^*(Y, F)$ is an injection; hence there is at most one $\phi' \in C^*(Y, F')$ such that $j(\phi') = \phi$. Iterating these, for any scion $\mathbf{D}_s$ of $\mathbf{D}$ we get a partially defined map, called the *restriction,*

$$\text{rest} \colon C^*(Y, F) \dashrightarrow C^*(Y_s, F_s) \quad \text{denoted by } \phi \mapsto \phi|_{Y_s} \text{ or } \phi \mapsto \phi|_{\mathbf{D}_s}.$$

The restriction map sits in a commutative square:

$$\begin{array}{ccc} C^*(Y, F) & \overset{\text{rest}}{\dashrightarrow} & C^*(Y_s, F_s) \\ \uparrow & & \uparrow \\ C^*(X, E) & = & C^*(X, E). \end{array}$$

If the structure map $r_s \colon Y_s \to Y$ is surjective then the restriction map rest: $C^*(Y, F) \dashrightarrow C^*(Y_s, F_s)$ is injective (on its domain). In this case, understanding the image of $f \circ p^* \colon C^*(X, E) \to C^*(Y, F)$ is pretty much equivalent to understanding the image of $f_s \circ p_s^* \colon C^*(X, E) \to C^*(Y_s, F_s)$.

As long as C^* satisfies properties (7.1)–(7.3), we can follow Definition 5 to obtain

$$C^*(\mathrm{Sci}(\mathbf{D}), F), \tag{9.1}$$

the space of C^*-valued global sections of F over $\mathrm{Sci}(\mathbf{D})$. We have natural maps

$$C^*(X, E) \to C^*(\mathrm{Sci}(\mathbf{D}), F) \hookrightarrow C^*(Y^{\mathrm{sn}}, F^{\mathrm{sn}}) \ (= C^*(Y, F)). \tag{9.2}$$

Note further that

$$H^0(\mathrm{Sci}(\mathbf{D}), F) = C^*(\mathrm{Sci}(\mathbf{D}), F) \cap H^0(Y^{\mathrm{sn}}, F^{\mathrm{sn}}). \tag{9.3}$$

To see this we need to show that if $\Phi \in C^*(\mathrm{Sci}(\mathbf{D}), F)$ and $\Phi|_Y$ is algebraic then every other restriction of Φ is also algebraic. This is clear for the steps (3.1) and (3.2). For scions as in (3.3), let ϕ be an algebraic section of F. We assume that ϕ is a C^*-valued section of F'. It is also a rational section over a Zariski dense open set; thus, by (7.4), ϕ is also an algebraic section of F'.

The restriction map on $C^*(Y, F)$ gives a restriction map on global sections of scions that also sits in a commutative diagram:

$$\begin{array}{ccc} C^*(\mathrm{Sci}(\mathbf{D}), F) & \xrightarrow{\text{rest}} & C^*(\mathrm{Sci}(\mathbf{D}_s), F_s) \\ \uparrow & & \uparrow \\ C^*(X, E) & = & C^*(X, E). \end{array} \tag{9.4}$$

Note that the restriction map on global sections of scions is everywhere defined. In essence, we defined $C^*(\mathrm{Sci}(\mathbf{D}), F)$ to ensure this.

Finitely Determined Descent Problems

The notion of a finitely determined descent problem (2.5) admits an obvious generalization to the C^*-valued case. We also need the following more general version.

DEFINITION 10. Let $\mathbf{D} = (p\colon Y \to X,\ f\colon p^*E \to F)$ be a descent problem and $Z \subset X$ a closed algebraic subvariety. Then $\mathbf{D}$ is called *finitely determined* relative to Z if, for every $\phi_Y \in C^*(Y, F)$ that vanishes on $p^{-1}(Z)$, the following are equivalent.

(10.1) There is a $\phi_X \in C^*(X, E)$ such that $\phi_Y = f \circ p^*(\phi_X)$.

(10.2) For every finite subset $\{y_1, \dots, y_m\} \subset Y(K)$ there is a $\phi_{X, y_1, \dots, y_m} \in C^*(X, E)$ such that $\phi_Y(y_i) = f \circ p^*(\phi_{X, y_1, \dots, y_m})(y_i)$ for $i = 1, \dots, m$.

We show in Lemma 11 that these are also equivalent to the following precise form.

(10.3) Condition (10.2) holds for all $m \le \operatorname{rank} E + 1$.

Although condition (10.2) asks about all possible finite sets of points in $Y(K)$, the conditions imposed by points in different fibers of p are independent. Thus the only interesting case is when all the y_i are in the same fiber. Working in a fiber, we have the following general abstract test.

LEMMA 11 (Wronskian test). *Let Y be a set and let $\phi, f_1, \dots, f_r$ be functions on Y with values in a field K. Assume that the f_i are linearly independent. Then the following are equivalent.*

(11.1) *ϕ is a K-linear combination of the f_i.*

(11.2) *For every $r+1$ points $y_1, \dots, y_{r+1}$ there are $c_1, \dots, c_r \in K$ (possibly depending on the y_i) such that $\phi(y_i) = \sum_j c_j f_j(y_i)$ for $i = 1, \dots, r+1$.*

(11.3) *The determinant*

$$\begin{vmatrix} f_1(y_1) & \cdots & f_1(y_r) & f_1(y_{r+1}) \\ \vdots & & \vdots & \vdots \\ f_r(y_1) & \cdots & f_r(y_r) & f_r(y_{r+1}) \\ \phi(y_1) & \cdots & \phi(y_r) & \phi(y_{r+1}) \end{vmatrix}$$

is identically zero as a function on Y^{r+1}.

Proof. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). We prove (3) $\Rightarrow$ (1) by induction on r. Since the f_i are linearly independent, there are $y_1, \dots, y_r \in Y$ such that the upper left $r \times r$ subdeterminant is nonzero. Fix these $y_1, \dots, y_r$ and solve the linear system

$$\phi(y_i) = \sum_j \lambda_j f_j(y_i) \quad \text{for } i = 1, \dots, r.$$

Replace ϕ by $\psi := \phi - \sum_i \lambda_i f_i$ and let y_{r+1} vary. Then our determinant is

$$\begin{vmatrix} f_1(y_1) & \cdots & f_1(y_r) & f_1(y_{r+1}) \\ \vdots & & \vdots & \vdots \\ f_r(y_1) & \cdots & f_r(y_r) & f_r(y_{r+1}) \\ 0 & \cdots & 0 & \psi(y_{r+1}) \end{vmatrix}.$$

The whole determinant vanishes iff $\psi(y_{r+1})$ is identically zero—that is, when $\phi \equiv \sum_j \lambda_j f_j$. □

If a descent problem is not finitely determined, we can still study the conditions imposed by (10.2). This leads to the following definition.

DEFINITION 12. Given a descent problem $\mathbf{D} = (p\colon Y \to X,\ f\colon p^*E \to F)$, let $\mathrm{Sci}^0(\mathbf{D}) \subset \mathrm{Sci}(\mathbf{D})$ denote all 0-dimensional scions and $\mathbf{D}$ itself. We can now define

$$H^0(\mathrm{Sci}^0(\mathbf{D}), F) \quad \text{and} \quad C^*(\mathrm{Sci}^0(\mathbf{D}), F)$$

as (respectively) the collection of sections $\{\phi_i \in H^0(Y_i^{\mathrm{sn}}, F_i^{\mathrm{sn}})\}$ and $\{\phi_i \in C^*(Y_i, F_i)\}$ that satisfy the compatibility conditions as in Definition 5, where now Y_i runs through only the scions in $\mathrm{Sci}^0(\mathbf{D})$.

Thus $\mathbf{D}$ is finitely determined iff

$$\mathrm{im}[C^*(X, E) \to C^*(Y, F)] = C^*(\mathrm{Sci}^0(\mathbf{D}), F).$$

An advantage of $H^0(\mathrm{Sci}^0(\mathbf{D}), F)$ is that it can be easily computed algebraically.

13 (Computation of $H^0(\mathrm{Sci}^0(\mathbf{D}), F)$). Let X be an affine scheme of finite type over a field and let $\mathbf{D} = (p\colon Y \to X,\ f\colon p^*E \to F)$ be a descent problem. We inductively construct descent problems $\mathbf{D}_i = (p_i\colon Y_i \to X_i,\ f_i\colon p_i^*E_i \to F_i)$ as follows. Set $\mathbf{D}_0 := \mathbf{D}$ and assume that $\mathbf{D}_i$ is already constructed.

By cohomology and base change [Ha, III.12.11] there is a largest open dense subset $X_i^0 \subset X_i$ over which the following hold:

(13.1) $p_i^{\mathrm{sn}}\colon Y_i^{\mathrm{sn}} \to X_i$ is flat,
(13.2) the $R^j(p_i^{\mathrm{sn}})_* F_i^{\mathrm{sn}}$ are locally free and commute with base change; and
(13.3) $E_i \to (p_i^{\mathrm{sn}})_* F_i^{\mathrm{sn}}$ has locally constant rank.

Set $X_{i+1} := X_i \setminus X_i^0$ and let $\mathbf{D}_{i+1}$ be the restriction of $\mathbf{D}_i$ to X_{i+1}.

Set $Q_i^0 := \mathrm{coker}[E_i \to (p_i^{\mathrm{sn}})_* F_i^{\mathrm{sn}}]$ and let Q_i be the push-forward of Q_i^0 by the locally closed embedding $X_i^0 \hookrightarrow X$. The Q_i are quasi-coherent sheaves on X. We get natural sheaf maps $q_i\colon E \to E_i \to Q_i$.

By construction, if $x \in X_i^0$ and $\phi \in H^0(Y^{\mathrm{sn}}, F^{\mathrm{sn}})$ then ϕ satisfies (10.2) for all subsets of $p^{-1}(x)$ iff $q_i(\phi) \in H^0(X, Q_i)$ vanishes at x. Since $X = \bigcup_i X_i^0$, this implies that

$$H^0(\mathrm{Sci}^0(\mathbf{D}), F) = \ker\bigl[H^0(Y^{\mathrm{sn}}, F^{\mathrm{sn}}) \to \textstyle\bigoplus_i H^0(X, Q_i)\bigr]. \tag{13.4}$$

This implies important functoriality properties of $H^0(\mathrm{Sci}^0(\mathbf{D}), F)$, but first we need a definition.

DEFINITION 14 (Pulling back descent problems). Let $\mathbf{D} = (p\colon Y \to X,\ f\colon p^*E \to F)$ be a descent problem over a base field k. We consider two ways of obtaining new descent problems by base change.

First, every field extension $k' \supset k$ gives a descent problem over k':

$$\mathbf{D}_{k'} := (p_{k'}\colon Y_{k'} \to X_{k'},\ f_{k'}\colon p_{k'}^* E_{k'} \to F_{k'}).$$

Second, let $b\colon X' \to X$ be a flat, finite-type morphism with reduced fibers. Let Y be a reduced scheme and $p\colon Y \to X$ a morphism. Then $b_Y\colon Y' := X' \times_X Y \to Y$ is flat with reduced fibers, hence Y' is also reduced. Thus

$$b^*\mathbf{D} := (p'\colon Y' \to X',\ f'\colon (p')^*E \to b_Y^* F)$$

is also a descent problem. All the constructions in Definition 3 commute with pull-back by flat morphisms with reduced fibers. Thus we get a pull-back map $b^*\colon \mathrm{Sci}(\mathbf{D}) \to \mathrm{Sci}(b^*\mathbf{D})$.

Note that it is not obvious that there is a pull-back map $b^*\colon H^0(\mathrm{Sci}(\mathbf{D}), F) \to H^0(\mathrm{Sci}(b^*\mathbf{D}), b_Y^* F)$. (Indeed, $b^*\mathbf{D}$ may have scions that are not pulled back from $\mathrm{Sci}(\mathbf{D})$, and these could pose additional restrictions on sections.) Corollary 20 (to follow) shows that such problems do not arise.

PROPOSITION 15. *Let X be an affine scheme of finite type over a field, and let $\mathbf{D} = (p\colon Y \to X,\ f\colon p^*E \to F)$ be a descent problem. Then the formation of $H^0(\mathrm{Sci}^0(\mathbf{D}), F)$ commutes with flat, seminormal base changes and with base field extensions.*

Proof. Note that in Section 13 the formation of the X_i and Q_i commutes with flat, seminormal base changes and with base field extensions. Using (13.4), this implies that $H^0(\mathrm{Sci}^0(\mathbf{D}), F)$ also commutes with flat, seminormal base changes and with base field extensions. □

The Main Theorem and Its Consequences

DEFINITION 16 (Universal properties). Let $\mathbf{P}$ be a property of descent problems. Let $\mathbf{D}$ be a descent problem over a field k. We say that $\mathbf{D}$ is *universally* $\mathbf{P}$ if $b^*\mathbf{D}_{k'}$ satisfies $\mathbf{P}$ for every base field extension $k' \supset k$ followed by any flat, finite-type, seminormal base change $b\colon X'_{k'} \to X_{k'}$.

The main technical result of this paper is the following.

THEOREM 17. *Let* $\mathbf{D} = (p\colon Y \to X,\ f\colon p^*E \to F)$ *be a descent problem over a field of characteristic 0. Then it has a universally finitely determined scion* $\mathbf{D}_s = (p_s\colon Y_s \to X,\ f_s\colon p_s^*E \to F_s)$ *whose structure map* $r_s\colon Y_s \to Y$ *is surjective.*

Before giving a proof, let us consider some consequences. First we have the following property, which was the very reason for our definition of scions.

COROLLARY 18. *Let* $\mathbf{D} = (p\colon Y \to X,\ f\colon p^*E \to F)$ *be a descent problem. Assume that* C^* *satisfies properties* (7.1)–(7.7). *Then*

$$C^*(\mathrm{Sci}(\mathbf{D}), F) = \mathrm{im}[C^*(X, E) \to C^*(Y, F)].$$

Proof. Note that, by (9.2), the containment

$$C^*(\mathrm{Sci}(\mathbf{D}), F) \supset \mathrm{im}[C^*(X, E) \to C^*(Y, F)]$$

always holds. To see the converse, let $\mathbf{D}_s = (p_s\colon Y_s \to X,\ f_s\colon p_s^*E \to F_s)$ be a finitely determined scion of $\mathbf{D}$ whose structure map $r_s\colon Y_s \to Y$ is surjective. We have the obvious inclusions

$$C^*(\mathrm{Sci}(\mathbf{D}), F) \subset C^*(\mathrm{Sci}(\mathbf{D}_s), F_s) \subset C^*(\mathrm{Sci}^0(\mathbf{D}_s), F_s)$$

and

$$C^*(\mathrm{Sci}^0(\mathbf{D}_s), F_s) = \mathrm{im}[C^*(X, E) \to C^*(Y_s, F_s)],$$

since $\mathbf{D}_s$ is finitely determined. Note further that $C^*(X, E) \to C^*(Y_s, F_s)$ factors through $C^*(Y, F)$ and through $C^*(\mathrm{Sci}(\mathbf{D}), F)$. Because the structure map $r_s\colon Y_s \to Y$ is surjective, $C^*(Y, F) \to C^*(Y_s, F_s)$ is injective. These statements show that

$$C^*(\mathrm{Sci}(\mathbf{D}), F) \subset \mathrm{im}[C^*(X, E) \to C^*(Y, F)].$$ □

We can now see that the two definitions of the continuous closure—namely, Definition 6 and the obvious generalization of Definition 1—agree with each other.

COROLLARY 19. *Let* X *be a reduced affine scheme and* $f\colon E \to F$ *a map between locally free sheaves. Set* $J = \mathrm{im}(f)$ *as a subsheaf of* F *and let* J^C *be as in Definition 6. Then*

$$J^C = \mathrm{im}[C^*(X, E) \to C^*(X, F)] \cap F^{\mathrm{sn}}.$$

Proof. By definition $J^C = H^0(\mathrm{Sci}(\mathbf{D}), F)$, and by (9.3)

$$H^0(\mathrm{Sci}(\mathbf{D}), F) = C^*(\mathrm{Sci}(\mathbf{D}), F) \cap H^0(X^{\mathrm{sn}}, F^{\mathrm{sn}}).$$

By Corollary 18, $C^*(\mathrm{Sci}(\mathbf{D}), F) = \mathrm{im}[C^*(X, E) \to C^*(X, F)]$. □

As another consequence, we obtain that global sections of scions are unchanged by surjective structure maps.

COROLLARY 20. *Let* $\mathbf{D}$ *be a descent problem with a scion* $\mathbf{D}_s$ *whose structure map* $r_s\colon Y_s \to Y$ *is surjective. Then the restriction maps*

$$C^*(\mathrm{Sci}(\mathbf{D}), F) \to C^*(\mathrm{Sci}(\mathbf{D}_s), F_s) \quad \textit{and} \quad H^0(\mathrm{Sci}(\mathbf{D}), F) \to H^0(\mathrm{Sci}(\mathbf{D}_s), F_s)$$

are isomorphisms.

Proof. Since r_s is surjective, the restriction maps are injective. By Corollary 18, $C^*(X, E) \to C^*(\mathrm{Sci}(\mathbf{D}_s), F_s)$ is surjective and it factors through $C^*(\mathrm{Sci}(\mathbf{D}), F)$. Thus the restriction map is surjective with C^*-coefficients.

The algebraic case also follows once we prove that, if $\phi \in C^*(\mathrm{Sci}(\mathbf{D}), F)$ and its restriction to Y_s is algebraic, then ϕ itself is algebraic. This is a local question on Y; hence we need to show that if $\phi \in C^*(Y)$ and $r_s^*\phi$ is a regular function then ϕ is a regular function on Y^{sn}. We can view ϕ as a morphism to $\mathbb{A}^1_Y$; let Y' be its image. Since $Y_s \to Y$ is proper, $Y' \to Y$ is proper and $Y'_K \to Y_K$ is a homeomorphism. Thus Y' is dominated by the seminormalization. □

The next result is an important invariance property of global sections of descent problems.

COROLLARY 21. *Let* X *be an affine scheme of finite type over* k *and let* $\mathbf{D} = (p\colon Y \to X,\ f\colon p^*E \to F)$ *be a descent problem. Then taking algebraic global sections commutes with base field extensions and with flat, finite-type, seminormal base changes.*

In particular, taking the continuous closure commutes with base field extensions and with flat, finite-type, seminormal base changes.

Proof. By Theorem 17, $\mathbf{D}$ has a universally finitely determined scion $\mathbf{D}_s = (p_s\colon Y_s \to X,\ f_s\colon p_s^*E \to F_s)$ whose structure map $r_s\colon Y_s \to Y$ is surjective.

By Corollary 20,

$$H^0(\mathrm{Sci}(\mathbf{D}), F) = H^0(\mathrm{Sci}(\mathbf{D}_s), F_s)$$

and the equality continues to hold after every base change. Thus it is sufficient to prove Corollary 21 in case $\mathbf{D}$ is universally finitely determined. For such descent problems,

$$H^0(\mathrm{Sci}(\mathbf{D}), F) = H^0(\mathrm{Sci}^0(\mathbf{D}), F),$$

and we saw in Proposition 15 that $H^0(\mathrm{Sci}^0(\mathbf{D}), F)$ commutes with base field extensions and with flat, finite-type, seminormal base changes. □

Since open embeddings are flat with seminormal fibers, we can sheafify the notion of continuous closure.

DEFINITION 22. Let $\mathbf{D} = (p\colon Y \to X,\ f\colon p^*E \to F)$ be a descent problem. By Corollary 21, as $\{U : U \hookrightarrow X\}$ runs through all affine open subsets, the rule

$$U \mapsto H^0(\mathrm{Sci}(\mathbf{D}|_U), F|_U)$$

defines a coherent sheaf in the Zariski topology, denoted by

$$R^0 p_*(\mathrm{Sci}(\mathbf{D}), F) \tag{22.1}$$

and called the *push-forward* of $\mathrm{Sci}(\mathbf{D})$.

As in (5.1), there are natural maps

$$E \to R^0 p_*(\mathrm{Sci}(\mathbf{D}), F) \hookrightarrow p_* F^{\mathrm{sn}}. \tag{22.2}$$

Finally, let us show that Definition 6 is independent of the auxiliary choices.

PROPOSITION 23. *The continuous closure is independent of the choice of* $f\colon E \to F$.

Proof. Pick $f\colon E \to F$ such that $J \cong \operatorname{im} f$. Composing a surjection $E' \to E$ and an injection $F \hookrightarrow F'$, we get another map $f'\colon E' \to F'$ such that $J \cong \operatorname{im} f'$. We get two descent problems, $\mathbf{D}$ and $\mathbf{D}'$. We claim that

$$C^*(\mathrm{Sci}(\mathbf{D}), F) = C^*(\mathrm{Sci}(\mathbf{D}'), F').$$

This follows from Corollary 18 and the obvious maps

$$C^*(X, E') \twoheadrightarrow C^*(X, E) \to C^*(X, F) \hookrightarrow C^*(X, F'). \qquad \square$$

Proof of Theorem 17

In order to get an idea of the proof, assume first that X, Y are normal and let $Y \to W \to X$ denote the Stein factorization. We first study which sections over Y descend to W and then try to descend them to X.

If we look over a single point $w \in W$, then the question is answered by Lemma 11. Working in our family, this means passing from $Y \to W$ to the $(n+1)$-fold fiber product $Y \times_W \times \cdots \times_W Y$. The fiber product can be rather singular in general, so this will work only over a dense open subset of W.

Going from W to X is easy if we work locally analytically. In this case $W \to X$ is a local isomorphism over an open subset of W; thus every question over W can be rewritten as a question over X. This will not work well algebraically, but there are no problems if $W \to X$ is Galois.

The point of Proposition 24 is to show that, by passing to a suitable scion, the foregoing considerations apply—at least over a dense open subset of X. We shall then finish by a straightforward dimension induction (Section 26).

PROPOSITION 24. *Let* $\mathbf{D} = (p\colon Y \to X,\ f\colon p^*E \to F)$ *be a descent problem. Then there is a closed algebraic subvariety* $Z \subset X$ *with* $\dim Z < \dim X$ *and a* *scion* $\tilde{\mathbf{D}} = (\tilde{p}\colon \tilde{Y} \to X,\ \tilde{f}\colon \tilde{p}^*E \to \tilde{F})$ *with surjective structure map* $\tilde{r}\colon \tilde{Y} \to Y$ *and with the following properties.*

Let $X = \bigcup_{i\in I} X_i$ *be the irreducible components. For every* $i \in I$ *let* $\tilde{Y}_i \subset \tilde{Y}$ *be the closure of* $\tilde{p}^{-1}(X_i \setminus Z)$ *and let* $\tilde{\mathbf{D}}_i$ *be the restriction of* $\tilde{\mathbf{D}}$ *to* $\tilde{Y}_i$. *Then, for every* $i \in I$:

(24.1) *a finite group* G_i *acts on* $\tilde{\mathbf{D}}_i$;
(24.2) *there is a* G_i*-equivariant factorization* $\tilde{p}_i\colon \tilde{Y}_i \xrightarrow{\tilde{q}_i} \tilde{W}_i \xrightarrow{\tilde{w}_i} X_i$;
(24.3) *over* $X_i \setminus Z$, *the map* $\tilde{w}_i\colon \tilde{W}_i \to X_i$ *is finite and Galois with group* G_i;
(24.4) *there is a* G_i*-equivariant quotient bundle* $\tilde{w}_i^*E \to \tilde{E}_i$ *such that* $\tilde{f}_i$ *factors as* $\tilde{p}_i^*E \twoheadrightarrow \tilde{q}_i^*\tilde{E}_i \cong \tilde{F}_i$.

Proof. We may harmlessly assume that $p(Y)$ is Zariski dense in X.

After we construct $\tilde{\mathbf{D}}$, the plan is to make sure that Z contains all of its "singular" points. In the original setting, Z is the set where the map $(f_1,\dots,f_r)\colon \mathcal{O}_X^r \to \mathcal{O}_X$ has rank 0. In the general case, we need to include points over which $\tilde{f}$ drops rank and also points over which $\tilde{p}$ drops rank. During the proof we gradually add more and more irreducible components to Z as needed.

Step 0. To start with, we add to Z the locus where X is not normal and the $p(Y_j)$ for which $Y_j \subset Y$ is an irreducible component that does not dominate any of the irreducible components of X. In the conclusions, the different $\tilde{\mathbf{D}}_i$ have no effect on each other; hence we can work with them one at a time. We construct each $\tilde{\mathbf{D}}_i$ separately and then let $\tilde{\mathbf{D}}$ be the disjoint union of $\mathbf{D}|_Z$ and the $\tilde{\mathbf{D}}_i$ for $i \in I$.

For simplicity of notation, we drop the index i. We thus assume that X is irreducible and that every irreducible component of Y dominates X. We may assume that Y is normal, take the Stein factorization $p\colon Y \xrightarrow{q} W \xrightarrow{s} X$, and set $m = \deg(W/X)$. In several steps we construct the following diagram:

$$\begin{array}{ccccccc}
(\tilde{q}^{(m)})^*\bar{E}^{(m)} \cong & \tilde{F}^{(m)} & & \bar{F}^{(m)} & & F^{(m)} & & F \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
(t_W^{(m)} \circ s_W^{(m)})^*E \twoheadrightarrow \bar{E}^{(m)} & \tilde{Y}_X^{(m)} & & \bar{Y}_X^{(m)} & \xrightarrow{t_Y^{(m)}} & Y_X^{(m)} & \xrightarrow{\pi_i^{(m)}} & Y \\
\quad\searrow & \downarrow{\scriptstyle \tilde{q}^{(m)}} & & \downarrow{\scriptstyle \bar{q}^{(m)}} & & \downarrow{\scriptstyle q^{(m)}} & & \downarrow{\scriptstyle p} \\
& \tilde{W}^{(m)} & = & \bar{W}^{(m)} & \xrightarrow{t_W^{(m)}} & W^{(m)} & \xrightarrow{s^{(m)}} & X.
\end{array}$$

Step 1: Constructing $W^{(m)}$ *and its column.* Let $s\colon W \to X$ be a finite morphism of (possibly reducible) varieties.

Consider the m-fold fiber product $W_X^m := W \times_X \cdots \times_X W$ with coordinate projections $\pi_i\colon W_X^m \to W$. For every $i \neq j$, let $\Delta_{ij} \subset W_X^m$ be the preimage of the diagonal $\Delta \subset W \times_X W$ under the map (π_i, π_j). Let $W_X^{(m)} \subset W_X^m$ be the union of the dominant components in the closure of $W_X^m \setminus \bigcup_{i\neq j} \Delta_{ij}$ with projection $s^{(m)}\colon W_X^{(m)} \to X$. The symmetric group S_m acts on $W_X^{(m)}$ by permuting the factors.

Let $X^0 \subset X$ be the largest Zariski open subset over which s is smooth. If $x \in X^0$ then $(s^{(m)})^{-1}(x)$ consists of ordered m-element subsets of $s^{-1}(x)$, so S_m acts transitively on $(s^{(m)})^{-1}(x)$ if $|s^{-1}(x)| = m$.

Let $p\colon Y \to X$ be as before but with Stein factorization $p\colon Y \xrightarrow{q} W \xrightarrow{s} X$. Let Y_X^m denote the m-fold fiber product $Y \times_X \cdots \times_X Y$ with coordinate projections $\pi_i\colon Y_X^m \to Y$. Let $Y_X^{(m)} \subset Y_X^m$ denote the dominant parts of the preimage of $W_X^{(m)}$ under the natural map $q^m\colon Y_X^m \to W_X^m$ with projection $p^{(m)}\colon Y_X^{(m)} \to X$. Note that, for general $x \in X$, S_m acts transitively on the irreducible components of $(p^{(m)})^{-1}(x)$.

Let F be a locally free sheaf on Y. Then $\bigoplus_i \pi_i^* F$ is a locally free sheaf on Y_X^m. Its restriction to $Y_X^{(m)}$ is denoted by $F^{(m)}$.

The S_m-action on $Y_X^{(m)}$ naturally lifts to an S_m-action on $F^{(m)}$. From $f\colon p^*E \to F$ we get an S_m-invariant map of locally free sheaves $f^{(m)}\colon (p^{(m)})^*E \to F^{(m)}$. For each m we get a scion of $\mathbf{D}$,

$$\mathbf{D}^{(m)} := \left(p^{(m)}\colon Y_X^{(m)} \to X,\ f^{(m)}\colon (p^{(m)})^*E \to F^{(m)}\right).$$

Step 2: Constructing $\bar{W}^{(m)}$ and its column. More generally, let $\mathbf{D} = (p\colon Y \to W, f\colon p^*E \to F)$ be a descent problem. (Note that the base is W instead of X.) Assume that W is irreducible. Consider the coherent sheaf $E' := \operatorname{im}[E \to p_* F]$.

Let $\operatorname{Grass}(d, E) \to X$ be the universal Grassmann bundle of rank-d quotients of E, where d is the rank of E' at a general point. At a general point $x \in X$, $E(x) \twoheadrightarrow E'(x)$ is such a quotient. Thus E' gives a rational map $X \dashrightarrow \operatorname{Grass}(d, E)$, which is defined on a dense open subset. Let $\bar{X} \subset \operatorname{Grass}(d, E)$ denote the closure of its image and $\tau_X\colon \bar{X} \to X$ the projection. Then τ_X is a proper birational morphism and we have a decomposition

$$\tau_X^* q\colon \tau_X^* E \overset{s}{\twoheadrightarrow} \bar{E} \overset{j}{\hookrightarrow} \tau_X^* E',$$

where $\bar{E}$ is a locally free sheaf of rank d on $\bar{X}$, s is a rank-d surjection everywhere, and j is a rank-d injection on a dense open subscheme.

Applying this to $\mathbf{D}^{(m)}$, with $W^{(m)}$ playing the role of the base, we obtain $\bar{\mathbf{D}}^{(m)}$.

Step 3: Constructing $\tilde{W}^{(m)}$ and its column. More generally, let $\mathbf{D} = (p\colon Y \to W, f\colon p^*E \to F)$ be a descent problem. Assume that W and the generic fiber of p are irreducible and that $E \to p_* F$ is an injection. We construct a scion

$$\tilde{\mathbf{D}} = (\tilde{p}\colon \tilde{Y} \to W,\ \tilde{f}\colon \tilde{p}^*E \to \tilde{F})$$

with surjective structure map such that $\tilde{f}$ is an isomorphism.

Set $n = \operatorname{rank} E$ and let Y_W^{n+1} be the union of the dominant components of the $(n+1)$-fold fiber product of $Y \to W$ with coordinate projections π_i. Let $\tilde{p}\colon Y_W^{n+1} \to W$ be the map given by any of the $p \circ \pi_i$. Consider the diagonal map

$$\tilde{f}\colon \tilde{p}^*E \to \sum_{i=1}^{n+1} \pi_i^* F,$$

which is an injection over a dense open set $Y^0 \subset Y_W^{n+1}$ by assumption. Using (3.3), we can replace $\sum_{i=1}^{n+1} \pi_i^* F$ by $\tilde{p}^* E$.

Applying this to $\bar{\mathbf{D}}^{(m)}$, we obtain $\tilde{\mathbf{D}}^{(m)}$. □

PROPOSITION 25. *Let $\tilde{\mathbf{D}} = (\tilde{p}: \tilde{Y} \to X, \tilde{f}: \tilde{p}^* E \to \tilde{F})$ be a descent problem, and let $Z \subset X$ be a closed algebraic subvariety. Let $X = \bigcup_i X_i$ be the irreducible components and assume that $X_i \cap X_j \subset Z$ for every $i \neq j$. Let $\tilde{Y}_i \subset Y$ be the closure of $\tilde{p}^{-1}(X_i \setminus Z)$ and let $\tilde{\mathbf{D}}_i$ be the restriction of $\tilde{\mathbf{D}}$ to $\tilde{Y}_i$. Assume that* (24.1)–(24.4) *hold for every i. Then $\tilde{\mathbf{D}}$ is finitely determined relative to Z.*

Proof. Let $\Psi_Y \in C^*(Y, F)$ be a section that vanishes on $p^{-1}(Z)$ such that (10.2) holds. We can uniquely write $\Psi_Y = \sum \Psi_i$, where $\operatorname{Supp} \Psi_i \subset \tilde{Y}_i$. It is thus enough to write $\Psi_i = f \circ p^*(\psi_{i,X})$ for each i. For a fixed i, we need to do this over X_i and then extend $\psi_{i,X}$ to X by setting it to zero on the complement. Thus it is sufficient to work with one $\tilde{\mathbf{D}}_i$ at a time.

Using the isomorphism $\tilde{q}_i^* \tilde{E}_i \cong \tilde{F}_i$, Ψ_i can be identified with a section $\tilde{\Psi}_i$ of $\tilde{q}_i^* \tilde{E}_i$. The conditions (10.2) now imply that $\tilde{\Psi}_i$ is constant on the fibers of $\tilde{Y}_i \to \tilde{W}_i$ and is G_i-invariant. Thus $\tilde{\Psi}_i$ is the pull-back of a G_i-invariant section $\tilde{\Psi}_{W,i}$ of $\tilde{E}_i$ that vanishes on the preimage of Z. Using a G_i-invariant C^*-splitting of $\tilde{w}_i^* E \twoheadrightarrow \tilde{E}_i$, we can think of $\tilde{\Psi}_{W,i}$ as a G_i-invariant section of $\tilde{w}_i^* E$. Therefore, $\tilde{\Psi}_{W,i}$ descends to a section $\psi_{X,i} \in C^*(X_i, E)$ that vanishes on Z. □

26 (Proof of Theorem 17). We use induction on the dimension of X. If $\dim X = 0$ then we are done by Lemma 11.

In general, construct Z and $\tilde{\mathbf{D}}$ as in Proposition 24. Let $\tilde{\mathbf{D}}_Z$ denote the restriction of $\tilde{\mathbf{D}}$ to Z. By induction, it has a finitely determined scion whose structure map is surjective; we denote it $(\tilde{\mathbf{D}}_Z)^\sim$. Let $\mathbf{D}_s$ be the disjoint union of $\tilde{\mathbf{D}}$ and of $(\tilde{\mathbf{D}}_Z)^\sim$.

Pick $\Phi_s \in C^*(Y_s, F_s)$ and assume that it satisfies the conditions (10.2). Its restriction to $(\tilde{\mathbf{D}}_Z)^\sim$ also satisfies the conditions (10.2); hence there is a section $\phi_Z \in C^*(Z, E|_Z)$ whose pull-back to $p_s^{-1}(Z)$ equals the restriction of Φ_s. (A priori this holds only over $(\tilde{Y}_Z)^\sim$, but since the structure map $(\tilde{Y}_Z)^\sim \to \tilde{Y}_Z$ is surjective, it also holds over $\tilde{Y}_Z$.)

According to Section 9, we can lift ϕ_Z to a section $\phi_X \in C^*(X, E)$. Consider next

$$\Psi_s := \Phi_s - f_s(p_s^* \phi_X) \in C^*(Y_s, F_s).$$

By construction, it vanishes along $p_s^{-1}(Z)$. By Proposition 25, $\tilde{\mathbf{D}}$ is finitely determined relative to Z; hence we can write $\Psi_s = f_s \circ p_s^*(\psi_X)$ for some $\psi_X \in C^*(X, E)$. Thus $\Phi_s = f_s \circ p_s^*(\phi_X + \psi_X)$. □

ACKNOWLEDGMENTS. I learned about continuous closures from a lecture of M. Hochster. I benefited from the comments and examples of H. Brenner and M. Hochster. The questions and comments of Ch. Fefferman, A. Némethi, and T. Szamuely were helpful in formulating many of the definitions and results.

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