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# Minimal Anisotropic Groups of Higher Real Rank

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(with an Appendix by V. Chernousov & A. Merkurjev)

## 1. Introduction

Throughout this paper, $G$ will be a connected, simple algebraic group over an algebraic number field $F$. Let $V^F_{\infty,\mathbb{R}}$ be the set of all real places of $F$ and let $F_v \simeq \mathbb{R}$ be the completion of $F$ with respect to the place $v \in V^F_{\infty,\mathbb{R}}$.

In [CLM], Chernousov, Lifschitz, and Morris define $S_G$ to be the elements of $V^F_{\infty,\mathbb{R}}$ such that $\operatorname{rank}_{F_v}(G) \geq 2$, and they introduce the following definition.

Definition [CLM, Def. 3.3]. Let $G$ be isotropic. We say $G$ is *minimal* if $S_G \neq \emptyset$ and there does not exist a proper, isotropic, almost simple $F$-subgroup $H$ of $G$ such that $\operatorname{rank}_{F_v}(H) \geq 2$ for all $v \in S_G$.

Under this definition they classified minimal isotropic groups over number fields and found that they had absolute type $A_2$, $A_2 \times A_2$, or $A_1^n$ for some $n \geq 2$. Taking the particular case of $F = \mathbb{Q}$ and applying the Margulis arithmeticity theorem [Ma, Thm. IX.1.16 and Rem. IX.1.6(iii)] and the Margulis superrigidity theorem [Ma, Thm. IX.5.12(ii) and Rem. IX.1.6(iv)], they were able to translate this into the following result regarding lattices in Lie groups.

Theorem 1.1 [CLM, Thm. 1.13]. *Every nonuniform lattice of higher rank contains a subgroup that is isomorphic to a finite-index subgroup of a lattice contained in either* $\mathrm{SL}_3(\mathbb{R})$, $\mathrm{SL}_3(\mathbb{C})$, *or a direct product* $\mathrm{SL}_2(\mathbb{R})^m \times \mathrm{SL}_2(\mathbb{C})^n$ *with* $m + n \geq 2$.

The theorem is very useful in examining properties of nonuniform lattices of higher rank that transfer to sublattices. For example, Ghys conjectured that no lattice of higher rank has a total order that is invariant under right translation [Gh]. Theorem 1.1 reduces the problem of proving Ghys's conjecture for nonuniform lattices to considering lattices of the form above, which was done by Lifschitz and Morris [LM].

For arithmetic lattices, the dichotomy between uniform and nonuniform lattices translates exactly into the dichotomy between anisotropic and isotropic algebraic

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groups over number fields [Ma, Rem. IX.1.6(vii)]; thus it is natural to attempt to classify minimal anisotropic groups with *appropriate real rank*.

DEFINITION 1.1 (Appropriate Real Rank). Let $G$ be a group over a number field $F$. Let

$$S'_G = \{v \in V^F_{\infty,\mathbb{R}} \mid \operatorname{rank}_{F_v}(G) = 1\} \quad \text{and} \quad S''_G = \{v \in V^F_{\infty,\mathbb{R}} \mid \operatorname{rank}_{F_v}(G) \geq 2\}.$$

We say that a subgroup $H \leq G$ has *appropriate real rank* if $\operatorname{rank}_{F_v}(H) = 1$ for all $v \in S'_G$ and $\operatorname{rank}_{F_v}(H) \geq 2$ for all $v \in S''_G$. Define $S_G = S'_G \cup S''_G$.

Given this definition of appropriate real rank, the following is the natural generalization of minimality to anisotropic groups.

DEFINITION 1.2 (Minimal). A group $G$ as before is said to be *minimal* if $S''_G \neq \emptyset$ and $G$ contains no proper $F$-simple subgroups of appropriate real rank.

It is useful to break the classification into the absolutely simple and non–absolutely simple cases.

THEOREM 1.2. *If $G$ is an absolutely simple, minimal, anisotropic group over an algebraic number field $F$, then $G$ is isomorphic to one of the following groups* (*up to isogeny*):

(1) $\mathrm{SU}_3(L, f)$ *for $L/F$ quadratic and $f$ anisotropic hermitian on $L^3$ with at least one $v \in V^F_{\infty,\mathbb{R}}$ such that $L \otimes F_v \simeq F_v \times F_v$; or*
(2) $\mathrm{SU}(D, \tau)$ *a central division algebra of prime degree $p \geq 3$ over $L$ quadratic over $F$ with involution of the second kind $\tau$; or*
(3) $\mathrm{SL}_1(D)$ *for a central division algebra $D$ over $F$ of prime degree $p > 2$.*

It is well known that every simple group that is not absolutely simple is isogenous to the restriction of scalars of an absolutely simple group [BOI, (28.8)], and subgroups of such groups are closely related to the concept of descent.

DEFINITION 1.3 (Descent). Given an object $A$ (an algebraic group, a central simple algebra, etc.) over a field $K$, we say that $A$ *descends* to $P \subset K$ if there exists an object $A'$ of the same kind defined over $P$ such that, when we extend scalars, we have $A'_K \simeq A$.

THEOREM 1.3. *If $G$ is a minimal anisotropic group over an algebraic number field $F$ that is not absolutely simple, then $G$ is isomorphic to one of the following groups, up to isogeny* (*let $\varepsilon = \pm 1$*).

(1) $R_{K/F}(\mathrm{SL}_1(D))$ *for a central division algebra $D$ of odd prime degree over an extension $K$ such that $D$ does not descend to any $P$ with $F \subset P \subset K$.*

(2) $R_{K/F}(\mathrm{SU}(D, \tau))$, *where $D$ is a central division algebra of prime degree $p \geq 3$ over a quadratic extension $K'/K$ with involution of the second kind $\tau$ such that, if $(D, \tau)$ descends to $P'$ with $F \subset P \subset K$ and $P'/P$ quadratic, then $P_{w_i} \simeq \mathbb{R}$ and $P_{w_i} \otimes P' \simeq \mathbb{C}$ for all $w_i \in V^P_{\infty,\mathbb{R}}$ lying over at least one $v_0 \in S_G$ and:*

(a) *if* $v_0 \in S'_G$ *then* $(D' \otimes_P P_{w_i}, \tau' \otimes 1) \simeq (M_n(\mathbb{C}), \varepsilon\langle 1, \ldots, 1\rangle)$ *for all* $w_i \in V^P_{\infty,\mathbb{R}}$ *lying over* $v_0$; *or*
(b) *if* $v_0 \in S''_G$ *then* $(D' \otimes_P P_{w_i}, \tau' \otimes 1) \simeq (M_n(\mathbb{C}), \varepsilon\langle 1, -1, 1, \ldots, 1\rangle)$ *for at most one i and* $(D' \otimes_P P_{w_i}, \tau' \otimes 1) \simeq (M_n(\mathbb{C}), \varepsilon\langle 1, \ldots, 1\rangle)$ *for all others.*

(3) $R_{K/F}(\mathrm{SL}_1(D))$ *for D a quaternion algebra over K such that, for every* $F \subset P \subset K$ *such that D descends to P, there exists a* $v_0 \in S_G$ *satisfying*:

(a) *if* $v_0 \in S'_G$, *then* $P_{w_i} \simeq \mathbb{R}$ *and* $D' \otimes_P P_{w_i} \simeq \mathbb{H}$ *for all* $w_i \in V^P_{\infty,\mathbb{R}}$ *lying over* $v_0$; *and*
(b) *if* $v_0 \in S''_G$, *then there is at most one* $w_i \in V^P_{\infty,\mathbb{R}}$ *lying over* $v_0$ *such that either* $P_{w_i} \simeq \mathbb{C}$ *or* $D' \otimes_P P_{w_i} \simeq M_2(\mathbb{R})$ *but not both.*

(4) $R_{K/F}(\mathrm{SU}_3(K', f))$ *for* $K'/K$ *quadratic and f hermitian over* $K'^3$ *such that*:

(a) *for any* $F \subset P \subset K$ *such that* $\mathrm{SU}_3(K', f)$ *descends to P, there exists a* $v_0 \in S_G$ *such that* $P_{w_i} \simeq \mathbb{R}$ *for all* $w_i \in V^P_{\infty,\mathbb{R}}$ *lying over* $v_0$ *and such that*
   (i) *if* $\mathrm{SU}_3(K', f)$ *descends to* $\mathrm{SU}_3(P', f')$, *where* $f' = \langle 1, a_2, a_3\rangle$, *then* $P_{w_i} \otimes P' \simeq \mathbb{C}$ *for every i and*
      (A) *if* $v_0 \in S'_G$ *then the image of* $a_j$ *in* $P_{w_i}$ *is positive for all i or*
      (B) *if* $v_0 \in S''_G$ *then the image of* $a_j$ *in* $P_{w_i}$ *is negative for at most one i or*
   (ii) *if* $\mathrm{SU}_3(K', f)$ *descends to* $\mathrm{SU}(D, \tau)$, *where D is a central division algebra of degree 3 over* $P'/P$ *quadratic with involution* $\tau$ *of the second kind, then* $P' \otimes P_{w_i} \simeq \mathbb{C}$ *for every* $w_i \in V^P_{\infty,\mathbb{R}}$ *lying over* $v_0$ *and*
      (A) *if* $v_0 \in S'_G$ *then* $(D \otimes P_{w_i}, \tau \otimes 1) \simeq (M_3(\mathbb{C}), \sigma)$, *where* $\sigma(X) = \bar{X}^T$ *for every* $w_i \in V^P_{\infty,\mathbb{R}}$, *or*
      (B) *if* $v_0 \in S''_G$ *then* $(D \otimes P_{w_i}, \tau \otimes 1) \simeq (M_3(\mathbb{C}), \sigma)$ *for all but at most one* $w_i \in V^P_{\infty,\mathbb{R}}$ *and, for at most one* $w_i$, $(D \otimes P_{w_i}, \tau \otimes 1) \simeq (M_3(\mathbb{C}), \sigma \circ \mathrm{Int}(\varepsilon\, \mathrm{diag}(1, -1, 1)))$; *and*

(b) *for any* $F \subset P \subseteq K$ *such that some subgroup* $\mathrm{SL}_1(D') \leq \mathrm{SU}_3(K', f)$ *descends to* $\mathrm{SL}_1(D)$ *over P, there exists a* $v_0 \in S_G$ *such that*
   (i) *if* $v_0 \in S'_G$ *then* $P_{w_i} \simeq \mathbb{R}$ *and* $D \otimes P_{w_i} \simeq \mathbb{H}$ *for all* $w_i \in V^P_{\infty,\mathbb{R}}$ *over* $v_0$, *or*
   (ii) *if* $v_0 \in S''_G$ *then either* $P_{w_i} \simeq \mathbb{C}$ *or* $D \otimes P_{w_i} \simeq M_2(\mathbb{R})$ *for at most one* $w_i \in V^P_{\infty,\mathbb{R}}$ *over* $v_0$.

Using the Margulis arithmeticity and superrigidity theorems, one can show that the next result gives a classification of the minimal semisimple real Lie groups with no compact factors containing uniform irreducible lattices of higher rank.

THEOREM 1.4. *Every uniform lattice of higher rank contained in a semisimple Lie group with no compact factors contains a subgroup that is isomorphic to a finite-index subgroup of a lattice contained either in* $\mathrm{SL}_p(\mathbb{R})^\ell \times \mathrm{SL}_p(\mathbb{C})^m \times \mathrm{SU}_p(\mathbb{C}, f_1) \times \cdots \times \mathrm{SU}_p(\mathbb{C}, f_n)$, *where the* $f_i$ *are hermitian forms of index at least 1, or in* $\mathrm{SL}_2(\mathbb{R})^n \times \mathrm{SL}_2(\mathbb{C})^m$ *with* $n + m \geq 2$.

This theorem has immediate applications to the theory of discrete subgroups of semisimple Lie groups. For example, to prove Ghys's conjecture for cocompact lattices it suffices to examine lattices contained in Lie groups of the form just described.

The rest of this paper will provide a proof of Theorems 1.2 and 1.3.

## 2. Groups of Classical Type

### *2.1. Orthogonal Groups*

Assume that $G = \mathrm{SO}(f)$, where $f$ is a quadratic form of dimension at least 5.

PROPOSITION 2.1. *The group $G$ contains a $F$-simple subgroup $H$ of type $A_1 \times A_1$ that has appropriate real rank.*

*Proof.* By [PrR, Lemma 6.2] there exists a 4-dimensional subform $f'$ of $f$ that has Witt index 1 over $F_v$ for every $v \in S'_G$ and Witt index at least 2 over $F_v$ for every $v \in S''_G$. It remains to show that we can choose $f'$ such that $\mathrm{disc}(f') \neq 1$. Assume that $\mathrm{disc}(f') = 1$ and let $f$ have diagonalization $\langle a_1, \ldots, a_n \rangle$ chosen so that $f' = \langle a_1, \ldots, a_4 \rangle$. Let $\alpha = a_1 \cdot a_2 \cdot a_3$ and note that $\mathrm{disc}(f') = 1$ implies that $\alpha \equiv a_4 \bmod F^{\times^2}$. Using the weak approximation property and arguing as in [PrR, Lemma 6.2], we can choose $a'_4$ such that $\langle a_4, a_5 \rangle$ represents $a'_4$, $a'_4 \not\equiv \alpha \bmod F^{\times^2}$, and $\langle a_1, a_2, a_3, a'_4 \rangle$ has the same Witt index as $f'$ over $F_v$ for all $v \in S_G$. Replacing $f'$ by $\langle a_1, a_2, a_3, a'_4 \rangle$ allows us to assume that $\mathrm{disc}(f') \not\equiv 1$. Let $H = \mathrm{SO}(f') \leq \mathrm{SO}(f)$; then $H$ has appropriate real rank and $H$ is $F$-simple because $\mathrm{disc}(f') \not\equiv 1$ [BOI, Thm. 15.7]. □

### *2.2. Type $C_n$*

For this section, $D$ is a nonsplit quaternion algebra over $F$, $\tau$ is a canonical involution on $D$, $f$ is a $\tau$-hermitian form on $D^n$, and $G = \mathrm{SU}_n(D, f, \tau)$. Thus $f = \sum_{i=1}^{n} x_i^\tau a_i y_i$, where $a_i \in D^\tau = F$. If $n = 2$ then $G$ has type $C_2 = B_2$, which was covered in the last section, so assume that $n \geq 3$. After normalizing, we can choose $a_1 = 1$. For each $v \in S_G$ such that $D \otimes_F F_v = D_v$ is nonsplit we have that at least one of $0 > a_i \in F_v$. Using the weak approximation property and a continuity argument, assume that $a_2$ has been chosen such that $0 > a_2 \in F_v$ for all $v \in S_G$. If $n = 3$, then $H = \mathrm{SU}_2(D, \langle 1, a_2 \rangle, \tau) \leq G$ has appropriate real rank. Therefore we can assume that $n \geq 4$.

If $n > 4$ then, applying the same reasoning as in [PrR, Lemma 6.2,], we can find a 4-dimensional subform of $f$ (say $f'$) such that $\mathrm{SU}_4(D, f', \tau)$ has rank 1 over $F_v$ for every $v \in S'_G$ and rank 2 over $F_v$ for every $v \in S''_G$ and such that $D_v$ is nonsplit. Then $H = \mathrm{SU}_4(D, f', \tau)$ is absolutely simple and of appropriate real rank.

If $n = 4$ then consider $H = \mathrm{SO}(f) \leq G$. By multiplying $a_1$ by elements of $\mathrm{Nrd}(D^\times)$ if necessary, we may assume that $\mathrm{disc}(f) \neq 1$ and so $H$ is $F$-simple. By applying the weak approximation property to $a_i$ we may also assume that $H$ has appropriate real rank.

### 2.3. Type $D_n$

Because the case of orthogonal groups has already been treated, we may assume that $G \simeq \mathrm{SU}_n(D, f, \tau)$ for $D$ a nonsplit quaternion algebra over $F$, $\tau$ the canonical involution on $D$, and $f$ a $\tau$-skew-hermitian form on $D^n$. If $n = 2$ then $G$ is of type $D_2 \simeq A_1 \times A_1$, which will be covered in a later section, so assume that $n \geq 3$.

Before I handle this case, I recall some basic facts about skew-hermitian forms. The following is a special case of Morita equivalence. It is actually a collection of results that can be best summarized in the following lemma.

LEMMA 2.1 [S, pp. 361–362]. *Given a skew-hermitian $h$ on $D^n$ as before, if $F \subset K$ is a field extension splitting $D$ then $h \otimes 1\colon (D \otimes_F K)^n \to (D \otimes_F K)$ corresponds to a unique bilinear form $b_h$ on $K^{2n}$, up to isometry, and* $\mathrm{disc}(b_h) = \mathrm{disc}(h)$. *Also, $h$ is isotropic over $K$ if and only if $b_h$ has Witt index $\geq 2$. This correspondence respects direct sums (i.e., $b_{h \oplus h'} = b_h \oplus b_{h'}$) and, on 1-dimensional forms $\langle d \rangle$, if we choose an isomorphism $D \otimes_F K \simeq \mathrm{M}_2(K)$ and if, under this isomorphism, $d$ corresponds to*

$$\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix},$$

*then there exists a basis of $K^2$ such that $b_{\langle d \rangle}$ has matrix*

$$\begin{pmatrix} \gamma & -\alpha \\ -\alpha & -\beta \end{pmatrix}.$$

We separate the examination of groups of type $D_n$ into three cases: $n = 3$, $n = 4$, and $n \geq 5$.

PROPOSITION 2.2. *With notation as before, if $n = 3$ then we can choose a diagonalization of $f = \langle c_1, c_2, c_3 \rangle$ such that $\mathrm{SU}_2(D, \langle c_1, c_2 \rangle, \tau) \leq G$ has appropriate real rank and* $\mathrm{disc}(\langle c_1, c_2 \rangle) \not\equiv 1 \bmod F^{\times 2}$.

*Proof.* For every $v \in V^F_{\infty,\mathbb{R}}$ such that $D_v$ is nonsplit, [S, Thm. 3.7] gives that any two 2-dimensional skew hermitian forms over $D_v$ are isometric; hence we ignore those valuations. Let $\{v_1, \dots, v_m\}$ be the elements of $S'_G$ for which $D_{v_i}$ is split, and notice that $D_v$ is split for every $v \in S''_G$ (by the same theorem). Let $S''_G = \{v_{m+1}, \dots, v_\ell\}$.

Let $f_{v_i} = f \otimes 1\colon D^n_{v_i} \to D_{v_i}$. The fact that $G_{F_{v_i}}$ is isotropic gives that $f_{v_i}$ represents some $c_{v_i}$ such that the 1-dimensional skew-hermitian form $\langle c_{v_i} \rangle$ corresponds to a hyperbolic plane under Morita equivalence. Using weak approximation and the continuity of Morita equivalence, we see that there exists a $c_1 \in D$ such that $f$ represents $c_1$ and $\langle c_1 \rangle_{v_i}$ corresponds to $\langle 1, -1 \rangle$ under Morita equivalence for all $v_i$. Choose $d_2, d_3$ so that $f = \langle c_1, d_2, d_3 \rangle$. Repeating the same arguments for $\langle d_2, d_3 \rangle$ yields $c_2$ such that $\langle d_2, d_3 \rangle$ represents $c_2$ and $\langle c_2 \rangle_{v_i}$ corresponds to an isotropic form over $F^2_{v_i}$ for all $v_i \in S''_G$. Choose $c_3$ such that $f = \langle c_1, c_2, c_3 \rangle$.

If $G$ is of type ${}^1D_3$ and if $\operatorname{disc}(\langle c_1, c_2\rangle) = 1$, then $c_1^2 c_2^2 c_3^2 \equiv c_3^2 \equiv 1 \bmod F^{\times^2}$. This contradicts the assumption that $D$ is a division algebra over $F$. Let $G$ be of type ${}^2D_3$ and assume that $c_1^2 c_2^2 \equiv 1 \bmod F^{\times^2}$. I claim that $\langle c_2, c_3\rangle$ then represents some $d \in D$ such that $\langle d\rangle_{v_i} \simeq \langle c_2\rangle_{v_i}$ for all $v_i$ and there exists some place $v_0$ such that $d^2 \not\equiv c_1^2 \bmod F^{\times^2}$. If this is true, then replacing $c_2$ by $d$ completes the proof.

It suffices to show that there exists some $p$-adic place $v_0$ on $F$ such that $\langle c_2, c_3\rangle_{v_0}$ represents $d_{v_0} \in D_{v_0}$ with $d_{v_0}^2 \not\equiv c_1^2 \bmod F_{v_0}^{\times^2}$. Indeed, once this is shown we can replace $c_2$ by some $d$ with $\operatorname{disc}(\langle c_1, d\rangle) \neq 1$ without changing the behavior over $F_v$ for all $v \in S_G$ by weak approximation. Choose any $p$-adic ($p \neq 2$) place $v_0$ such that $D_{v_0}$ is split. Suppose that $b_{\langle c_2, c_3\rangle_{v_0}} = \langle \beta_1, \beta_2, \beta_3, \beta_4\rangle$. We then have that $\langle \beta_1, \beta_2, \beta_3, \beta_4, -1\rangle \simeq \langle 1, -1\rangle \oplus \langle r, s, t\rangle$ because any 5-dimensional quadratic form over a $p$-adic field is isotropic. From [La] we have that $\langle r, s, t\rangle$ represents at least three square classes in $F_{v_0}^\times / F_{v_0}^{\times^2}$; thus we can choose $y \in F_{v_0}^\times$ such that $\langle r, s, t\rangle$ represents $-y$ and $y \not\equiv c_1^2 \bmod F_{v_0}^{\times^2}$. Then $\langle \beta_1, \beta_2, \beta_3, \beta_4\rangle \oplus \langle -1, y\rangle$ has Witt index at least 2 and thus, by Lemma 2.1, $h_{v_0}$ represents some $d_{v_0}$ such that $\langle d_{v_0}\rangle$ corresponds to $\langle 1, -y\rangle$ under Morita equivalence. Then $d_{v_0}^2 \equiv y \not\equiv c_1^2 \bmod F_{v_0}^{\times^2}$, as required. □

The restriction that $\operatorname{disc}(\langle c_1, c_2\rangle) \not\equiv 1 \bmod F^{\times^2}$ implies that $H = \mathrm{SU}_2(D, \langle c_1, c_2\rangle, \tau)$ is $F$-simple. Since $H$ has appropriate real rank by construction, it follows that $G$ is not minimal when $n = 3$.

Assume now that $n \geq 5$. I claim that there exists a diagonalization $\langle d_1, \dots, d_n\rangle$ of $f$ such that, if $h = \langle d_1, \dots, d_4\rangle$, then $H = \mathrm{SU}_4(D, h, \tau) \leq G$ is of appropriate real rank. Arguing as in the proof of Proposition 2.2, we can choose a diagonalization $\langle d_1, d_2, \dots, d_n\rangle$ of $f$ such that $b_{\langle d_1, d_2\rangle_v}$ has appropriate Witt index over $F_v$ for every $v \in S_G$ such that $D_v$ is split. Then, by [S, Thm. 3.7], the subgroup $H = \mathrm{SU}_4(D, \langle d_1, d_2, d_3, d_4\rangle, \tau)$ has rank 2 over $F_v$ for every $v \in V_{\infty,\mathbb{R}}^F$ such that $D_v$ is nonsplit, so $H$ has appropriate real rank by the choice of $d_1, d_2$.

Finally, we must consider the case $n = 4$. In both the inner and outer cases, I require the following lemma due to Chernousov and Merkurjev (see the Appendix).

LEMMA 2.2. *If $K$ is a maximal subfield of $D$ and if $f$ is a skew-hermitian form such that $b_f$ is isotropic over $K$, then there exists a $v \in D^n$ such that $F(f(v)) \simeq K$.*

2.3.1. *Type ${}^2D_4$*

PROPOSITION 2.3. *Up to isogeny, we have that $G$ contains a subgroup of the form $R_{F(\sqrt{a})/F}(\mathrm{SO}_4(f'))$ for some $a \not\equiv 1 \bmod F^{\times^2}$ that is of appropriate real rank.*

The proof is broken into a series of lemmas as follows.

LEMMA 2.3. *There exists an $\alpha \in F$ such that*:

(1) $\mathrm{Sign}_v(\alpha) = -1$ *for all $v \in V_{\infty,\mathbb{R}}^F$*;
(2) $-\alpha \notin F^{\times^2}$; *and*
(3) *$G$ is quasi-split over $F(\sqrt{\alpha})$.*

*Proof.* Let $K = F(\sqrt{c})$ be the unique quadratic extension of $F$ such that $G$ becomes of type ${}^1D_4$ over $K$, $f_q = \langle 1, -\sqrt{c}, 1, 1, 1, -1, -1, -1\rangle$, and consider the exact sequence

$$H^1(F, \mathrm{Spin}(f_q)) \to H^1(F, \mathrm{PSO}(f_q)) \to H^2(F, Z(\mathrm{Spin}(f_q))).$$

We have that $Z(\mathrm{Spin}(f_q)) = R_{K/F}(\mu_2)$ and so $H^2(F, Z(\mathrm{Spin}(f_q))) \simeq {}_2\mathrm{Br}(K)$ by Shapiro's lemma. Suppose that $G$ corresponds to $[\xi] \in H^1(F, \mathrm{PSO}(f_q))$ and that $[\xi] \mapsto [T] \in \mathrm{Br}(K)$. Let $s_1, \dots, s_m$ be the elements of $V^K_{\infty,\mathbb{R}}$ such that $\mathrm{Res}_{K_{s_i}/K}([T]) \neq 1$, and let $t_1, \dots, t_\ell$ be the set of non-Archimedean places of $K$ such that $\mathrm{Res}_{K_{t_i}/K}([T]) \neq 1$. For every $v \in V^F_{\infty,\mathbb{R}}$, choose $0 > \alpha_v \in F_v$. Then we have that $[T]$ splits over $K_{s_i}(\sqrt{\alpha_{v_i}})$, where $v_i$ is the restriction of $s_i$ to $F$. For the non-Archimedean valuations $t_i$, I claim that there exist $\alpha_{w_i} \in F_{w_i}$ (where $w_i$ is the restriction of $t_i$ to $F$) such that the image of $\alpha_{w_i}$ in $K_{t_i}$ is nonsquare. If $F_{w_i} = K_{t_i}$ then this is trivial. If $F_{w_i} \neq K_{t_i}$, then $F^\times_{w_i}/F^{\times 2}_{w_i} \to K^\times_{t_i}/K^{\times 2}_{t_i}$ has kernel of order 2 and $|F^\times_{w_i}/F^{\times 2}_{w_i}| = 4$, so $\alpha_{w_i}$ exists. Once we have made such a choice of $\alpha_{w_i}$, the fact that any nonsplit quaternion algebra over a $p$-adic field splits over any proper quadratic extension gives that $\mathrm{Res}_{K_{t_i}/K}([T])$ splits over $K_{t_i}(\sqrt{\alpha_{w_i}})$. Finally, choose some non-Archimedean valuation $r$ on $K$ such that $\mathrm{Res}_{K_r/K}([T])$ is split, and let $\alpha_r \in F_r$ be such that $-\alpha_r \notin F^{\times 2}_r$.

Applying the weak approximation property, we find $\alpha \in F$ such that $|\alpha_x - \alpha|_x < \varepsilon_x$ for all valuations $x$ on $F$ described previously, where $\varepsilon_x$ is chosen such that $\alpha < 0$ is in $F_v$ for all $v$ Archimedean, $\alpha$ is not square in $K_{t_i}$ for any $t_i$, and $-\alpha$ is not square in $F^\times_r$. Let $L = F(\sqrt{\alpha})$.

It remains to show that $G$ is quasi-split over $L$. For $s_i$, $\mathrm{Res}_{K_{s_i}/K}([T])$ splits over $L \cdot K_{s_i}$ because $L \cdot K_{s_i} \simeq \mathbb{C}$. For $t_i$, because $L \cdot K_{t_i}$ is a quadratic field extension of $K_{t_i}$, $\mathrm{Res}_{K_{t_i}/K}([T])$ splits over $L \cdot K_{t_i}$ [PR, Thm. 1.7]. Since $\{s_i, t_i\}$ was the collection of all valuations on $K$ such that $\mathrm{Res}_{K_{x_i}/K}([T]) \neq 1$, the Hasse principle yields that $[T]$ splits over $L \cdot K$. This means that $[\xi]_L$ lies in the image of $H^1(L, \mathrm{Spin}(f_q))$; but $V^L_{\infty,\mathbb{R}} = \emptyset$ and so, by Kneser's theorem, $H^1(L, \mathrm{Spin}(f_q)) = \{1\}$. Thus $[\xi]_L$ is trivial. □

Note that, because $G_L$ is quasi-split, $\mathrm{Res}_{L/F}([D])$ is trivial and so $L$ is a maximal subfield of $D$. Choose an embedding $L \hookrightarrow D$ and let $i$ be the image of $\sqrt{\alpha}$ under this embedding. Applying Lemma 2.2, we see that $h$ has a diagonalization $\langle \beta_1 i_1, \beta_2 i_2, \beta_3 i_3, d\rangle$ for some $d \in D^0$ and $i_j \in D^0$ such that $F(i_j) \simeq F(i) \subset D$ for each $j$. By the Skolem–Noether theorem [BOI, Thm. 1.4] we have that each of the $i_j$ are conjugate to $i$, say $d_j^{-1} i_j d_j = i$. If $h(v_j) = i_j$ then $h(v_j \cdot d_j) = \mathrm{Nrd}(d_j) \cdot i$; hence replacing $v_j$ with $v_j \cdot d_j$ gives that $h$ has diagonalization $\langle \beta_1 i, \beta_2 i, \beta_3 i, d\rangle$, where $d \in D^0$. Note that the subspaces

$$V_1 = \{d' \in D^0 \mid id' = -d'i\} \quad \text{and} \quad V_2 = \{d' \in D^0 \mid dd' = -d'd\}$$

both have dimension at least 2 and $D^0$ has dimension 3, so $\{0\} \neq V_1 \cap V_2 \subset D^0$. Choose $0 \neq d' \in D^0$ such that $id' = -d'i$ and $dd' = -d'd$, so that $i^{-1}d$ commutes with $d'$ and thus $i^{-1}d \in F(d')$.

LEMMA 2.4. *At least one of the groups*

$$R_{F(d')/F}(\mathrm{SO}(\langle\beta_1,\beta_2,\beta_3,i^{-1}d\rangle)) \quad \textit{and} \quad R_{F(d')/F}(\mathrm{SO}(\langle-\alpha\beta_1,\beta_2,\beta_3,i^{-1}d\rangle))$$

*is $F$-simple.*

*Proof.* It suffices to prove that $\mathrm{SO}(\langle\beta_1,\beta_2,\beta_3,i^{-1}d\rangle)$ or $\mathrm{SO}(\langle-\alpha\beta_1,\beta_2,\beta_3,i^{-1}d\rangle)$ is $F(d')$-simple. Assume that $\beta_1\cdot\beta_2\cdot\beta_3\cdot i^{-1}d \equiv 1 \bmod F(d')^{\times 2}$. Then $\beta_1\cdot\beta_2\cdot\beta_3\cdot i^{-1}d \equiv -\alpha \bmod F(d')^{\times 2}$. By Lemma 2.3(2) we have that $-\alpha \notin F^{\times 2}$, which yields $-\alpha \equiv (d')^2 \bmod F^{\times 2}$. By the assumption that $d'$ is purely imaginary and $d'i = -id'$, we have that $i, d'$ is a quaternion basis for $D$. Thus the norm form of $D$ is given by $\langle 1,-\alpha,\alpha,\alpha^2\rangle$; but then $D$ is split over $F$, a contradiction. □

Note that both groups sit inside $G$, since $\langle\beta_1 i,\beta_2 i,\beta_3 i,d\rangle$ and $\langle-\alpha\beta_1 i,\beta_2 i,\beta_3 i,d\rangle$ are both diagonalizations of $h$. Let $H \le G$ be $R_{F(d')/F}(\mathrm{SO}(\langle\beta_1,\beta_2,\beta_3,i^{-1}d\rangle))$ if $\beta_1\cdot\beta_2\cdot\beta_3 i^{-1}d \not\equiv 1 \bmod F(d')^{\times 2}$ and $R_{F(d')/F}(\mathrm{SO}(\langle-\alpha\beta_1,\beta_2,\beta_3,i^{-1}d\rangle))$ if $\beta_1\cdot\beta_2\cdot\beta_3 i^{-1}d \equiv 1 \bmod F(d')^{\times 2}$.

LEMMA 2.5. *The subgroup $H$ has appropriate real rank.*

First, I need the following.

LEMMA 2.6. *Suppose we are given $H = \mathrm{SO}_4(f_1)\times\mathrm{SO}_4(f_2) \le \mathrm{SO}_8(f)$. Then $f \simeq \langle c_1\rangle\cdot f_1 \oplus \langle c_2\rangle\cdot f_2$.*

*Proof.* Because $H$ is standard of type $A_1^4$ in $G$ of type $D_4$, we have that, over $\bar F$, $H$ is conjugate to $\mathrm{SO}(f|_{V_1})\times\mathrm{SO}(f|_{V_2})$ for $V_1 \perp V_2$ such that $V_1\oplus V_2 = V$ (say $gHg^{-1} = \mathrm{SO}(f|_{V_1})\times\mathrm{SO}(f|_{V_2})$). This means that, if we let $W_1 = \{v \in V \mid g_2 v = v\ \forall g_2 \in \mathrm{SO}(f_2)\}$ and $W_2 = \{v \in V \mid g_1 v = v\ \forall g_1 \in \mathrm{SO}(f_1)\}$, then over $\bar F$ we have $g(W_i\otimes\bar F) = V_i\otimes\bar F$ and hence $W_1\cap W_2 = \{0\}$ and $W_1 \perp W_2$. Now $\mathrm{SO}(f_i) \le \mathrm{SO}(f|_{V_i})$, each connected of equal dimension, gives that $\mathrm{SO}(f_i) = \mathrm{SO}(f|_{V_i})$. It is well known that this yields $f_i = \langle c\rangle\cdot f|_{V_i}$, which completes the proof. □

Consider a $v \in V^F_{\infty,\mathbb{R}}$ such that $D\otimes F_v = D_v$ is split. By Lemma 2.1 we then have that

$$G_{F_v} \simeq \mathrm{SO}(\langle\beta_1,\beta_1,\beta_2,\beta_2,\beta_3,\beta_3\rangle \oplus \beta_4\langle 1,-d^2\rangle).$$

Because $i, d'$ form a quaternion basis for $D$ and we chose $i$ such that $i^2$ is negative in every $F_v$ for $v \in V^F_{\infty,\mathbb{R}}$, we have that $F(d')$ splits over $F_v$ and

$$H_{F_v} \simeq \mathrm{SO}_4(\langle\beta_1,\beta_2,\beta_3,i^{-1}d\rangle)\times\mathrm{SO}_4(\langle\beta_1,\beta_2,\beta_3,\overline{i^{-1}d}\rangle),$$

where $\bar{\cdot}$ represents conjugation in $F(d')$.

*Proof of Lemma 2.5.* Let $D = (\alpha,\gamma)$, and note first that $(d')^2 = \gamma\cdot N_{F(\sqrt{\alpha})/F}(x)$ for some $x$; hence $(d')^2 < 0$ is in $F_v$ if and only if $D_v$ is nonsplit. We break the valuations $v \in S_G$ into four cases as follows.

*Case 1: $D_v$ is nonsplit.* Then $F(d') \otimes_F F_v$ is a subfield of $\mathbb{H} = (-1,-1)_{F_v}$; thus $F(d') \otimes_F F_v \simeq \mathbb{C}$ and $H_{F_v} \simeq R_{\mathbb{C}/\mathbb{R}}(\mathrm{SL}_2 \times \mathrm{SL}_2)$ has $F_v$-rank 2.

*Case 2: $v \in S'_G$.* In this case $D_v$ is split, $\beta_i$ all have the same sign, and $d^2 > 0$ is in $F_v$. Applying Lemma 2.6 and Witt cancellation then gives that $\langle 1, -d^2\rangle \simeq \langle 1, -1\rangle \simeq \langle i^{-1}d, \overline{i^{-1}d}\rangle$. Thus one of $i^{-1}d, \overline{i^{-1}d}$ is positive in $F_v$ and the other negative, so $\mathrm{rank}_{F_v}(H) = 1$.

*Case 3:* $\mathrm{rank}_{F_v}(G) \geq 3$ *and $D_v$ is split.* In this case, two of $\beta_1, \beta_2, \beta_3$ have different signs in $F_v$ and so $\mathrm{rank}_{F_v}(H) \geq 2$.

*Case 4:* $\mathrm{rank}_{F_v}(G) = 2$ *and $D_v$ is split.* Because $\mathrm{disc}(\langle \beta_1, \beta_1, \beta_2, \beta_2, \beta_3, \beta_3\rangle \oplus \beta_4\langle 1, -d^2\rangle) = -d^2$ and $\mathrm{disc}(\langle 1,1,1,1,1,1,-1,-1\rangle) = 1$ is in $F_v^\times/F_v^{\times 2}$, we have that $d^2 \equiv -1 \bmod F_v^{\times 2}$ in this case. If two of $\beta_1, \beta_2, \beta_3$ have different signs then $\mathrm{rank}_{F_v}(H) \geq 2$, so assume that $\beta_1, \beta_2, \beta_3$ are all positive in $F_v$ (the case where $\beta_1, \beta_2, \beta_3$ are all negative is handled analogously). In this case, Lemma 2.6 gives

$$\langle 1,1,1,1,1,1,\beta_4,\beta_4\rangle \simeq c_1\langle 1,1,1,i^{-1}d\rangle \oplus c_2\langle 1,1,1,\overline{i^{-1}d}\rangle.$$

By inspection, the only possibility is that $c_1 = c_2 = 1$ and $\langle -1,-1\rangle \simeq \langle \beta_4, \beta_4\rangle \simeq \langle i^{-1}d, \overline{i^{-1}d}\rangle$ by Witt cancellation. Then $H_{F_v} \simeq \mathrm{SO}(\langle 1,1,1,-1\rangle) \times \mathrm{SO}(\langle 1,1,1,-1\rangle)$ has $F_v$-rank 2. □

2.3.2. *Type* $^1D_4$

The proof that $G$ is not minimal in this case is completely analogous to the case that $G$ is of type $^2D_4$.

## 2.4. *Type* $^{1,2}A_n$

2.4.1. *Type* $^1A_n$

All groups of this form are isogenous to $\mathrm{SL}_m(D)$ for some $m$ and central division algebra $D$ over $F$. By the restriction that $G$ is anisotropic, $m = 1$.

PROPOSITION 2.4. *The group $G$ is minimal if and only if* $\deg(D) = p$ *for $p$ prime, $p \geq 3$.*

*Proof.* Assume that $\deg(D) = d$ is not prime. Let $d = p_1^{n_1} \cdots p_m^{n_m}$ be the degree of $D$, where $p_i$ are listed in increasing order. Using a construction analogous to the one on page 127 in the proof of [PrR, Thm. 4.1], we can find a subgroup $H \leq G$ of the form $R_{K_0/F}(\mathrm{SL}_1(T))$ for a field extension $K_0$ of $F$ of degree $p_1$ and a central simple algebra of degree $p_1^{n_1-1} \cdots p_m^{n_m}$ over $K_0$. We immediately have that $H$ is $F$-simple, and after checking the small number of possibilities for $T \otimes K_{w_i}$ for $w_i$ lying over $v_i \in S_G$ we see that $H$ is automatically of appropriate real rank unless $d = 4$. In this case we have that $G$ is of type $A_3 = D_3$, which was handled previously.

If $\deg(D)$ is prime, then $G$ contains no semisimple subgroups [GGi, Prop. 4.1]. This means that $\mathrm{SL}_1(D)$ is minimal for any central division algebra $D$ of prime degree $p \geq 3$. □

2.4.2. *Type* $^2A_n$
First we handle those groups of type $^2A_n$ that are minimal.

PROPOSITION 2.5. *If $G$ is of type $^2A_{p-1}$ for any prime $p \geq 3$ and if $G$ corresponds to a division algebra of degree $p$, then $G$ is minimal.*

*Proof.* Let $L$ be the unique quadratic extension of $F$ over which $G$ becomes inner type. Then $G_L$ contains no semisimple subgroups [GGi, Prop. 4.1]; thus $G$ contains no semisimple subgroups and therefore $G$ is minimal. □

I claim that these are all of the possible minimal groups of type $^2A_n$ for $n \neq 2$.

LEMMA 2.7. *If $G \simeq \mathrm{SU}_m(L, f)$ for a hermitian form $f$ over $L$, then $G$ is minimal if and only if $m = 3$ and $L \otimes F_v \simeq F_v \times F_v$ for some $v \in V^F_{\infty,\mathbb{R}}$.*

NOTE. By the assumption that $S''_G \neq \emptyset$, we have $m \geq 3$.

*Proof of Lemma 2.7.* After normalizing, we may assume that $f = \langle 1, a_2, \ldots, a_m \rangle$. If $m \geq 5$, I claim that we can choose a diagonalization of $f$ such that $\langle 1, a_2, a_3, a_4 \rangle$ corresponds to a subgroup of $G$ that has appropriate real rank. To see this, we use the same arguments as in the skew-hermitian case—namely that, for any completions $F_v$ such that $L \otimes F_v \simeq \mathbb{C}$, the form $f_{F_v}$ is isotropic and so represents any $a \in F_v$. Hence we may use the weak approximation property to replace $a_2, a_3, a_4$ if necessary so that:

- $a_2 < 0$ in $F_v$ for all $v \in S'_G$;
- $a_3 > 0$ and $a_4 < 0$ in $F_v$ for all $v \in S''_G$ such that $L \otimes F_v \simeq \mathbb{C}$.

After this replacement, we have that $\mathrm{SU}_4(L, \langle 1, a_2, a_3, a_4 \rangle)$ is a simple, proper subgroup of $G$ that has appropriate real rank over every $F_v$; hence $G$ is not minimal. If $G \simeq \mathrm{SU}_4(L, f)$, then $G$ has type $^2A_3 = {}^2D_3$ and so $G$ is isomorphic to a group handled in the skew-symmetric section.

Finally, assume $m = 3$. Recall that any subgroup of appropriate real rank must have absolute rank at least 2 (since $S''_G \neq \emptyset$). Assume that $G$ contains a proper simple subgroup $H$ of appropriate real rank. We would then have that $H$ is standard, because the absolute rank of $G$ is equal to that of $H$ and so the root system of $H$ corresponds to a subroot system of $A_2$. Because all the roots of $G$ have equal length, the only possibility is that $H$ is of type $A_1 \times A_1$; but $A_2$ does not contain two orthogonal roots, a contradiction. □

PROPOSITION 2.6. *If $(A, \tau)$ is a central simple algebra with involution $\tau$ of the second type over a quadratic extension $L/F$ such that $L^\tau = F$ and if $\deg(A) = n \geq 5$ is not prime, then $\mathrm{SU}(A, \tau)$ is not minimal.*

*Proof.* Let $\{v_1, \ldots, v_\ell\} = V^F_{\infty,\mathbb{R}}$ and let $w_1, \ldots, w_t$ be the non-Archimedean valuations on $F$ such that $G$ is not split or quasi-split over $F_{w_i}$. The first step will be to construct towers of algebras $J_{x_i} \subset K_{x_i}$ for maximal commutative étale $F_{x_i}$-subalgebras $K_{x_i}$ of $(A \otimes_F F_{x_i})^{\tau \otimes 1}$ that are linearly disjoint from $L_{x_i}$ for $x_i = w_i$ or $v_i$. We consider two cases.

*Case I: $n = 2m$ is even.* For the Archimedean valuations, $A \otimes_F F_{v_i}$ is isomorphic to either $M_n(\mathbb{C})$, $M_n(\mathbb{R}) \times M_n(\mathbb{R})^{op}$ or $M_m(\mathbb{H}) \times M_m(\mathbb{H})^{op}$.

- If $A \otimes_F F_{v_i} \simeq M_{2m}(\mathbb{R}) \times M_{2m}(\mathbb{R})$ with exchange involution, let $J_{v_i} = \mathbb{R}^2 \subset \mathbb{R}^{2m} = K_{v_i}$. Let

$$F_{v_i} \hookrightarrow M_{2m}(\mathbb{R}) \overset{\Delta}{\hookrightarrow} M_{2m}(\mathbb{R}) \times M_{2m}(\mathbb{R}),$$

  let $K_{v_i}$ embed as diagonal matrices in $M_{2m}(\mathbb{R})$, and compose this embedding with the diagonal embedding of $M_{2m}(\mathbb{R})$ in $A \otimes_F F_v$. If $e_1$ is the matrix consisting of 1s along each first $m$ diagonal entries in each component and 0s elsewhere and if $e_2 = (I_{2m\times 2m}, I_{2m\times 2m}) - e_1$, then $J_{v_i}$ embeds in $K_{v_i}$ via $\mathbb{R} \cdot e_1 + \mathbb{R} \cdot e_2$.
- If $A \otimes_F F_{v_i} \simeq M_{2m}(\mathbb{C})$ with involution $\tau(X) = f\bar{X}^T f$, which corresponds to the hermitian form $r \cdot \langle 1, -1\rangle \oplus (2m - 2r)\langle 1\rangle$, then let $K_{v_i} = \mathbb{R}^{2m}$ embed in $A \otimes_F F_{v_i}^{\tau\otimes 1}$ via diagonal matrices. Let $e_1$ be the diagonal matrix with first $m$ entries equal to 1 and last $m$ entries equal to 0, and let $e_2 = I_{2m\times 2m} - e_1$. Then $J_{v_i} = \mathbb{R}^2$ embeds in $K_{v_i}$ via $\mathbb{R}e_1 + \mathbb{R}e_2$.
- If $A \otimes_F F_{v_i} \simeq M_m(\mathbb{H}) \times M_m(\mathbb{H})^{op}$, then let $K_{v_i} = \mathbb{C}^m$ embed in $A \otimes_F F_{v_i}$ as diagonal matrices in each component and let $J_{v_i} = \mathbb{C}$ embed in $K_{v_i}$ as scalar matrices in each component.
- If $L \otimes_F F_{w_i} = L_{w_i}$ is a field then by [T] we have that $G_{F_{w_i}} \simeq \mathrm{SU}_{2m}(L_{w_i}, f)$, where $f$ is the sum of $m-1$ hyperbolic hermitian forms and one anisotropic form $\langle \alpha, \beta\rangle$. By rank considerations, $\mathrm{SU}_2(L_{w_i}, \langle -1, 1\rangle) \simeq \mathrm{SL}_2$ and $\mathrm{SU}_2(\langle \alpha, \beta\rangle) \simeq \mathrm{SL}(Q)$ for some nonsplit quaternion algebra $Q$ over $F_{w_i}$. Choose any quadratic extension $J_{w_i}$ of $F_{w_i}$ disjoint from $L_{w_i}$. By [La, Rem. 2.7] we have that $Q$ is split over $J_{w_i}$; thus we can embed $R^{(1)}_{J_{w_i}/F_{w_i}}(\mathrm{G_m})$ in $\mathrm{SL}(Q)$ and $\mathrm{SU}_2(L_{w_i}, \langle -1, 1\rangle)$. This is equivalent to finding embeddings of $J_{w_i} \cdot L_{w_i}$ in $M_2(L_{w_i})$ such that the involutions corresponding to $\langle 1, -1\rangle$ and $\langle \alpha, \beta\rangle$ fix $J_{w_i}$. Use the diagonal product of these embeddings to construct an embedding $L_{w_i} \cdot J_{w_i} \hookrightarrow M_{2m}(L_{w_i})$ such that $(L_{w_i} \cdot J_{w_i})^{\tau\otimes 1} = J_{w_i}$.

  The double centralizer theorem gives that $C := C_{A\otimes F_{w_i}}(L_{w_i} \cdot J_{w_i})$ is a central simple algebra over $L_{w_i} \cdot J_{w_i}$ of degree $m$. The fact that $\tau \otimes 1$ fixes $J_{w_i}$ means that $\tau \otimes 1|_C$ is an involution of the second kind on $C$ fixing $J_{w_i}$. Consider an arbitrary subfield $E_{w_i}$ of $C$ such that $[L_{w_i} \cdot J_{w_i} : E_{w_i}] = m$; then $K_{w_i} = E_{w_i}^{\tau\otimes 1|_C}$ is a degree-$m$ extension of $J_{w_i}$ disjoint from $L_{w_i}$.
- If $L \otimes_F F_{w_i} \simeq F_{w_i} \times F_{w_i}$, then $A \otimes_F F_{w_i} \simeq A'_{w_i} \times A'^{op}_{w_i}$ with the exchange involution, so we can choose a maximal subfield $K_{w_i}$ of $A'_{w_i}$ and let $J_{w_i} \subset K_{w_i}$ be such that $[K_{w_i} : J_{w_i}] = m$. Then $E_{w_i} = K_{w_i} \times K_{w_i}^{op} \simeq K_{w_i}^2 \subset A_{w_i}$ and $E_{w_i}^{\tau_{w_i}} = K_{w_i}$.
- Finally, if $L \otimes_F F_{w_i} \simeq F_{w_i} \times F_{w_i}$ for all $i$ and $L \otimes_F F_{v_j} \simeq F_{v_j} \times F_{v_j}$ for all $j$, choose a (non-Archimedean) valuation $s$ on $F$ such that $L \otimes_F F_s = L_s$ is a field. Choose an arbitrary subfield $E_s \subset A \otimes_F F_s$ such that $\dim_{F_s}(E_s^{\tau_s}) = 2m$ and $E_s \simeq E_s^{\tau_s} \otimes_{F_s} L_s$, and let $K_s = E_s^{\tau_s}$ for $J_s \subset K_s$ an arbitrary subfield with $[K_s : J_s] = m$.

*Case II: n is odd.* In this case, let $p$ be the smallest prime dividing $n$ and let $n = mp$. For the Archimedean valuations, either $A \otimes_F F_{v_i} \simeq M_n(\mathbb{C})$ or $M_n(\mathbb{R}) \times M_n(\mathbb{R})^{op}$.

- If $A \otimes_F F_{v_i} \simeq M_{pm}(\mathbb{R}) \times M_{pm}(\mathbb{R})$ with exchange involution, let $J_{v_i} \simeq \mathbb{R}^p \subset \mathbb{R}^n = K_{v_i}$. Here $K_{v_i}$ embeds as in the even case, but now let $e_i$ be the matrix with 1s in the $(i-1)m+1$ to $im$ diagonal entries and 0s elsewhere, and let $J_{v_i}$ embed in $K_{v_i}$ via $\sum \mathbb{R}e_i$.
- If $A \otimes_F F_{v_i} \simeq M_{2m}(\mathbb{C})$ with involution $\tau(X) = f\bar{X}^T f$, which corresponds to the hermitian form $r \cdot \langle 1, -1\rangle \oplus (pm-2r)\langle 1\rangle$, then let $K_{v_i} = \mathbb{R}^n$ embed in $A \otimes_F F_{v_i}^{\tau\otimes 1}$ via diagonal matrices. Let $e_i$ be the matrix with 1s in the $(i-1)m+1$ to $im$ diagonal entries and 0s elsewhere. Then $J_{v_i} = \mathbb{R}^p$ embeds in $K_{v_i}$ via $\sum \mathbb{R}e_i$.
- For the non-Archimedean valuations, choose $K_{w_i}$, $J_{w_i}$ (and $K_s$, $J_s$, if necessary) as in the case for $n$ even.

Choose a tower of field extensions $F \subset J \subset K$ having the local behavior just prescribed. For every valuation $x$ for which $G_{F_x}$ is not quasi-split or split, there is a local embedding $K_v \otimes L \hookrightarrow A_v$ that respects involution by construction. Hence there is an embedding

$$(K \otimes_F L, 1 \otimes \gamma) \overset{\iota}{\hookrightarrow} (A, \tau)$$

such that the tower of field extensions $F \subset J \subset K$ has the prescribed local behavior (see the proof of [PrR, Thm. 5.1] and [PrR, Apx. A, pp. 176–178]). I claim that the two algebras are conjugate by an element of $G_{F_v}$ for every $v$ Archimedean. Indeed, since $\iota(K \otimes L) \otimes F_v$ and $E_v$ both correspond to unique maximal tori in $G_{F_v}$, it suffices to show that the corresponding tori are conjugate.

If $A \otimes_F F_v \simeq M_n(\mathbb{C})$, then $\iota(K \otimes L) \otimes F_v$ and $E_v$ both correspond to maximal anisotropic tori in $G_{F_v}$ and hence are conjugate in $G_{F_v}$. If $A \otimes_F F_v \simeq M_n(\mathbb{R}) \times M_n(\mathbb{R})^{op}$, then both $\iota(K \otimes L) \otimes F_v$ and $E_v$ correspond to tori of maximal $F_v$-rank; hence they are also conjugate by an element of $G_{F_v}$. Finally, if $A \otimes_F F_v \simeq M_{n/2}(\mathbb{H}) \times M_{n/2}(\mathbb{H})$, then $\iota(K \otimes L) \otimes F_v$ and $E_v$ both correspond to maximal tori of maximal rank over $F_v$ in $G_{F_v}$; hence they are conjugate as well. By considering eigenvalues with multiplicity, we must have that this conjugation takes $\iota(J \otimes L) \otimes F_v$ to $J_v$.

Let $P = J \otimes_F L$ and consider $H = R_{J/F}(\mathrm{SU}(C_A(P), \tau|_{C_A(P)})) \leq G$. Then $H$ is a proper simple subgroup, and I claim that $H$ has appropriate real rank. To see this, note that if $v \in V^F_{\infty,\mathbb{R}}$ is such that $J \otimes_F F_v \simeq \prod J_v^{(i)}$, where $J_v^{(i)}$ are field extensions of $F_v$, then

$$\begin{aligned} H_{F_v} &\simeq \prod R_{J_v^{(i)}/F_v}(\mathrm{SU}(C_A(P), \tau|))_{J_v^{(i)}} \\ &\simeq \prod R_{J_v^{(i)}/F_v}(\mathrm{SU}(C_A(P) \otimes_J J_v^{(i)}, \tau| \otimes 1)). \end{aligned}$$

First, consider the case that $J \otimes_F F_v \simeq \mathbb{C}$. This implies that $A \otimes_F F_v \simeq M_{n/2}(\mathbb{H}) \times M_{n/2}(\mathbb{H})$, and because $J \otimes_F F_v$ is conjugate to $J_v$, we have that $C_{A\otimes_F F_v}(J_v \otimes L)$ consists of scalar matrices in each component. Thus

$$\mathrm{SU}(C_{A\otimes_F F_v}(J\otimes_F L\otimes F_v), \tau|\otimes 1) \simeq \mathrm{SL}_{n/2}(\mathbb{C})$$

and so $H_{F_v} \simeq R_{\mathbb{C}/\mathbb{R}}(\mathrm{SL}_{n/2}(\mathbb{C}))$ has rank $\frac{n}{2} - 1 \geq 2$, as required.

Next, assume that $J \otimes_F F_v$ is not a field. Then $J \otimes_F F_v \simeq \mathbb{R}^p$ if $n = pm$, where $p$ is the smallest prime dividing $n$ and where, up to conjugation (and possibly renumbering), $J_v^{(i)} = \mathbb{R}e_i$. To calculate $\mathrm{SU}(C_A(P)\otimes_J J_v^{(i)}, \tau|\otimes 1)$, consider the following chain of isomorphisms:

$$\begin{aligned}\bigoplus C_A(P)\otimes_J J_v^{(i)} &\simeq C_A(P)\otimes_J \Big(\prod J_v^{(i)}\Big) \simeq C_A(P)\otimes_J J\otimes_F F_v\\ &\simeq C_A(P)\otimes_F F_v \simeq C_{A\otimes_F F_v}(P\otimes_F F_v)\\ &\simeq C_{A\otimes_F F_v}(J_v) \simeq \prod_i C_{e_i\cdot A\otimes_F F_v e_i}(\mathbb{R}e_i\cdot L_v).\end{aligned}$$

All of the isomorphisms respect components and involutions (because we conjugate by an element of $G_{F_v}$), so $H_{F_v} \simeq \prod \mathrm{SU}(C_{e_i A\otimes_F F_v e_i}(\mathbb{R}e_i\cdot L_v))$.

If $A\otimes_F F_v \simeq M_n(\mathbb{R})\times M_n(\mathbb{R})$ then this means that $H_{F_v} \simeq \prod_{i=1}^p \mathrm{SL}_m(\mathbb{R})$, which has higher rank. If $A\otimes_F F_v \simeq M_n(\mathbb{C})$ and if $\tau\otimes 1$ corresponds to the hermitian form with diagonalization $r\cdot\langle 1,-1\rangle \oplus (pm-2r)\langle 1\rangle$, then $H_{F_v} \simeq \mathrm{SU}_m(\mathbb{C}, f_1)\times\cdots\times \mathrm{SU}_m(\mathbb{C}, f_p)$, where $f = f_1\oplus\cdots\oplus f_p$ and $f_1$ is taken from the first $m$ coefficients of the diagonalization of $f$, $f_2$ from the second, and so on. If $r=1$, then both $G_{F_v}$ and $\mathrm{SU}_m(\mathbb{C}, f_1)$ have rank 1; therefore, $H_{F_v}$ has rank 1. If $r\geq 2$ and $m>3$, then $\mathrm{SU}_m(\mathbb{C}, f_1)$ has rank $\geq 2$ and so $H_{F_v}$ is of higher rank. If $r\geq 2$ and $m=3$, then $\mathrm{SU}_m(\mathbb{C}, f_1)$ has rank 1, as does $\mathrm{SU}_m(\mathbb{C}, f_2)$; hence $H_{F_v}$ is of higher rank as well.

Combining these cases shows that $H$ has appropriate real rank and thus $G$ is not minimal. □

## 3. Exceptional Groups Splitting over Quadratic Extensions

The purpose of this section is to prove that absolutely simple groups of type $E_7$, $E_8$, $F_4$, and $G_2$ are not minimal. Unless otherwise stated, $G$ will be simply connected throughout this section. The approach for these four cases will rely on the following observation.

LEMMA 3.1. *Any group of type $E_7$, $E_8$, or $F_4$ over $F$ becomes split over a purely imaginary quadratic extension $K$.*

This follows from Kneser's theorem, which states that $H^1(F_v, G) = \{0\}$ for $v$ non-Archimedean and $G$ simply connected, and the Hasse principle for simply connected groups.

REMARK 3.1. Note that if $G$ has type $G_2$ then we can choose $K = F(\sqrt{a})$ with $a$ positive in $F_v$ for all $v\in S_G$ such that $G$ splits over $K$. Recall from Tits's classification [T] that in this case $S_G = S''_G$ (i.e., $G$ is split over $F_v$ for all $v\in S_G$). By the weak approximation property, we may choose $a\in F$ such that the image of $a$

in $F_v$ is positive for all $v \in S_G = S''_G$ and the image of $a$ in $F_v$ is negative for all $v \in V^F_{\infty,\mathbb{R}} \backslash S_G$. Let $K = F(\sqrt{a})$; now if $w \in V^K_{\infty,\mathbb{R}}$ lies over $v \in S_G$ then

$$\mathrm{Res}_{K_w/K} \circ \mathrm{Res}_{K/F}([\xi]) = \mathrm{Res}_{K_w/F_v} \circ \mathrm{Res}_{F_v/F}([\xi]) = \mathrm{Res}_{K_w/F_v}(1) = 1,$$

and if $w \in V^K_{\infty,\mathbb{R}}$ lies over $v \in V^F_{\infty,\mathbb{R}} \backslash S_G$ then $K_w$ is algebraically closed, so

$$\mathrm{Res}_{K_w/K} \circ \mathrm{Res}_{K/F}([\xi]) = 1$$

automatically. Applying Lemma 3.1 gives that $G$ splits over $K$ and that $K \hookrightarrow F_v$ for all $v \in S_G$.

I introduce some notions developed by Weisfeiler in [W] relating to groups splitting over quadratic extensions. Let $G$ be an $F$-defined group splitting over a quadratic extension $K/F$, let $\tau$ be the nontrivial element of $\mathrm{Gal}(K/F)$, let $B$ be an $F$-defined Borel subgroup of $G$ splitting over $K$ such that $B \cap B^\tau = T$, and let $G_\alpha$ be the root subgroup of $G$ corresponding to $\alpha \in \Sigma(G,T)$ (see [W, Lemma 3]).

LEMMA 3.2 [W, Lemma 5]. *$G_\alpha \simeq \mathrm{SL}_1(D_\alpha)$, where $D_\alpha = (d, c_\alpha)$.*

The numbers $c_\alpha \in F^\times$ are called the *structure constants of G with respect to T.*

PROPOSITION 3.1. *Every anisotropic group G of type $G_2$ over F contains an absolutely simple subgroup H of type $A_2$ of appropriate real rank.*

*Proof.* Choose $a$ as in Remark 3.1 and $T = B \cap B^\tau$ splitting over $K = F(\sqrt{a})$. Given a subroot system $\Sigma' \subset \Sigma(G,T)$, let $G_{\Sigma'}$ be the standard subgroup of $G$ generated by $G_\alpha$ for $\alpha \in \Sigma'$. Let $\Sigma'$ be the root subsystem of long roots in $\Sigma(G,T)$, and let $H = G_{\Sigma'}$. For any $v \in S_G$ we have that $T$ is split over $F_v$, so $H$ is split over $F_v$. □

Assume again that $G$ is any simple group splitting over quadratic extension and that $\tau$ and $T$ are as before. The structure constants defined previously are very useful in determining the isotropy of $G$ over $F_v$ for $v \in V^F_{\infty,\mathbb{R}}$. The following statement is immediate from Lemma 3.2.

LEMMA 3.3. *Given $v \in V^F_{\infty,\mathbb{R}}$ such that $K \otimes_F F_v \simeq \mathbb{C}$:*

(1) *G is anisotropic over $F_v$ if and only if the $c_\alpha$ are negative in $F_v$ for all $\alpha \in \Sigma(G,T)$;*
(2) *if $\langle \alpha, \beta \rangle = 0$ and $c_\alpha, c_\beta > 0$ in $F_v$, then G has higher rank over $F_v$.*

By [T], there are three possibilities for the rank of a group $G$ of type $F_4$ over any field. Over a completion $F_v$ for $v \in V^F_{\infty,\mathbb{R}}$, I claim that the sign of the structure constants completely determines the rank of $G$ over $F_v$.

LEMMA 3.4. *Suppose that G is anisotropic over F of type $F_4$, that $T \leq G$ is a maximal F-defined torus splitting over K as in Lemma 3.1, and that $\{c_\alpha\}$ are the structure constants of G with respect to T. Then, for $v \in V^F_{\infty,\mathbb{R}}$:*

(1) $c_\alpha < 0$ *in* $F_v$ *for all* $\alpha$ *if and only if* $G_{F_v}$ *is anisotropic*;
(2) *over* $F_v$, $c_\alpha < 0$ *for all long roots* $\alpha$ *and* $c_\beta > 0$ *for some short root* $\beta$ *if and only if G has* $F_v$*-rank 1*;
(3) *at least one long root* $\alpha$ *has* $c_\alpha > 0$ *in* $F_v$ *if and only if G is* $F_v$*-split.*

*Proof.* The first statement is Lemma 3.3(1). Assume that for some $\alpha \in \Sigma(G,T)$ with length 2 we have $c_\alpha > 0$ in $F_v$. I claim that $G_{F_v}$ is then split.

Let $\Sigma' \leq \Sigma(G,T)$ be the subroot system generated by the long roots, so that $\Sigma'$ has type $D_4$, and let $H = G_{\Sigma'}$. Then, since $\mathrm{Gal}(K/F)$ stabilizes $\{\pm\alpha\}$ for each $\alpha \in \Sigma(G,T)$, it follows that $H$ is of type ${}^1D_4$. By the assumption that $c_\alpha > 0$ for some long root $\alpha$, we also have that $H$ is $F_v$-isotropic. From [T], we therefore have that $\mathrm{rank}_{F_v}(H) \geq 2$; thus $\mathrm{rank}_{F_v}(G) \geq 2$ and so $G$ is split over $F_v$.

To complete the proof of the lemma, it suffices to prove that if $G$ is split over $F_v$ then $c_\alpha > 0$ for some long root $\alpha \in \Sigma(G,T)$. Assume that $G$ is split over $F_v$ and let $T'$ be a maximal torus in $G$ split over $F_v$. If $c_\alpha < 0$ in $F_v$ for all $\alpha \in \Sigma'$, then $H$ is anisotropic over $F_v$. Let $B$ be a Borel subgroup of $G$ containing $T'$. By dimension considerations, $(B \cap H)^0$ is nontrivial over $F_v$. If we choose an $F_v$-rational point $x$ of $(B \cap H)^0$, then the closure of $\langle x \rangle$ is a connected diagonalizable subgroup of $H$—contradicting the fact that $H$ is anisotropic. □

### 3.1. Modification of Structure Constants

The structure constants are not unique, and [W, Prop. 8] tells us that we can choose another maximal torus $T'$ to get a new set of structure constants $c'_\alpha$ related to $c_\alpha$ by $c'_\alpha = \nu^{\langle\alpha,\beta\rangle} c_\alpha$ for any $\nu \in \mathrm{Nrd}(D_\alpha)$ and $\beta \in \Sigma(G,T)$.

Given that Lemma 3.3 is concerned with the sign of $c_\alpha \in F_v$ only for $v \in V^F_{\infty,\mathbb{R}}$ (which I denote by $\mathrm{Sign}_v(c_\beta)$), this is all we seek to change when modifying structure constants. We can do this for each $v \in V^F_{\infty,\mathbb{R}}$ independently, as follows.

Lemma 3.5. *Given* $\alpha \in \Sigma(G,T)$ *and* $v \in V^F_{\infty,\mathbb{R}}$ *such that* $\mathrm{Sign}_v(c_\alpha) = 1$, *we can choose* $g_\alpha \in G_\alpha(K)$ *such that, if* $\{c'_\beta\}$ *are the structure constants of G with respect to* $g_\alpha T g_\alpha^{-1}$, *then*:

(1) $\mathrm{Sign}_w(c'_\beta) = \mathrm{Sign}_w(c_\beta)$ *for all* $w \neq v \in V^F_{\infty,\mathbb{R}}$; *and*
(2) $\mathrm{Sign}_v(c'_\beta) = (-1)^{\langle\beta,\alpha\rangle}\,\mathrm{Sign}_v(c_\beta)$ *for all* $\beta$.

*Proof.* By the weak approximation property, we can choose $y \in F$ such that $|y^2|_w < |c_\alpha|_w$ for all $w \neq v \in V^F_{\infty,\mathbb{R}}$ and $|c_\alpha|_v < |y^2|_v$. Define $g_\alpha$ as before. Replacing $T$ by $T' = g_\alpha T g_\alpha^{-1}$, we get that $c'_\beta = \left(\frac{c_\alpha}{c_\alpha - y^2}\right)^{\langle\beta,\alpha\rangle} c_\beta$. Our choice of $y$ gives that $c'_\beta$ has the desired sign in $F_v$ for all $v \in V^F_{\infty,\mathbb{R}}$. □

We call a modification of the form just described a *modification of T by* $\alpha$ *with respect to* $v$.

Proposition 3.2. *Every anisotropic group G of type* $F_4$ *over F contains an absolutely simple subgroup H of type* $B_3$ *of appropriate real rank.*

*Proof.* Let $\Sigma'$ be the root subsystem of $\Sigma(G,T)$ generated by $\{\alpha_1,\alpha_2,\alpha_3\}$ and let $H = G_{\Sigma'}$. (Throughout the proof, I use Bourbaki's explicit realization of root systems [B, Plates I–IX] and the same notation.) Then $H$ is a proper, absolutely simple subgroup of $G$, so it suffices to show that $H$ has appropriate real rank.

*Claim. We can choose* $T$ *in such a way that* $\mathrm{Sign}_v(c_{\alpha_3}) = 1$ *for all* $v \in S_G$ *and* $\mathrm{Sign}_v(c_{\alpha_1}) = 1$ *for all* $v \in S''_G$.

First I claim that we can modify $T$ so that $\mathrm{Sign}_v(c_{\alpha_1}) = 1$ for all $v \in S''_G$. If $v \in S''_G$, then by Lemma 3.3 we have that $\mathrm{Sign}_v(c_\alpha) = 1$ for some long root $\alpha \in \Sigma(G,T)$. The possibilities for $\langle\alpha_1,\alpha\rangle$ are 0, $\pm 1$, and $\pm 2$. If there exists a long root $\alpha$ such that $\mathrm{Sign}_v(c_\alpha) = 1$ and $\langle\alpha_1,\alpha\rangle = \pm 2$, then $\alpha = \pm\alpha_1$; so assume no such $\alpha$ exists. If there exists such an $\alpha$ such that $\langle\alpha_1,\alpha\rangle = \pm 1$, then modifying $T$ by $\alpha$ with respect to $v$ yields $T$ as desired.

If there does not exist an $\alpha$ with $\mathrm{Sign}_v(c_\alpha) = 1$ and $\langle\alpha_1,\alpha\rangle = \pm 1$ but there does exist an $\alpha$ with $\mathrm{Sign}_v(c_\alpha) = 1$ and $\langle\alpha_1,\alpha\rangle = 0$, then $\alpha$ must be of the form $\pm(\varepsilon_1+\varepsilon_2)$ or $\pm\varepsilon_3 \pm \varepsilon_4$. If $\alpha = \pm\varepsilon_3 \pm \varepsilon_4$ let $\alpha' = \varepsilon_2 + \varepsilon_4$, and if $\alpha = \pm(\varepsilon_1+\varepsilon_2)$ let $\alpha' = \varepsilon_2 + \varepsilon_3$. In either case, we have that $\langle\alpha',\alpha\rangle = \pm 1$ and $\langle\alpha_1,\alpha'\rangle = \pm 1$, so modifying $T$ by $\alpha'$ with respect to $v$ returns us to the case that there exists a long root $\alpha$ with $\mathrm{Sign}_v(c_\alpha) = 1$ and $\langle\alpha_1,\alpha\rangle = \pm 1$.

Assume that $v \in S''_G$ and we have already made the preceding modifications, so that $\mathrm{Sign}_v(c_{\alpha_1}) = 1$. If $\mathrm{Sign}_v(c_{\alpha_3}) = 1$, then $T$ is as required. If $\mathrm{Sign}_v(c_{\alpha_3}) = -1$ and there exists a short root $\beta$ such that $\mathrm{Sign}_v(c_\beta) = 1$ and $\langle\alpha_3,\beta\rangle = \pm 1$, then modifying $T$ by $\beta$ with respect to $v$ gives $T$ as required. If no such $\beta$ exists, let $\beta' = \frac{1}{2}(\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4)$; then $\langle\beta',\alpha_1\rangle = 1 = \langle\alpha_3,\beta'\rangle$ and $\langle\alpha_1,\beta'\rangle = 2$. Modifying $T$ by $\alpha_1$ with respect to $v$ gives a new $T$ such that $\mathrm{Sign}_v(c_{\beta'}) = 1$. Next, modifying $T$ by $\beta'$ with respect to $v$ gives another $T$ such that $\mathrm{Sign}_v(c_{\alpha_3}) = 1$ and $\mathrm{Sign}_v(c_{\alpha_1})$ is unchanged (because $\langle\alpha_1,\beta'\rangle = 2$). This new $T$ is such that $\mathrm{Sign}_v(c_{\alpha_1}) = 1 = \mathrm{Sign}_v(c_{\alpha_3})$ for all $v \in S''_G$.

Assume now that $v \in S'_G$. If $\mathrm{Sign}_v(c_\beta) = 1$ for $\beta = \pm\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$, then $\langle\beta,\alpha_3\rangle = \pm 1$; hence we can modify $T$ by $\beta$ with respect to $v$ to obtain $\mathrm{Sign}_v(c'_{\alpha_3}) = 1$. If $\mathrm{Sign}_v(c_\beta) = -1$ for all $\beta$ of the form above, then we must have that $\mathrm{Sign}_v(\varepsilon_i) = 1$ for some $i \neq 4$ by the assumption that some short root has positive associated structure constant. Fix $\beta = \frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)$. Then $\mathrm{Sign}_v(c_\beta) = -1$ by assumption, and for all $i$ we have $\langle\varepsilon_i,\alpha_3\rangle = 0$ and $\langle\beta,\varepsilon_i\rangle = 1$. This means that, if we modify $T$ first by $\varepsilon_i$ and then by $\beta$ with respect to $v$, the result will be $\mathrm{Sign}_v(c''_{\alpha_3}) = 1$. This proves the claim.

Combining Lemma 3.3 with this claim yields that $H$ has appropriate real rank, so $H$ is not minimal. □

PROPOSITION 3.3. *Any anisotropic group* $G$ *of type* $E_7$ *over* $F$ *contains an absolutely simple subgroup* $H$ *of type* $A_3$ *of appropriate real rank.*

NOTE. By [T], $S_G = S''_G$ for $G$ of type $E_7$.

*Proof of Proposition 3.3.* For a maximal $F$-defined torus $T$ of $G$, define $\Sigma' \subset \Sigma(G,T)$ to be the subroot system generated by $\{\alpha_5,\alpha_6,\alpha_7\}$ and let $H = G_{\Sigma'}$.

Clearly, $H$ is an absolutely simple proper subgroup of type $A_3$, and it remains to show that $H$ has appropriate real rank. By Lemma 3.3, it suffices to prove the following.

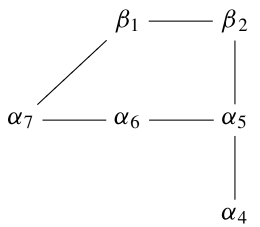
*Claim. We can choose $T$ so that $c_{\alpha_5}, c_{\alpha_7} > 0$ in $F_v$ for all $v \in S_G$.*

By Lemma 3.3, we may always choose some $\alpha \in \Sigma(G,T)$ such that $\operatorname{Sign}_v(c_\alpha) = 1$. After modification, we can say that $\operatorname{Sign}_v(c_{\alpha_7}) = 1$. Indeed, assume that $\operatorname{Sign}_v(c_{\alpha_7}) = -1$. If there exists an $\alpha$ with $\langle \alpha_7, \alpha \rangle = \pm 1$, then modification of $T$ by $\alpha$ with respect to $v$ reverses the sign of $c_{\alpha_7}$. If $\langle \alpha_7, \alpha \rangle \in \{0, \pm 2\}$ for all $\alpha \in \Sigma(G,T)$ with $\operatorname{Sign}_v(c_\alpha) = 1$, then choose an $\alpha$ with $\operatorname{Sign}_v(c_\alpha) = 1$ and let

$$\alpha' = \begin{cases} \varepsilon_j + \varepsilon_6 & \text{if } \alpha = \pm\varepsilon_j \pm \varepsilon_k,\ j < k \in \{1,2,3,4\}, \\ \varepsilon_4 + \varepsilon_6 & \text{if } \alpha = \pm(\alpha_5 + \alpha_6), \\ \tfrac{1}{2}\big(\varepsilon_7 - \varepsilon_8 + \varepsilon_6 - \varepsilon_5 + \sum_{i=1}^{4} \varepsilon_i\big) & \\ & \text{if } \alpha = \pm(\varepsilon_7 - \varepsilon_8), \\ \tfrac{1}{2}\big(\varepsilon_7 - \varepsilon_8 + \varepsilon_6 - \varepsilon_5 + (-1)^{\nu(4)} + \sum_{i=1}^{3} (-1)^{1-\nu(i)} \varepsilon_i\big) & \\ & \text{if } \alpha = \tfrac{1}{2}\big(\varepsilon_7 - \varepsilon_8 \pm (\varepsilon_5 + \varepsilon_6) + \sum_{i=1}^{4} (-1)^{\nu(i)}\big). \end{cases}$$

Then modifying $T$ by $\alpha$ with respect to $v$ returns us to the case where there exists an $\alpha'$ with $\operatorname{Sign}_v(c_{\alpha'}) = 1$ and $\langle \alpha_7, \alpha' \rangle = \pm 1$; hence we can modify $T$ again so that $\operatorname{Sign}_v(c_{\alpha_7}) = 1$.

Now, assuming that we have modified $T$ so that $\operatorname{Sign}_v(c_{\alpha_7}) = 1$, I claim that we can modify $T$ further so that $\operatorname{Sign}_v(c_{\alpha_5}) = 1$ as well. To see this, let $\beta_1 = \varepsilon_1 - \varepsilon_6$ and $\beta_2 = \frac{1}{2}(\varepsilon_8 - \varepsilon_7 + \varepsilon_6 + \varepsilon_5 + \varepsilon_4 - \varepsilon_3 - \varepsilon_2 - \varepsilon_1)$. Recall that if $\operatorname{Sign}_v(c_\alpha) = 1$, then modifying $T$ by $\alpha$ with respect to $v$ affects $\operatorname{Sign}_v(\beta)$ only for those $\beta$ with $\langle \beta, \alpha \rangle$ odd. In the following graph, the nodes correspond to roots and the edges connect roots such that $\langle \alpha, \beta \rangle$ is odd:




If $\operatorname{Sign}_v(c_{\alpha_5}) = 1$, then no modification is necessary. If $\operatorname{Sign}_v(c_{\alpha_5}) = -1$ but $\operatorname{Sign}_v(c_{\beta_2})$ or $\operatorname{Sign}_v(c_{\alpha_4}) = 1$, then modify $T$ by $\beta_2$ or $\alpha_4$ with respect to $v$ in order to change the sign of $c_{\alpha_5}$ in $F_v$. Assume then that $\operatorname{Sign}_v(c_{\alpha_5}) = \operatorname{Sign}_v(c_{\alpha_4}) = \operatorname{Sign}_v(c_{\beta_2}) = -1$. If $\operatorname{Sign}_v(c_{\alpha_6}) = \operatorname{Sign}_v(c_{\beta_1}) = 1$, then modifying $T$ first by $\alpha_6$ and then by $\beta_1$ with respect to $v$ reverses $\operatorname{Sign}_v(c_{\alpha_7})$ twice and $\operatorname{Sign}_v(c_{\alpha_5})$ once; so, after modification, $\operatorname{Sign}_v(c_{\alpha_7}) = 1 = \operatorname{Sign}_v(c_{\alpha_5})$. If $\operatorname{Sign}_v(c_{\alpha_6}) = \operatorname{Sign}_v(c_{\beta_1}) = -1$, then modifying by $\alpha_7$ with respect to $v$ returns us to the case where $\operatorname{Sign}_v(c_{\alpha_6}) = \operatorname{Sign}_v(c_{\beta_1}) = 1$.

If $\mathrm{Sign}_v(c_{\beta_1}) = 1$ and $\mathrm{Sign}_v(c_{\alpha_6}) = -1$, then modifying $T$ by $\alpha_7$ with respect to $v$ gives $\mathrm{Sign}_v(c_{\beta_1}) = -1$ and $\mathrm{Sign}_v(c_{\alpha_6}) = 1$. Therefore the only case left to consider is the one where

$$\mathrm{Sign}_v(c_{\alpha_7}) = \mathrm{Sign}_v(c_{\alpha_6}) = 1,$$
$$\mathrm{Sign}_v(c_{\beta_1}) = \mathrm{Sign}_v(c_{\beta_1}) = \mathrm{Sign}_v(c_{\beta_2}) = \mathrm{Sign}_v(c_{\alpha_5}) = \mathrm{Sign}_v(c_{\alpha_4}) = -1.$$

In this case, if we modify $T$ with respect to $v$ by roots in the order $\alpha_6, \alpha_5, \beta_2, \beta_1, \alpha_4$, then $(\mathrm{Sign}_v(c_{\alpha_7}), \mathrm{Sign}_v(c_{\alpha_5}))$ changes as follows:

$$(1,-1) \xrightarrow{\alpha_6} (-1,1) \xrightarrow{\alpha_5} (1,-1) \xrightarrow{\beta_2} (-1,-1) \xrightarrow{\beta_1} (1,-1) \xrightarrow{\alpha_4} (1,1).$$

After modification, then, $\mathrm{Sign}_v(c_{\alpha_7}) = 1 = \mathrm{Sign}_v(c_{\alpha_5})$ as required. □

PROPOSITION 3.4. *Any anisotropic group $G$ of type $E_8$ over $F$ contains an absolutely simple subgroup $H$ of type $A_3$ of appropriate real rank.*

*Proof.* As in the previous case, define $\Sigma'$ to be the subsystem of $\Sigma(G,T)$ generated by $\{\alpha_5, \alpha_6, \alpha_7\}$. Also as in the previous case, from [T] we have $S_G = S''_G$ for groups of type $E_8$. It therefore suffices to prove that we can choose some maximal $F$-torus $T$ of $G$ so that $\mathrm{Sign}_v(c_{\alpha_5}) = \mathrm{Sign}_v(c_{\alpha_7}) = 1$ for all $v \in S_G$.

Let $\Sigma'' \subset \Sigma(G,T)$ be the subsystem of $\Sigma(G,T)$ of type $E_7$ generated by $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$. To reduce the proof to the previous case, it suffices to show that it is possible to choose a maximal $F$-torus $T$ of $G$ so that $\mathrm{Sign}_v(\alpha) = 1$ for some root $\alpha \in \Sigma''$. Indeed, if we can show that some $\alpha \in \Sigma''$ has $\mathrm{Sign}_v(c_\alpha) = 1$, then we can modify $T$ with respect to each $v$ by roots in $\Sigma''$ as described in the previous proof to obtain $\mathrm{Sign}_v(c_{\alpha_5}) = \mathrm{Sign}_v(c_{\alpha_7}) = 1$ for all $v \in S''_G$.

Let $\beta_1 = \varepsilon_6 + \varepsilon_8$ and $\beta_2 = \varepsilon_6 - \varepsilon_8$; then $\left\langle \sum_{i=1}^{8} (-1)^{\nu(i)} \varepsilon_i, \beta_j \right\rangle \neq 0 \bmod 2$ for $j = 1$ or $2$. Next, if $\alpha = \pm\varepsilon_i \pm \varepsilon_j$ and $\langle \beta_j, \alpha \rangle \equiv 0 \bmod 2$, then $\langle \alpha, \alpha_i \rangle \neq 0 \bmod 2$ for some $1 \leq i \leq 7$. This means that, no matter what, for every $\alpha \in \Sigma(G,T)$ there exist a $\gamma \in \Sigma(G,T)$ and a $\delta \in \Sigma''$ such that $\langle \alpha, \gamma \rangle \equiv \langle \gamma, \delta \rangle \equiv 1 \bmod 2$.

If $\mathrm{Sign}_v(c_\delta) = 1$ then we are done. If $\mathrm{Sign}_v(c_\gamma) = 1$, modify $T$ by $\gamma$ with respect to $v$ to obtain that $\mathrm{Sign}_v(c'_\delta) = 1$. If $\mathrm{Sign}_v(c_\gamma) = -1$, modify $T$ by $\alpha$ with respect to $v$. This either reverses the sign of $c_\delta$ with respect to $v$ or returns us to the previous case. In any event, $\mathrm{Sign}_v(c_\delta) = 1$ with $\delta \in \Sigma''$. □

## 4. Type $^{3,6}D_4$

The purpose of this section is to prove the following result.

PROPOSITION 4.1. *No group of type $^{3,6}D_4$ is minimal.*

### *4.1. Preliminaries*

4.1.1. *Groups of Type $D_4$ over $\mathbb{R}$*
Because there exist no cubic field extensions of $\mathbb{R}$, any group $G$ of type $D_4$ over $\mathbb{R}$ is of type $^{1,2}D_4$. By Tits's classification, any simply connected group of type $^1D_4$ over $\mathbb{R}$ is isomorphic to a group of the form $\mathrm{Spin}(f_i)$, where $f_i$ is one of

$$f_0 = \sum_{i=1}^{8} x_i^2,$$

$$f_2 = \sum_{i=1}^{6} x_i^2 - y_1^2 - y_2^2, \quad \text{or}$$

$$f_4 = \sum_{i=1}^{4} x_i^2 - \sum_{i=1}^{4} y_i^2$$

up to multiplication by $\pm 1$. Let $G_0$ be the split, simply connected group of type ${}^1D_4$, so $G_0 \simeq \mathrm{Spin}(f_4)$. Note that $f_4$ is a Pfister form over $\mathbb{R}$ and recall that a Pfister form over $\mathbb{R}$ is either split or anisotropic. This gives that $\mathrm{Spin}(f_0)$ and $G_0$ are the two distinct strongly inner forms of $G_0$ and that $\mathrm{Spin}(f_2)$ corresponds to a cocycle in $H^1(K, \bar{G}_0)$ not contained in the image of $H^1(K, G_0)$.

If $G$ has type ${}^2D_4$ then $G$ is also isomorphic to a group of the form $\mathrm{Spin}(f_i)$, except now $f_i$ has discriminant $-1$; thus, $f_i$ is either

$$f_1 = \sum_{i=1}^{7} x_i^2 - y_1^2 \quad \text{or}$$

$$f_3 = \sum_{i=1}^{5} x_i^2 - \sum_{i=1}^{3} y_i^2$$

up to multiplication by $\pm 1$.

4.1.2. *Tori in* $\mathrm{SL}_2$ *and Quaternion Algebras*
Given an element $a \in F$, we can embed $T = R^{(1)}_{F(\sqrt{a})/F}(\mathrm{G_m})$ in $\mathrm{SL}_2$ via the regular representation. Let $\bar{T}$ be is its image in $\mathrm{PSL}_2$. We have the exact sequence

$$1 \to \mu_2 \to T \to \bar{T} \to 1,$$

which gives a map $H^1(F, \bar{T}) \to H^2(F, \mu_2)$. The following is not difficult to prove.

LEMMA 4.1. *If* $[\delta] \in H^2(F, \mu_2)$ *corresponds to* $D \in {}_2\mathrm{Br}(F)$ *and if* $D$ *is split by* $F(\sqrt{a})$, *then* $[\delta]$ *is in the image of* $H^1(F, \bar{T}) \to H^2(F, \mu_2)$.

4.1.3. *Modification of Cocycles*
Let $G_0$ be a simple, simply connected algebraic group with adjoint $\bar{G}_0$ and let $T \leq G_0$ be a maximal torus. Given a $[\xi] \in H^1(F, \bar{G}_0)$ with $[\mu] \in H^1(F, \bar{T})$ such that $[\xi]$ and $[\mu]$ have the same image in $H^2(F, Z(G_0))$ under the commuting diagram

$$\begin{array}{ccccc} H^1(F, G_0) & \xrightarrow{\pi_1} & H^1(F, \bar{G}_0) & \xrightarrow{\delta_1} & H^2(F, Z(G_0)) \\ \uparrow{\scriptstyle \iota_1} & & \uparrow{\scriptstyle \iota_2} & & \uparrow{\scriptstyle =} \\ H^1(F, T) & \xrightarrow[\pi_2]{} & H^1(F, \bar{T}) & \xrightarrow[\delta_2]{} & H^2(F, Z(G_0)) \end{array} \qquad (*)$$

with exact rows, we wish to "modify" $[\mu] \in H^1(F, \bar{T})$ by an element $[\alpha] \in H^1(F, T)$ to get $[\mu] \cdot \pi_2([\alpha]) \in H^1(F, \bar{T})$ so that $\iota_2([\mu] \cdot \pi_2([\alpha])) = [\xi]$. More precisely, we have the following statement.

LEMMA 4.2 (Modification of Cocycles). *Given $G_0, \bar{G}_0, T, \bar{T}, [\xi]$ as before, if there exist*

(1) $[\mu] \in H^1(F, \bar{T})$ *with* $\delta_2([\mu]) = \delta_1([\xi])$ *and*
(2) $[\nu_v] \in H^1(F_v, \bar{T})$ *with* $\iota_2([\nu_v]) = [\xi_v]$ *for each Archimedean place* $v$

*then there exists a* $[\gamma] \in H^1(F, \bar{T})$ *such that* $\iota_2([\gamma]) = [\xi]$.

*Proof.* We retain the notation of diagram $(*)$. By the Hasse principle for $H^1(F, \bar{G}_0)$ [PR], it suffices to show that we can choose $[\gamma] \in H^1(F, \bar{T})$ such that $\iota_2([\gamma_v]) = [\xi_v]$ for any valuation $v$ on $F$.

First, I claim that $\iota_2([\mu_v]) = [\xi_v]$ for any non-Archimedean place $v$. From the condition that $\delta_2([\mu]) = \delta_1([\xi])$, we see that $\iota_2([\mu_v]) \in \delta_1^{-1}(\delta_1([\xi_v]))$. By [Se, Chap. 1, Sec. 5], $\delta_1^{-1}(\delta_1([\xi_v]))$ is in bijective correspondence with $H^1(F_v, {}_\xi G_0)/\sim$ for some equivalence relation $\sim$. Because we assume that ${}_\xi G_0$ is simply connected and $v$ is non-Archimedean, Kneser's theorem gives that $H^1(F_v, {}_\xi G_0) = \{1\}$ and so $\delta_1^{-1}(\delta_1([\xi_v])) = \{[\xi_v]\}$; that is, $\iota_2([\mu_v]) = [\xi_v]$.

Next, given $v \in V^F_{\infty,\mathbb{R}}$, condition (2) gives that $\delta_2([\nu_v]) = \delta_1([\xi_v])$ and condition (1) that $\delta_2([\mu_v]) = \delta_1([\xi_v])$, so $\delta_2([\nu_v]) = \delta_2([\mu_v])$. By the exactness of the bottom row in $(*)$, we have $[\mu_v] = [\nu_v] \cdot \pi_2([\lambda_v])$ for some $[\lambda_v] \in H^1(F_v, T)$. From [PR], the map $H^1(F, T) \xrightarrow{\prod \mathrm{Res}_{F_v}} \prod_{v \in V^F_{\infty,\mathbb{R}}} H^1(F_v, T)$ is surjective. This means that we can choose $[\alpha] \in H^1(F, T)$ such that $[\alpha_v] = [\lambda_v]$ for all $v \in V^F_{\infty,\mathbb{R}}$.

I claim that $[\gamma] := [\mu] \cdot \pi_2([\alpha])$ has $\iota_2([\gamma_v]) = [\xi_v]$ for every $v$. For $v$ non-Archimedean, note that

$$\delta_1(\iota_2([\gamma_v])) = \delta_2([\gamma_v]) = \delta_2([\mu_v]) \cdot \delta_2(\pi_2([\alpha_v])) = \delta_2([\mu_v]) = \delta_1([\xi_v]);$$

however, we have shown that the fibre of $[\xi_v]$ under $\delta_1$ is just $\{[\xi_v]\}$, so $\iota_2([\gamma_v]) = [\xi_v]$ for every non-Archimedean $v$. Finally, for $v \in V^F_{\infty,\mathbb{R}}$ we have

$$\iota_2([\gamma_v]) = \iota_2([\mu_v] \cdot \pi_2([\alpha_v])) = \iota_2([\mu_v] \cdot \pi_2([\lambda_v])) = \iota_2([\nu_v]) = [\xi_v]$$

by construction. □

### *4.2. Construction of a Special Torus T*

Let $G$ now be a simply connected group of type ${}^{3,6}D_4$ corresponding to $[\xi] \in H^1(F, \bar{G}_0)$, where $G_0$ is now the simply connected quasi-split group of type ${}^{3,6}D_4$. Let $E$ be a cubic extension of $F$ over which $G$ has type ${}^{1,2}D_4$. Then $Z(G_0) \simeq R^{(1)}_{E/F}(\mu_2)$ and so $H^2(F, Z(G_0)) \simeq \ker({}_2\mathrm{Br}(E) \xrightarrow{N} {}_2\mathrm{Br}(F))$, where $N$ is the norm map. Let $[(a,b)_E]$ be the image of $[\xi]$ in $H^2(F, Z(G_0))$. By [BOI] we can choose $a, b$ such that $a \in F$, $F(\sqrt{a})$ has no real completions, and $N_{E/F}(b) = 1$.

The following result is proved in [CLM].

THEOREM 4.1. *There exists a subgroup $H < G_0$ of type $A_1 \times A_1 \times A_1 \times A_1$ that is isogenous to $R_{P/F}(\mathrm{SL}_2)$ for some quartic field extension $P/F$ contained in $E\big(\sqrt{b}, \sqrt{\sigma(b)}, \sqrt{\sigma^2(b)}\big)$, where $\sqrt{\sigma^i(b)}$ are the Galois conjugates of $\sqrt{b}$ in the normal closure of $E$ over $F$.*

Let $\tilde{H} = R_{P/F}(\mathrm{SL}_2)$, $H$ be the image of $\tilde{H}$ in $G_0$, $\bar{H}$ be the image of $\tilde{H}$ in $\bar{G}_0$, and $\bar{H}' = \tilde{H}/Z(\tilde{H})$. If we consider the sequence of projections

$$\tilde{H} \xrightarrow{\phi_1} H \xrightarrow{\phi_2} \bar{H} \xrightarrow{\phi_3} \bar{H}',$$

then $\ker(\phi_1)$ is the diagonal embedding of $\mu_2$ into $Z(\tilde{H})$ over the algebraic closure, $\ker(\phi_2) = Z(G_0)$, and $\ker(\phi_3) = Z(\bar{H}) = Z(\tilde{H})/Z(G_0) \simeq \mu_2$.

We need some notation from the proof in [CLM]. Let

$$\tilde{T}_0 = \mathrm{G_m} \times R_{E/F}(R^{(1)}_{E(\sqrt{b})/E}(\mathrm{G_m})),$$

let $T_0$ be its image in $G_0$, and let $\bar{T}_0$ be its image in $\bar{G}_0$. If $\alpha_1, \ldots, \alpha_4$ are a basis of $\Sigma(G_0, T_0)$, then $R_{E/F}(R^{(1)}_{E(\sqrt{b})/E}(\mathrm{G_m})) = T_0 \cap G_{\alpha_1,\alpha_3,\alpha_4}$ and $H = G_\Phi$, where $\Phi = \{\alpha_2, \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_1 + \alpha_4, \alpha_2 + \alpha_1 + \alpha_3\}$.

LEMMA 4.3. *There exists a cocycle* $[\mu_{\bar{T}_0}] \in H^1(F, \bar{T}_0)$ *such that* $[\mu_{\bar{T}_0}] \mapsto [(a,b)_E]$ *under* $H^1(F, \bar{T}_0) \to H^1(F, \bar{G}_0) \to H^2(F, Z(G_0))$.

*Proof.* Consider the subtorus $S \leq T_0$ given by $S = R_{E/F}R^{(1)}_{E(\sqrt{b})/E}\mathrm{G_m}$, and let $\bar{S}$ be the image of $S$ in $\bar{G}_0$. I claim that there exists a $[\mu_{\bar{S}}] \in H^1(F, \bar{S})$ that maps to $[(a,b)_E] \in H^2(F, Z(G_0))$. The image of $[\mu_{\bar{S}}] \in H^1(F, \bar{T}_0)$ is then the cocycle we are looking for.

To see that $[\mu_{\bar{S}}]$ exists, consider the $F$-defined subgroups $\tilde{Z}, Z \leq S$, where $Z$ is the center of $G_0$ and $\tilde{Z}$ is the 2-torsion part of $S$, which is also given by $R_{E/F}(\mu_2)$. Note that, over $\bar{F}$, $\tilde{Z}$ has the form $\mu_2 \times \mu_2 \times \mu_2$, the norm map is given by the product of the entries, and $Z$ is the kernel of this map. Using this, we have an interlocking diagram of exact sequences,

$$\begin{array}{ccccccccc}
 & & & & & & 1 & & \\
 & & & & & & \downarrow & & \\
 & & 1 & & & & \mu_2 & & \\
 & & \downarrow & & & & \downarrow & & \\
1 & \longrightarrow & Z & \longrightarrow & S & \longrightarrow & \bar{S} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{=} & & \downarrow & & \\
1 & \longrightarrow & \tilde{Z} & \longrightarrow & S & \longrightarrow & S/\tilde{Z} & \longrightarrow & \\
 & & \downarrow{N_{E/F}} & & & & \downarrow & & \\
 & & \mu_2 & & & & 1 & & \\
 & & \downarrow & & & & & & \\
 & & 1 & & & & & &
\end{array}$$

that induces the following exact sequences of Galois cohomology sets with corresponding morphisms:

$$
\begin{array}{ccccc}
H^1(F,S) & \longrightarrow & H^1(F,\bar{S}) & \longrightarrow & H^2(F,Z) \\
\big\downarrow{\scriptstyle =} & & \big\downarrow & & \big\downarrow \\
H^1(F,S) & \longrightarrow & H^1(F,S/\tilde{Z}) & \longrightarrow & H^2(F,\tilde{Z}) \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle N_{E/F}} \\
 & & H^2(F,\mu_2) & \xrightarrow{=} & H^2(F,\mu_2)
\end{array}
$$

Suppose a $[\mu_{S/\tilde{Z}}] \in H^1(F,S/\tilde{Z})$ maps to $[(a,b)_E]$ under $H^1(F,S/\tilde{Z}) \to H^2(F,\tilde{Z})$ in the preceding diagram. The norm of $(a,b)_E$ is trivial by assumption, so $[\mu_{S/\tilde{Z}}]$ is the image of some $[\mu_{\bar{S}}] \in H^1(F,\bar{S})$. We have a section $\lambda\colon \mu_2 \to \tilde{Z}$ given by the diagonal embedding, and so $H^2(F,Z) \to H^2(F,\tilde{Z})$ is injective. This, combined with the commutativity of the upper right-hand square, shows that $[\mu_{\bar{S}}] \mapsto [(a,b)_E] \in H^2(F,Z)$.

It remains to prove that there exists a $[\mu_{S/\tilde{Z}}] \in H^1(F,S/\tilde{Z})$ such that $[\mu_{S/\tilde{Z}}] \mapsto [(a,b)_E] \in H^2(F,\tilde{Z})$. Note that, by Shapiro's lemma, $H^1(F,S/\tilde{Z}) \to H^2(F,\tilde{Z})$ is equivalent to $H^1(E, R^{(1)}_{E(\sqrt{b})/E}(\mathrm{G_m})/\mu_2) \to H^2(E,\mu_2)$. Thus Lemma 4.1 gives the existence of $[\mu_{S/\tilde{Z}}] \in H^1(F,S/\tilde{Z})$. $\square$

Let $[\mu_{\bar{H}}]$ be the image of $[\mu_{\bar{T}_0}]$ in $H^1(F,\bar{H})$, let $[\mu_{\bar{H}'}]$ be its image in $H^1(F,\bar{H}')$, and let $[(a,b)_P]$ be the image of $[\mu_{\bar{H}'}]$ under the isomorphism $H^1(F,\bar{H}') \to H^2(F,R_{P/F}(\mu_2)) \simeq {}_2\mathrm{Br}(P)$. Choose $p \in P$ such that $[(a,b)_P]$ splits over $P(\sqrt{p})$, and define $\tilde{T} = R_{P/F}(R^{(1)}_{P(\sqrt{p})/P}(\mathrm{G_m}))$ embedded in $\tilde{H}$ via the regular representation. Let $T$ be the image of $\tilde{T}$ in $H$, $\bar{T}$ the image of $\tilde{T}$ in $\bar{H}$, and $\bar{T}'$ the image of $\tilde{T}$ in $\bar{H}'$.

LEMMA 4.4. *There exists a* $[\mu] \in H^1(F,\bar{T})$ *such that* $[\mu] \mapsto [\mu_{\bar{H}}]$ *under* $H^1(F,\bar{T}) \to H^1(F,\bar{H})$.

*Proof.* By Shapiro's lemma, the mapping $H^1(F,\bar{T}') \xrightarrow{\iota_4} H^1(F,\bar{H}')$ is isomorphic to $H^1(P, R^{(1)}_{P(\sqrt{p})/P}(\mathrm{G_m})) \to H^1(P,\mathrm{PSL}_2)$; hence, by Lemma 4.1, there exists a $[\mu'] \in H^1(F,\bar{T}')$ such that $\iota_4([\mu']) = [\mu_{\bar{H}'}]$. Consider the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & Z(\bar{H}) & \longrightarrow & \bar{T} & \longrightarrow & \bar{T}' & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle =} & & \big\downarrow & & \big\downarrow & & \\
1 & \longrightarrow & Z(\bar{H}) & \longrightarrow & \bar{H} & \longrightarrow & \bar{H}' & \longrightarrow & 1
\end{array}
$$

This induces the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} H^1(F, Z(\bar{H})) & \xrightarrow{\iota_1} & H^1(F, \bar{T}) & \xrightarrow{\pi_1} & H^1(F, \bar{T}') & \xrightarrow{\delta_1} & H^2(F, Z(\bar{H})) \\ \downarrow = & & \downarrow \iota_2 & & \downarrow \iota_4 & & \downarrow = \\ H^1(F, Z(\bar{H}) & \xrightarrow[\iota_3]{} & H^1(F, \bar{H}) & \xrightarrow[\pi_2]{} & H^1(F, \bar{H}') & \xrightarrow[\delta_2]{} & H^2(F, Z(\bar{H}) \end{array} \qquad (**)$$

By the assumption that $\iota_4([\mu']) = [\mu_{\bar{H}'}]$ we have that $\delta_1([\mu']) = \delta_2([\mu_{\bar{H}'}]) = 1$, so there exists a $[\mu''] \in H^1(F, \bar{H})$ such that $\pi_1([\mu'']) = [\mu']$. By the commutativity of diagram $(**)$ we have $\pi_2(\iota_2([\mu''])) = \pi_2([\mu_{\bar{H}}])$ and so, from [Se, Chap. 1, Sec. 5], we find that there exists a $[\theta] \in H^1(F, Z(\bar{H}))$ such that $\iota_3([\theta]) \cdot \iota_2([\mu'']) = [\mu_{\bar{H}}]$. If we define $[\mu] = \iota_1([\theta]) \cdot [\mu'']$, then $\iota_2([\mu]) = \iota_2\iota_1([\theta]) \cdot \iota_2([\mu'']) = \iota_3([\theta]) \cdot \iota_2([\mu'']) = [\mu_{\bar{H}}]$. □

## 4.3. *Modification of* $[\mu]$

By Lemma 4.4 and the commutativity of the diagram

$$\begin{array}{ccc} H^1(F, \bar{T}) & \longrightarrow & H^2(F, Z(G_0)) \\ \downarrow & \nearrow & \\ H^1(F, \bar{H}) & & \end{array}$$

we have that $[\mu] \mapsto [(a, b)_E]$ under $H^1(F, \bar{T}) \to H^2(F, Z(G_0))$. In this section we modify $[\mu]$ as in Section 4.1.2 to obtain a cocycle $[\gamma] \in H^1(F, \bar{T})$ such that $[\gamma] \mapsto [\xi]$ under $H^1(F, \bar{T}) \to H^1(F, \bar{G}_0)$. In order to do this, we need cocycles $[\nu_v] \in H^1(F_v, \bar{T})$ for each $v \in V^F_{\infty,\mathbb{R}}$ such that $[\nu_v] \mapsto [\xi_v]$ under $H^1(F_v, \bar{T}) \to H^1(F_v, \bar{G}_0)$. We break this into two cases.

### 4.3.1. $E \otimes_F F_v \simeq F_v \times F_v \times F_v$

In order to understand how $\bar{T}$ behaves over $F_v$, it is necessary to understand the structure of $P \otimes_F F_v$. Recall that $H$ is isogenous to $R_{P/F}(\mathrm{SL}_2)$ and so, in order to understand $P \otimes_F F_v$, it is instructive to examine $H$ over $F_v$. In order to examine $H$, we need to remember that $H = G_\Phi$, where $\Phi = \{\alpha_2, \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_1 + \alpha_3, \alpha_2 + \alpha_1 + \alpha_4\} \subset \Sigma(G, T_0)$ has Galois action described in [CLM]. I claim that the sign of $b$ under each of the maps $E \hookrightarrow E \otimes_F F_v \xrightarrow{\pi_i} F_v$ determines the Galois action of $\mathrm{Gal}(\mathbb{C}/F_v)$ on $\Phi$ and hence determines the structure of $H$ and thus the structure of $P \otimes_F F_v$.

LEMMA 4.5. *With notation as before, let* $b_1, b_2, b_3$ *be the images of* $b$ *under the maps* $E \hookrightarrow E \otimes_F F_v \xrightarrow{\pi_i} F_v$. *If at least one of* $b_1, b_2, b_3$ *is negative, then* $P \otimes_F F_v \simeq \mathbb{C} \times \mathbb{C}$; *if all of* $b_1, b_2, b_3$ *are positive, then* $P \otimes_F F_v \simeq F_v \times F_v \times F_v \times F_v$.

*Proof.* Suppose that $b_1, b_2, b_3$ are all positive in $F_v$. In this case,

$$\begin{aligned} &R_{E/F}(R^{(1)}_{E(\sqrt{b})/E}(\mathrm{G_m}))_{F_v} \\ &\quad \simeq R^{(1)}_{F_v(\sqrt{b_1})/F_v}(\mathrm{G_m}) \times R^{(1)}_{F_v(\sqrt{b_2})/F_v}(\mathrm{G_m}) \times R^{(1)}_{F_v(\sqrt{b_3})/F_v}(\mathrm{G_m}) \\ &\quad \simeq \mathrm{G_m} \times \mathrm{G_m} \times \mathrm{G_m} \end{aligned}$$

and so $T_0$ is split over $F_v$. This gives that all $\alpha \in \Sigma(G_0, T_0)$ are fixed under $\mathrm{Gal}(\mathbb{C}/F_v)$. Therefore, $\Phi$ is fixed under $\mathrm{Gal}(\mathbb{C}/F_v)$; hence $\tilde{H}_{F_v} \simeq \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$ and so $P \otimes_F F_v \simeq F_v \times F_v \times F_v \times F_v$.

Suppose now that one of $b_1, b_2, b_3$ is negative. Up to renumbering, we may assume that $b_1$ and $b_2$ are negative while $b_3$ is positive (because $N_{E/F}(b) = 1$). In this case,

$$\begin{aligned} &R_{E/F}(R^{(1)}_{E(\sqrt{b})/E}(\mathrm{G_m}))_{F_v} \\ &\quad\simeq R^{(1)}_{F_v(\sqrt{b_1})/F_v}(\mathrm{G_m}) \times R^{(1)}_{F_v(\sqrt{b_2})/F_v}(\mathrm{G_m}) \times R^{(1)}_{F_v(\sqrt{b_3})/F_v}(\mathrm{G_m}) \\ &\quad\simeq R^{(1)}_{\mathbb{C}/F_v}(\mathrm{G_m}) \times R^{(1)}_{\mathbb{C}/F_v}(\mathrm{G_m}) \times \mathrm{G_m} \end{aligned}$$

and thus (again, up to renumbering) $1 \neq \tau \in \mathrm{Gal}(\mathbb{C}/F_v)$ acts by

$$\alpha_1 \mapsto \alpha_1,$$
$$\alpha_3 \mapsto -\alpha_3,$$
$$\alpha_4 \mapsto -\alpha_4;$$

if $\tilde{\alpha}$ is a root of maximal height, then $\tilde{\alpha} \mapsto \tilde{\alpha}$ (since this was true over $F$). This means that $\alpha_2 \mapsto \alpha_2 + \alpha_1 + \alpha_3$ and $\alpha_1 + \alpha_2 + \alpha_4 \mapsto \alpha_2 + \alpha_3 + \alpha_4$; hence $\Phi$ has type $[A_1 \times A_1] \times [A_1 \times A_1]$, with $\mathrm{Gal}(\mathbb{C}/F_v)$ permuting the factors inside the brackets. This gives that $\tilde{H}_{F_v} \simeq R_{\mathbb{C}/F_v}(\mathrm{SL}_2) \times R_{\mathbb{C}/F_v}(\mathbb{C}/F_v)$, so $P \otimes_F F_v \simeq \mathbb{C} \times \mathbb{C}$. □

By our restriction that $E \otimes_F F_v \simeq F_v \times F_v \times F_v$, we have that $G_{F_v}$ is of type ${}^1D_4$. By the classification given in Section 4.1.1, we have that $G_{F_v}$ is of rank 0, 2, or 4. Recall that $[\xi]_v$ is in the image of $H^1(F_v, G_0) \to H^1(F_v, \bar{G}_0)$ if and only if $G$ has rank 0 or 4. This is true if and only if $(a, b_1)_{F_v}$, $(a, b_2)_{F_v}$, and $(a, b_3)_{F_v}$ are all split, which is equivalent to the condition that $b_1, b_2, b_3$ are all positive (since $F(\sqrt{a})$ is purely imaginary by assumption). When combined with Lemma 4.5, these remarks yield our next lemma.

LEMMA 4.6. *If $G_{F_v}$ has rank 2, then $\tilde{T}$ has the form*

$$R_{\mathbb{C}/F_v}(\mathrm{G_m}) \times R_{\mathbb{C}/F_v}(\mathrm{G_m})$$

*and at least one of $b_1, b_2, b_3$ is negative in $F_v$.*

*If $G_{F_v}$ is anisotropic or split, then $b_1, b_2, b_3$ are all positive in $F_v$. Moreover, if we let $\psi_{i,v}$ be the composition*

$$P \hookrightarrow P \otimes_F F_v \simeq F_v \times F_v \times F_v \times F_v \xrightarrow{\pi_i} F_v,$$

*then $\tilde{T}_{F_v}$ has the form*

$$\begin{aligned} &R^{(1)}_{F_v(\sqrt{\psi_{1,v}(p)})/F_v}(\mathrm{G_m}) \times R^{(1)}_{F_v(\sqrt{\psi_{2,v}(p)})/F_v}(\mathrm{G_m}) \\ &\qquad\qquad \times R^{(1)}_{F_v(\sqrt{\psi_{3,v}(p)})/F_v}(\mathrm{G_m}) \times R^{(1)}_{F_v(\sqrt{\psi_{4,v}(p)})/F_v}(\mathrm{G_m}). \end{aligned}$$

Notice that if $b_1, b_2, b_3$ are all positive in $F_v$, then the structure of $\bar{T}_{F_v}$ depends on the sign of $\psi_{i,v}(p)$. The following lemma allows us to control these signs.

LEMMA 4.7. *There exists a $p \in P$ such that $P(\sqrt{p})$ splits $(a,b)_P$ and $\psi_{i,v}(p) < 0$ in $F_v$ if and only if $[\xi]$ is trivial over $F_v$.*

*Proof.* Recall the definition of $[\mu_{\bar{H}}]$ and $[\mu_{\bar{H}'}]$ that was given immediately before Lemma 4.4.

Let $\Psi_1 \subset V^F_{\infty,\mathbb{R}}$ be the set of all places of $F$ such that $b_1, b_2, b_3$ are all positive in $F_v$ but $[\xi]_v$ is nontrivial. Let $([(\alpha_1,\beta_1)_{F_v}], [(\alpha_2,\beta_2)_{F_v}], [(\alpha_3,\beta_3)_{F_v}], [(\alpha_4,\beta_4)_{F_v}])$ be the image of $[(a,b)_P]$ under the isomorphism $H^1(F_v,\bar{H}') \simeq H^2(F_v,\mu_2) \times \cdots \times H^2(F_v,\mu_2)$. Given a quaternion algebra over the real numbers, it is always possible to find a pure quaternion $q$ such that $q^2 = -1$. For $v \in \Psi_1$, choose $x_{i,v}, y_{i,v}, z_{i,v} \in F_v$ such that

$$\alpha_i x_{i,v}^2 + \beta_i y_{i,v}^2 - \alpha_i\beta_i z_{i,v}^2 = -1.$$

Let $\Psi_2$ be the set of all places of $F$ such that $[\xi]_v$ is split. For every such $v$, I claim that $[(\alpha_1,\beta_1)_{F_v}]$, $[(\alpha_2,\beta_2)_{F_v}]$, $[(\alpha_3,\beta_3)_{F_v}]$, and $[(\alpha_4,\beta_4)_{F_v}]$ are split. To see this, recall the definition of $S$ from the proof of Lemma 4.3 and consider the short exact sequence

$$1 \to Z(G_0) \to S \to \bar{S} \to 1.$$

Recall also that $[\mu_{\bar{H}'}]$ was the image of a cocycle $[\mu_{\bar{S}}] \in H^1(F,\bar{S})$ that mapped to $[(a,b)_E]$ under $H^1(F,\bar{S}) \to H^2(F,Z(G_0))$. Because $[(a,b)_E]$ is split over $F_v$, this means that $[\mu_{\bar{S}}]$ is the image of some $[\mu_S] \in H^1(F_v,S)$; but by the definition of $S$, $S_{F_v} \simeq \mathrm{G_m} \times \mathrm{G_m} \times \mathrm{G_m}$. This means that $[\mu_{\bar{S}}]$ is split over $F_v$ by Hilbert 90. Hence $[\mu_{\bar{H}'}]$ is also split over $F_v$ and thus $[(\alpha_1,\beta_1)_{F_v}]$, $[(\alpha_2,\beta_2)_{F_v}]$, $[(\alpha_3,\beta_3)_{F_v}]$, and $[(\alpha_4,\beta_4)_{F_v}]$ are split as claimed.

Because $[(\alpha_i,\beta_i)_{F_v}]$ are split, there exist pure quaternions $q_i \in (\alpha_i,\beta_i)$ such that $q_i^2 = 1$. For $v \in \Psi_2$, choose $x_{i,v}, y_{i,v}, z_{i,v} \in \mathbb{R}$ such that

$$\alpha_i x_{i,v}^2 + \beta_i y_{i,v}^2 - \alpha_i\beta_i z_{i,v}^2 = 1.$$

Next, choose $\varepsilon > 0$ such that, if $|x'_{i,v} - x_{i,v}| + |y'_{i,v} - y_{i,v}| + |z'_{i,v} - z_{i,v}| < \varepsilon$, then

$$|\alpha_i x'^2_{i,v} + \beta_i y'^2_{i,v} - \alpha_i\beta_i z'^2_{i,v} - \alpha_i x^2_{i,v} - \beta_i y^2_{i,v} + \alpha_i\beta_i z^2_{i,v}| < \frac{1}{2}.$$

Applying the weak approximation property then provides $x, y, z \in P$ such that

$$|\psi_{i,v}(x) - x_{i,v}| + |\psi_{i,v}(y) - y_{i,v}| + |\psi_{i,v}(z) - z_{i,v}| < \varepsilon$$

and so, if we let $p = \alpha x^2 + \beta y^2 - \alpha\beta z^2$, then $p$ satisfies the conditions of the lemma. □

Recall that there are three possibilities for $G_{F_v}$: it can be split, anisotropic, or of rank 2. If $G_{F_v}$ is split then $[\xi]_v$ is trivial, so we can let $[\nu_v] = 1$ and then $[\nu_v] \mapsto [\xi]_v$. If $G_{F_v}$ is anisotropic then, by our choice of $p$, $T_{F_v}$ is anisotropic and thus $T_{F_v}$ is isomorphic to a maximal torus of $G_{F_v}$. By Steinberg's theorem, we therefore have an embedding $\phi\colon \bar{T}_{F_v} \hookrightarrow \bar{G}_{0,F_v}$ and $[\nu'_v] \in H^1(F_v, \phi(\bar{T}_{F_v}))$ such that $[\nu'_v] \mapsto [\xi]_v$.

Any two anisotropic tori in $\bar{G}_{0,F_v}$ are conjugate [Hu]. Hence the image of $H^1(F_v, \bar{T}_{F_v})$ and $H^1(F_v, \phi(\bar{T}_{F_v}))$ in $H^1(F_v, \bar{G}_{0,F_v})$ are the same and there exists a $[\nu_v] \in H^1(F_v, \bar{T}_{F_v})$ such that $[\nu_v] \mapsto [\xi]_v$.

Finally, we must consider the case in which $G_{F_v}$ has rank 2. In this case, Lemma 4.6 gives that $P \otimes_F F_v \simeq \mathbb{C} \times \mathbb{C}$ and $\tilde{T}_{F_v} \simeq R_{\mathbb{C}/\mathbb{R}}(\mathrm{G_m}) \times R_{\mathbb{C}/\mathbb{R}}(\mathrm{G_m})$. Recall the definition of $T_0$. The action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $\Sigma(G_0, T_0)$ is described in Lemma 4.5 and, up to renumbering, the subsets $\Phi_1 = \{\alpha_2, \alpha_2 + \alpha_1 + \alpha_3\}$ and $\Phi_2 = \{\alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_1 + \alpha_4\}$ are $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-stable. Let $G_i$ be the subgroup of $G_{0,F_v}$ generated by $G_\alpha$, where $\alpha \in \Phi_i$. Finally, recall that $G_0$ is split over $F_v$ in this case and hence $G_{0,F_v} \simeq \mathrm{Spin}(f_4)$ (with $f_4$ defined as in Section 4.1). The following is a slight rephrasing of Lemma 2.6 to suit our situation.

LEMMA 4.8. *Given $(V, f_4)$ as before, there exist $V_1, V_2 \subset V$ such that $V = V_1 \oplus V_2$ and $V_2 = V_1^\perp$ under $(\cdot,\cdot)_f$. Also, if $g_1 = f_4|_{V_1}$ and $g_2 = f_4|_{V_2}$ then $f = g_1 \oplus g_2$ and, up to isogeny, $G_i \le G_{0,F_v}$ is given by $\mathrm{SO}(g_i) \le \mathrm{SO}(f_4)$.*

For a given 2-dimensional quadratic form $g$ over a field $F$,

$$\mathrm{Spin}(g) \simeq R_{F(\sqrt{\mathrm{disc}(g)})/F}(\mathrm{SL}_1(T)),$$

where $T$ is a quaternion algebra over $F(\sqrt{\mathrm{disc}(g)})$. Recalling from Lemma 4.5 that $G_i \simeq R_{\mathbb{C}/\mathbb{R}}(\mathrm{SL}_2)$, this gives that the $g_i$ have nontrivial discriminant and so, up to multiplication by $\pm 1$, $g_1 = \langle 1,1,1,-1\rangle = g_2$. Lemma 4.8 gives that $g_1 \oplus g_2 = \langle 1,-1,1,-1,1,-1,1,-1\rangle$ and so, up to renumbering, $g_1 = \langle 1,1,1,-1\rangle$ and $g_2 = \langle 1,-1,-1,-1\rangle$.

Let $T'$ be the image of $T$ in $\mathrm{SO}(f_4)$. Consider $z = (1,-1) \in \mathrm{SO}(g_1) \times \mathrm{SO}(g_2) \le \mathrm{SO}(f_4)$. Let $[\nu_v'] \in H^1(F_v, \mathrm{PSO}(f_4)) = H^1(F_v, \bar{G}_{0,F_v})$ be given by $(\nu_v')_\tau = \bar{z} \in \mathrm{PSO}(f_4)$ if $\tau \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is nontrivial. By definition of $T$, we have that $T' \cap \mathrm{SO}(g_2)$ is a maximal torus in $\mathrm{SO}(g_2)$; thus $Z(\mathrm{SO}(g_2)) \le T' \cap \mathrm{SO}(g_2)$ and so $z \in T'$. Hence there exists $[\nu_v] \in H^1(F_v, \bar{T}_{F_v})$ such that $[\nu_v] \mapsto [\nu_v']$.

LEMMA 4.9. *Under $H^1(F_v, \bar{T}_{F_v}) \to H^1(F_v, \bar{G}_{0,F_v})$, $[\nu_v] \mapsto [\xi]_v$.*

*Proof.* It suffices to show that ${}_{\nu_v'}G_{0,F_v} \simeq G$. This property is invariant under taking quotients by a central subgroup, so it suffices to show that ${}_{\nu_v'}\mathrm{SO}(f_4) \simeq \mathrm{SO}\big(\sum_{i=1}^6 x_i^2 - x_7^2 - x_8^2\big)$, which can be verified by direct calculation. □

NOTE. From our choice of $p$ it follows that $\bar{T}_{F_v}$ has higher rank for all $v \in V^F_{\infty,\mathbb{R}}$ such that $F_v \otimes_F E \simeq F_v \times F_v \times F_v$ and $v \in S_G''$.

4.3.2. $E \otimes_F F_v \simeq F_v \times \mathbb{C}$

In this case, $[(a,b)_E]$ has norm $[(a,b_1)_{F_v}] \cdot \mathrm{Res}_{\mathbb{C}/\mathbb{R}}([\mathrm{M}_2(\mathbb{C})]) = [(a,b_1)_{F_v}]$, where $b_1$ is the image of $b$ under the map

$$E \hookrightarrow E \otimes_F F_v \xrightarrow{\pi_1} \mathbb{R} \times \mathbb{C}.$$

By the restriction that $N_{E/F}([(a,b)_E]) = 1$, we therefore get that $(a,b_1)_E$ becomes split over $F_v$. Because we chose $a$ such that $F(\sqrt{a})$ is purely imaginary,

$\mathrm{Sign}_v(a) = -1$ and so $\mathrm{Sign}_v(b_1) = 1$. The next lemma gives us the structure of $P \otimes_F F_v$.

LEMMA 4.10. *If $E \otimes_F F_v \simeq F_v \times \mathbb{C}$, then $P \otimes_F F_v \simeq \mathbb{R} \times \mathbb{R} \times \mathbb{C}$.*

*Proof.* First, recall that if $G_0$ is as in [CLM] then $G_{0,\alpha_1,\alpha_3,\alpha_4}$ has maximal torus $R_{E/F}(R^{(1)}_{E(\sqrt{b})/E}(\mathrm{G_m}))$, which becomes $\mathrm{G_m} \times R_{\mathbb{C}/F_v}(\mathrm{G_m})$ over $F_v$. So up to relabeling, $\mathrm{Gal}(\mathbb{C}/F_v)$ acts by fixing $\alpha_1$ and sending $\alpha_3 \mapsto \pm\alpha_4$. Next, by [CLM], $\tilde{\alpha}$ is fixed and so $\mathrm{Gal}(\mathbb{C}/F_v)$ acts on $\Phi = \{\alpha_2, \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_1 + \alpha_3, \alpha_2 + \alpha_1 + \alpha_4\}$ by fixing two elements and permuting the other two (which elements are fixed and which are permuted depends on the sign of $\alpha_3 \mapsto \pm\alpha_4$). This gives that $\tilde{H}_{F_v} \simeq \mathrm{SL}_2 \times \mathrm{SL}_2 \times R_{\mathbb{C}/F_v}(\mathrm{SL}_2)$; thus $P \otimes_F F_v \simeq \mathbb{R} \times \mathbb{R} \times \mathbb{C}$. □

As in the case $E \otimes_F F_v \simeq F_v \times F_v \times F_v$, it is necessary to understand the sign of $p$ under the maps $\psi_{i,v}\colon P \hookrightarrow P \otimes_F F_v \xrightarrow{\pi_i} F_v$ for $i = 1, 2$. How the sign of $\psi_{i,v}(p)$ is controlled will depend on the form that $\bar{G}$ takes over $F_v$. From the restriction that $E \otimes_F F_v \simeq F_v \times \mathbb{C}$ we have that $\bar{G}$ is of type ${}^2D_4$ over $F_v$, so Tits's classification gives two possibilities: either $\bar{G}_{F_v}$ is quasi-split of rank 3 or $\bar{G}_{F_v}$ has rank 1.

Let $\Psi_3 \subset V^F_{\infty,\mathbb{R}}$ be the set of all places of $F$ such that $E \otimes_F F_v \simeq F_v \times \mathbb{C}$ and $G$ becomes quasi-split over $F_v$, and let $\Psi_4 \subset V^F_{\infty,\mathbb{R}}$ be the set of all places of $F$ such that $E \otimes_F F_v \simeq F_v \times \mathbb{C}$ and $G$ has rank 1 over $F_v$.

LEMMA 4.11. *There exists a $p \in P$ satisfying the conditions of Lemma 4.7 and such that $\psi_{i,v}(p)$ is positive in $F_v$ if $v \in \Psi_3$ and is negative in $F_v$ if $v \in \Psi_4$.*

*Proof.* The proof is identical to the proof of Lemma 4.7 with one exception. Recall the definitions of $S$ and $\bar{S}$ from Lemma 4.3. Although we do not have that $S$ is split in this case, we do still have that $H^1(F_v, S) = H^1(F_v, \mathrm{G_m} \times R_{\mathbb{C}/\mathbb{R}}(\mathrm{G_m})) = 1$; and the same arguments as in Lemma 4.7 then give that $[(\alpha_1, \beta_1)_{F_v}]$ and $[(\alpha_2, \beta_2)_{F_v}]$ as defined there are split (here there are no $[(\alpha_3, \beta_3)_{F_v}]$ or $[(\alpha_4, \beta_4)_{F_v}]$, since $P \otimes_F F_v \simeq F_v \times F_v \times \mathbb{C}$). □

Now, choosing $p$ as in Lemma 4.11, I claim that there exist $[\nu_v] \in H^1(F_v, \bar{T}_{F_v})$ that map to $[\xi]_v$ for all $v \in \Psi_3 \cup \Psi_4$. This is proven in an analogous manner to the case where $E \otimes_F F_v \simeq F_v \times F_v \times F_v$, with a few exceptions. Namely, in this case $G_{0,F_v} \simeq \mathrm{Spin}(f_3)$. Recall the definition of $T_0 \le G_0$ and the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action described in Lemma 4.10. Up to renumbering, if we let $G_1$ be the subgroup of $G_0$ generated by the root subgroups corresponding to $\{\alpha_2, \alpha_2 + \alpha_3 + \alpha_4\}$ then $G_1 \simeq \mathrm{SL}_2 \times \mathrm{SL}_2$, and if we let $G_2$ be the subgroup generated by the root subgroups corresponding to $\{\alpha_2 + \alpha_1 + \alpha_4, \alpha_2 + \alpha_1 + \alpha_3\}$ then $G_2 \simeq R_{\mathbb{C}/\mathbb{R}}(\mathrm{SL}_2)$.

LEMMA 4.12. *Given $(V, f_3)$ with $f_3$ as defined in Section 4.1, there exist $V_1, V_2 \subset V$ such that $V = V_1 \oplus V_2$ and $V_2 = V_1^{\perp}$ under $(\cdot,\cdot)_{f_3}$. Also, if $g_1 = f_3|_{V_1}$ and $g_2 = f_3|_{V_2}$ then $f = g_1 \oplus g_2$ and, up to isogeny, $G_i \le G_{0,F_v}$ is given by $\mathrm{SO}(g_i) \le \mathrm{SO}(f_3)$.*

*Proof.* As in Lemma 4.8. □

Recall that we have $\mathrm{Spin}(g_i) \simeq R_{F_v(\sqrt{\mathrm{disc}(g_i)})/F_v}(\mathrm{SL}_1(T))$, where $T$ is a quaternion algebra over $F_v(\sqrt{\mathrm{disc}(g_i)})$. Because $G_1$ is split, $g_1$ is as well; in contrast, $G_2$ has no $F_v$-defined subgroups of type $A_1$ and so $g_2$ has nontrivial discriminant. This means that, up to multiplication by $\pm 1$, we have

$$g_1 = x_1^2 - x_2^2 + x_3^2 - x_4^2,$$
$$g_2 = y_1^2 + y_2^2 + y_3^2 - y_4^2,$$

and the criterion that $g_1 \oplus g_2 = f_3$ means that we can choose $g_i$ as above.

If $G_{F_v}$ has rank 3 then $G_{F_v} \simeq G_{0,F_v}$, so $[\xi]_v$ is trivial and $1 \in H^1(F_v, \bar{T})$ maps to $[\xi]_v$. If $G_{F_v}$ has rank 1 then recall that, by our choice of $p$, we have $T_1 = T \cap G_1 \simeq R^{(1)}_{\mathbb{C}/F_v}(\mathrm{G_m}) \times R^{(1)}_{\mathbb{C}/F_v}(\mathrm{G_m})$. Let $S_1 = \mathrm{Spin}(x_1^2 + x_3^2) \times \mathrm{Spin}(-x_2^2 - x_4^2) \le G_1$. Since any two anisotropic tori over $\mathbb{R}$ are conjugate, it follows that if $\bar{T}_1$ and $\bar{S}_1$ are the images of $T_1$ and $S_1$ in $\mathrm{PSO}(g_1)$ then the images of $H^1(F_v, \bar{T}_1)$ and $H^1(F_v, \bar{S}_1)$ in $H^1(F_v, \mathrm{PSO}(g_1))$ are the same. Let $T_1'$ and $S_1'$ be the images of $T_1$ and $S_1$ in $\mathrm{SO}(g_1)$, and let $z_1 = (1, -1) \in S_1'$. If we let $[\gamma_v] \in H^1(F_v, \bar{S}_1)$ be given by $(\gamma_v)_\tau = \bar{z}_1 \in \bar{S}_1$, let $[\gamma_v'] \in H^1(F_v, \bar{T}_1)$ be chosen such that $\mathrm{Im}([\gamma_v']) = \mathrm{Im}([\gamma_v]) \in H^1(F_v, \mathrm{PSO}(g_1))$.

Let $[\nu_v] \in H^1(F_v, \bar{T})$ be the image of $[\gamma_v']$ under the map $H^1(F_v, \bar{T}_1) \to H^1(F_v, \bar{T}_{F_v})$. Let $g_{11} = x_1^2 + x_2^2$ and $g_{12} = -x_2^2 - x_4^2$ so that $g_1 = g_{11} \oplus g_{12}$. As in Lemma 4.9, direct calculation shows that ${}_{\nu_v}\mathrm{SO}(f_3) \simeq \mathrm{SO}(f_1)$. We thus have our next lemma.

LEMMA 4.13. *In the situation just described,* $[\nu_v] \mapsto [\xi]_v$ *under* $H^1(F_v, \bar{T}_{F_v}) \to H^1(F_v, \bar{G}_{0,F_v})$.

NOTE. For every $v \in S_G$ such that $E \otimes_F F_v \simeq F_v \times \mathbb{C}$, we have that $T_{F_v}$ has rank 1 whenever $v \in S_G'$ but is of higher rank whenever $v \in S_G''$.

### *4.4. Concluding Argument*

*Proof of Proposition 4.1.* Thus far we have constructed a torus $\bar{T} \le \bar{G}_0$ such that:

(1) there exists a $[\gamma] \in H^1(F, \bar{T})$ that maps to $[\xi] \in H^1(F, \bar{G}_0)$;
(2) $T \le H$, where $H \le G_0$ is a simple group of type $A_1 \times A_1 \times A_1 \times A_1$; and
(3) $T$ has appropriate real rank.

If we let $[\chi]$ be the image of $[\gamma]$ in $H^1(F, \bar{H})$, then $[\chi] \mapsto [\xi]$ under $H^1(F, \bar{H}) \to H^1(F, \bar{G}_0)$ and so ${}_\chi H \le G$. Also, ${}_\chi H$ is a simple group and ${}_\lambda T = T \le {}_\chi H$. This means that ${}_\chi H$ is a proper simple subgroup of $G$ that is of appropriate real rank. □

NOTE. Allison [A] showed how to construct all central simple Lie algebras of type $D_4$ over an algebraic number field. It was pointed out by the referee that these results can also be used to obtain subgroups of $G$ of type $A_1 \times A_1 \times A_1 \times A_1$, at least one of which has appropriate real rank. We keep the original proof here because the same technique (i.e., modification of cocycles) is used to prove that groups of type ${}^{1,2}E_6$ are not minimal.

## 5. Type ${}^{1,2}E_6$

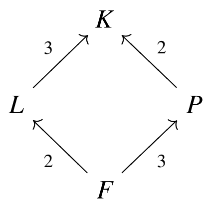
### *5.1. Type ${}^2E_6$*

PROPOSITION 5.1. *If $G$ is of type ${}^2E_6$, then $G$ contains a simple subgroup of type $A_5$ of appropriate real rank over real completions.*

5.1.1. *Construction of a Special Torus*

Let $G_0$ be the simply connected quasi-split group of type ${}^2E_6$, let $\bar{G}_0$ be the adjoint, and let $G = {}^{\xi}G_0$ for $[\xi] \in H^1(F, \bar{G}_0)$. Our strategy is to apply Lemma 4.2 to a torus that normalizes a subgroup of type ${}^2A_5$ with appropriate real rank. Recall that $Z(G_0) \simeq R^{(1)}_{L/F}(\mu_3)$, where $L/F$ is the unique quadratic extension of $F$ over which $G$ becomes inner, and recall that $H^2(F, Z(G_0)) \simeq \ker({}_3\mathrm{Br}(L) \xrightarrow{\mathrm{cor}} {}_3\mathrm{Br}(F))$. Let $[D]$ be the image of $[\xi]$ in $H^2(F, Z(G_0))$, and let $\tau$ be the involution of the second kind on $D$ fixing $F$. The first step in applying Lemma 4.2 is to show that we can choose $T \leq G_0$ such that $[D]$ is in the image of $H^1(F, \bar{T}) \to H^2(F, Z(G_0))$.

First, some notation. Let $K \subset D$ be any maximal subfield of $D$, and let $P = K^{\tau}$ (note that $K = P \otimes_F L$, with $\tau$ acting on the second component by [PrR, Proof of Prop. 2.1]). This means that we have the following diagram of extensions:




LEMMA 5.1. *Let $T_0$ be an $F$-defined maximal quasi-split torus of $G_0$ and let $\Sigma' \subset \Sigma(G_0, T_0)$ be the root subsystem of type ${}^2A_5$ generated by roots $\{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$. Let $H_0 \leq G_0$ be the subgroup generated by the root subgroups $G_\alpha \leq G_0$ for $\alpha \in \Sigma'$, and let $T_1$ be any maximal torus of $H' = \mathrm{SU}_2(D, \langle -1, 1\rangle)$. Then there exists an embedding $T_1 \hookrightarrow G_0$ such that $[D] \in \mathrm{Im}(H^1(F, \bar{T}_1) \to H^2(F, Z(G_0)))$.*

*Proof.* Let $\tilde{H}_0 = H_0/Z(H_0)$ and $\tilde{T}_1 = T_1/Z(H_0)$. Then $H'$ is a form of $H_0$ and so there exists a $[\lambda'] \in H^1(F, \tilde{H}_0)$ such that ${}^{\lambda'}H_0 = H'$. By Steinberg's theorem, there exists an embedding $T_1 \hookrightarrow H_0$ such that $[\lambda'] \in \mathrm{Im}(H^1(F, \tilde{T}_1) \to H^1(F, \tilde{H}_0))$. Let $[\mu'] \in H^1(F, \tilde{T}_1)$ be chosen such that $[\mu'] \mapsto [\lambda']$. Let $[\chi']$ be the image of $[\mu']$ in $H^2(F, Z(H_0))$. Note that $H'$ becomes quasi-split over $P$; hence $[\lambda']$ (and $[\chi']$) become split over $P$ as well. This means that $|[\chi']|$ divides 3 in $H^2(F, Z(H_0))$.

Note that $Z(H_0) = R^{(1)}_{L/F}(\mu_6)$ and $Z(G_0) = R^{(1)}_{L/F}(\mu_3)$ fit in the exact sequence

$$1 \to Z(G_0) \to Z(H_0) \to \mu_2 = R^{(1)}_{L/F}(\mu_2) \to 1 \qquad (\dagger)$$

and that this sequence splits. We can use these facts to construct the following diagram with exact columns:

$$\begin{array}{ccccc} H^2(F,\mu_2) & \xrightarrow{=} & H^2(F,\mu_2) & \xrightarrow{=} & H^2(F,\mu_2) \\ \uparrow & & \uparrow & & \uparrow \\ H^1(F,\tilde{T}_1) & \longrightarrow & H^1(F,\tilde{H}) & \longrightarrow & H^2(F,Z(H_0)) \\ \uparrow & & \uparrow & & \uparrow \\ H^1(F,\bar{T}_1) & \longrightarrow & H^1(F,\bar{H}_0) & \longrightarrow & H^2(F,Z(G_0)) \end{array}$$

Because $[\chi']$ has order dividing 3, its image in $H^2(F,\mu_2)$ is trivial; because the diagram commutes, there exist a $[\mu] \in H^1(F,\bar{T}_1)$ and a $[\lambda] \in H^1(F,\bar{H}_0)$ such that $[\mu] \mapsto [\mu']$ and $[\lambda] \mapsto [\lambda']$ under the maps in the diagram. Let $[\chi]$ be the image of $[\lambda]$ in $H^2(F,Z(G_0))$, and consider the following diagram:

$$\begin{array}{ccc} H^2(F,Z(G_0)) & \longrightarrow & H^2(F,Z(H_0)) \\ \text{Res}\downarrow & & \downarrow\text{Res} \\ H^2(L,Z(G_0)) & \longrightarrow & H^2(L,Z(H_0)) \end{array}$$

here the horizontal arrows are injections because the sequence (†) is exact. The vertical arrow on the left-hand side is injective because Cor $\circ$ Res is multiplication by $[L:F]=2$ and $H^2(F,Z(G_0))$ is a 3-torsion group. Thus, to prove that $[\lambda] \in H^1(F,\bar{H}_0)$ maps to $[D]$ in $H^2(F,Z(G_0))$, it suffices to show that $[\chi]_L = [D]_L$. Recall that if $[\alpha] \in H^1(F,\mathrm{PGL}_n)$ has ${}^{\alpha}\mathrm{SL}_n = \mathrm{SL}_1(A)$ for $A$ a central simple algebra of degree $n$ (not necessarily a division algebra), then $[A] = \mathrm{Im}([\alpha]) \in H^2(F,\mu_n) = {}_n\mathrm{Br}(F)$.

The proof is then completed by noticing that

$$^{\lambda}(H_0)_L = \mathrm{SL}_2(D) \quad \text{and} \quad H^2(L,Z(G_0)) \hookrightarrow H^2(L,Z(H_0)). \qquad \square$$

Let $\tilde{\alpha}$ be the root of maximal height in the root system of $G_0$, and let $G_{\tilde{\alpha}}$ be the corresponding root subgroup. Then $G_{\tilde{\alpha}}$ commutes with $H_0$. I aim to construct a torus $T$ that is the almost direct product of maximal tori $T_1 \le G_{\tilde{\alpha}}$ and $T_2 \le H_0$.

For $T_1$, choose $a \in F$ such that $a$ is positive in $F_v$ for all $v \in V^F_{\infty,\mathbb{R}}$ such that $G_{F_v}$ is split or quasi-split and negative otherwise, and let $T_1 = R^{(1)}_{F(\sqrt{a})/F}(\mathrm{G_m})$ be embedded in $G_{\tilde{\alpha}}$ via the regular embedding.

Next, we construct $T_2$. Let $\sigma$ be the involution on $M_2(D)$ corresponding to the $\tau$-hermitian form $\langle 1,-1\rangle$. Recall from the classification of minimal groups of type ${}^2A_n$ that, given local constructions $E_v \subset M_2(D)\otimes_F F_v$ such that $E_v^{\tau_v}$ has dimension $n$ for every $v \in V^F_{\infty,\mathbb{R}}$, there exists a subfield $E \subset M_2(D)$ such that $(E\otimes_F F_v, \tau\otimes 1) \simeq (E_v,\tau_v)$ (see [PrR, Proof of Thm. 5.1, p. 135] and [PrR, Apx. A, pp. 176–178]). We break the local construction into the following three cases.

(i) If $\mathrm{rank}_{F_v}(G) = 0$ then, by Tits's classification, $G$ remains outer over $F_v$ in this case; thus $(M_2(D) \otimes F_v, \langle 1, -1\rangle) \simeq (M_6(\mathbb{C}), \langle 1, -1, 1, -1, 1, -1\rangle)$. Let $E_v = \mathbb{C}^6$ embed via diagonal matrices, so $E_v^{\tau_v} = \mathbb{R}^6$ and the maximal torus of $\mathrm{SU}_6(\mathbb{C}, \langle 1, -1, 1, -1, 1, -1\rangle)$ corresponding to $E_v$ is anisotropic.

(ii) If $G_{F_v}$ is isotropic of outer type, we have that $(M_2(D) \otimes F_v, \langle 1, -1) \simeq (M_6(\mathbb{C}), \langle -1, -1, -1, 1, 1, 1\rangle)$. Note that $M_3(\mathbb{R}) \times M_3(\mathbb{R}) \subset M_6(\mathbb{C})^{\tau_v}$ in this case, so we can embed $F_v = (\mathbb{R} \times \mathbb{C}) \times (\mathbb{R} \times \mathbb{C}) \subset M_6(\mathbb{C})^{\tau_v}$ by first embedding $\mathbb{R} \times \mathbb{C} \subset M_3(\mathbb{R})$ via the regular representation along the diagonal and then taking the product of this embedding with itself. We then let $E_v = F_v \otimes_{\mathbb{R}} \mathbb{C} \hookrightarrow M_6(\mathbb{C})$ via $(M_3(\mathbb{R}) \times M_3(\mathbb{R}) \otimes \mathbb{C} \hookrightarrow M_6(\mathbb{C})$. Then

$$\{x \in E_v \mid x\tau_v(x) = 1 = \mathrm{Nrd}(x)\} = \{(z_1, z_2, z_2^{-1}, z_1^{-1}, z_4, z_4^{-1}) \mid N_{\mathbb{C}/\mathbb{R}}(z_1) = 1\},$$

so the maximal torus of $\mathrm{SU}_6(\mathbb{C}, \langle 1, -1, 1, -1, 1, -1\rangle)$ corresponding to $E_v$ in this case has $F_v$-rank 2.

(iii) If $G_{F_v}$ is isotropic of inner type, let $E_v = \mathbb{C}^3 \times \mathbb{C}^3 \hookrightarrow M_6(\mathbb{R}) \times M_6(\mathbb{R})^{op}$ with exchange involution (embedded via the regular embedding). Then the maximal torus of $\mathrm{SL}_6(\mathbb{R})$ corresponding to $E_v$ is

$$\{(z_1, z_2, z_3) \mid N_{\mathbb{C}/\mathbb{R}}(z_1 z_2 z_3) = 1\},$$

which has rank 2 over $\mathbb{R}$.

Let $E \subset \mathrm{M}_2(D)$ be a maximal subfield such that $(E \otimes_F F_v, \tau \otimes 1) \simeq (E_v, \tau_v)$ for each $v \in V^F_{\infty,\mathbb{R}}$, and let $T_2 = \{x \in E \mid x\tau(x) = 1 = \mathrm{Nrd}(x)\}$. By Lemma 5.1 there exists an embedding $\phi\colon T_2 \hookrightarrow G_0$ such that $[D] \in \mathrm{Im}(H^1(F, \bar{T}_2) \to H^2(F, Z(G_0)))$. Let $T = \phi(T_2) \cdot T_1$; then there exists a $[\mu] \in H^1(F, \bar{T})$ such that $[\mu] \mapsto [D]$.

5.1.2. *Modification of* $[\mu]$

In order to apply Lemma 4.2 to $[\mu]$, it suffices to show that $[\xi]_v$ is in the image of $H^1(F_v, T) \to H^1(F_v, G_0)$ for every $v \in V^F_{\infty,\mathbb{R}}$. When $G_v$ is split, we may choose the trivial cocycle in $H^1(F_v, T)$. When $G_v$ is anisotropic, $T$ is anisotropic over $F_v$ by construction and so $H^1(F_v, T) \twoheadrightarrow H^1(F_v, G_0)$ by [Bo, Thm. 1]. Thus it remains to address the cases where $G_v$ is isotropic but not split.

If $G_v$ is inner then $|H^1(F_v, G_0)| = 2$, so it suffices to prove that the image of $H^1(F_v, T)$ in $H^1(F_v, G_0)$ is nontrivial. Also, if $G_v$ is outer of rank 2 then $T_v$ is also rank 2; hence any twist by a cocycle in $T_v$ will also have rank at least 2. We have that $|H^1(F_v, G_0)| = 3$ by Tits's classification, where one element is trivial and another corresponds to the anisotropic group. If $1 \neq [\chi]$ is in the image of $H^1(F_v, T)$ in $H^1(F_v, G_0)$, then ${}^{\chi}G_0$ is neither split nor anisotropic and so must be equal to $[\xi]_v$. Thus it suffices to prove that the image of $H^1(F_v, T)$ in $H^1(F_v, G_0)$ is nontrivial as well.

Lemma 5.2. *If $T$ is nonsplit over $F_v$, then the image of $H^1(F_v, T) \to H^1(F_v, G_0)$ is nontrivial.*

*Proof.* If $G_0$ is inner over $F_v$, then $T$ has rank 2 over $F_v$; thus the anisotropic part of $T_a$ over $F_v$ has rank 4 and hence is maximal anisotropic (see Proposition 5.3 to follow). Therefore, $H^1(F_v, T_a) \twoheadrightarrow H^1(F_v, G_0)$ by [Bo]; in particular, the image of $H^1(F_v, T) \to H^1(F_v, G_0)$ is nontrivial.

If $G_0$ is outer over $F_v$ then let $T = T_1 \cdot T_2$, where $T_1$ is split of rank 2 over $F_v$ and $T_2$ is anisotropic of rank 4. Then $C_{G_0}(T_2)$ is a reductive group and so $C_{G_0}(T_2) = H \cdot S$, where $S$ is a torus in $G_0$ containing $T_2$ and $H$ is semisimple.

*Claim.* $S = T_2$.

Suppose not. If $H$ is trivial, then $C_{G_0}(T_2) = T$. But $G_0$ contains a maximal anisotropic torus containing $T_2$, and $T$ has rank 2—a contradiction.

If $H$ has rank 1, then $C_{G_0}(T_2) = \mathrm{SL}_2 \cdot S$. Let $T_a$ be a maximal torus of $G_0$ that is anisotropic over $F_v$ and contains $T_2$; then $T_a \subset \mathrm{SL}_2 \cdot S$ yields that $T_a \cap S$ has dimension 5 and $S$ is anisotropic. In particular: $C_{G_0}(T_2)$ has rank 1 but $T \subset C_{G_0}(T_2)$ has rank 2, a contradiction. This proves the claim.

Because $H$ is standard of rank 2, $H$ may be either of type $A_1 \times A_1$ or of type $A_2$ (if $H$ were of type $G_2$ or $B_2$ then $H$ would have roots of different lengths, which is impossible). In either case, $H$ contains a split subgroup of type $A_1$. If $\tilde{\alpha}$ is the root of maximal height in $E_6$, then we may assume (after conjugation) that $G_{\tilde{\alpha}} \leq H$. Then $T_2 \subset C_{G_0}(H) \subset C_{G_0}(G_{\tilde{\alpha}})$, so we can consider $C_{C_{G_0}(G_{\tilde{\alpha}})}(T_2)$. We have $C_{C_{G_0}(G_{\tilde{\alpha}})}(T_2) = H' \cdot S'$, where $H'$ is semisimple and $S'$ is a torus containing $T_2$, as before.

Note that $C = C_{G_0}(G_{\tilde{\alpha}})$ is standard in $G_0$ of type ${}^2A_5$. Thus $C$ contains an anisotropic torus of rank 5. Arguing as in the claim, we see that $S' = T_2$ and $H' \simeq \mathrm{SL}_2$. Let $\tilde{\beta}$ be the root of maximal height in $A_5$. After conjugation by an element of $C$, we may assume that $H' = G_{\tilde{\beta}}$. Then $C_C(H') = H'' \cdot S''$, where $H''$ is of type ${}^2A_3$ and $S''$ is anisotropic of dimension 1. Then $T_2 \cap H''$ is a maximal torus of $H''$, which is also maximal. By [Bo], it follows that there exists an element $[\alpha]$ of $H^1(F_v, T_2 \cap H'')$ such that ${}^{\alpha}H''$ is compact. It suffices to show that the image of $[\alpha]$ in $H^1(F_v, G_0)$ is nontrivial.

To see this, first note that because ${}^{\alpha}H'' \leq {}^{\alpha}C$ is standard, if ${}^{\alpha}H'' = \mathrm{SU}(\mathbb{C}, f_4)$ for a compact hermitian form $f_4$ then ${}^{\alpha}C = \mathrm{SU}(\mathbb{C}, f_4 \oplus f_2)$ for some hermitian 2-form $f_2$. Thus the maximum possible rank of ${}^{\alpha}C$ is 2, so the image of $[\alpha]$ in $H^1(F_v, C)$ is nontrivial.

To complete the proof, it suffices to show that, if $[\alpha] \in H^1(F_v, C)$ maps to the trivial cocycle in $H^1(F_v, G_0)$, then $[\alpha]$ is trivial. Recall that $C$ commutes with $G_{\tilde{\alpha}}$ by definition of $C$, so for any $[\alpha] \in H^1(F_v, C)$ we have ${}^{\alpha}G_{\tilde{\alpha}} = G_{\tilde{\alpha}}$. Let $T_0$ be a split torus sitting in $G_{\tilde{\alpha}}$, and consider $C_{{}^{\alpha}G_0}(T_0)$. Because ${}^{\alpha}C \leq C_{{}^{\alpha}G_0}(T_0)$ and $C_{{}^{\alpha}G_0}(T_0)$ is reductive, we have that $C_{{}^{\alpha}G_0}(T_0) = T_0 \cdot {}^{\alpha}C$. Thus the maximum possible rank of any torus containing $T_0$ is $1 + 2 = 3$, but if ${}^{\alpha}G_0$ is split then $T_0$ is contained in a maximal split torus in ${}^{\alpha}G_0$, which has rank 4—a contradiction. □

5.1.3. *Concluding Argument*

*Proof of Proposition 5.1.* Applying Lemma 4.2, we see that there exists a cocycle $[\gamma] \in H^1(F, \bar{T})$ such that $[\gamma] \mapsto [\xi]$. Since $T$ normalizes a group of type ${}^2A_5$ containing $T_2$, it follows that ${}^{\xi}G_0 = G$ contains a subgroup $H$ of type ${}^2A_5$ that contains $T_2$. Because $T_2$ has appropriate real rank by construction, so does $H$ and therefore $G$ is not minimal. □

### 5.2. *Type* $^1E_6$

For the duration of this section, $G_0$ will be a simply connected split group of type $^1E_6$ and $G$ will be a twist of $G_0$ corresponding to $[\xi] \in H^1(F, \bar{G}_0)$. We then have that $Z(G_0) = \mu_3$ over $F$ and thus $H^2(F, Z(G_0)) \simeq {}_3\mathrm{Br}(F)$. Note that, over a number field $F$, any element of ${}_3\mathrm{Br}(F)$ corresponds to a unique cyclic algebra of degree 3. Let $[D] \in {}_3\mathrm{Br}(F)$ be the image of $[\xi]$ in $H^2(F, Z(G_0))$ with $D$ a cyclic division algebra of degree 3, and let $L \subset D$ be a maximal subfield that is Galois over $F$.

Proposition 5.2. *$G$ contains a simple subgroup of type $A_5$ of appropriate real rank.*

5.2.1. *Construction of a Special Torus $T$*

Let $\{\alpha_1, \ldots, \alpha_6\}$ be a basis of the root system $\Sigma(G_0, T_0)$, where $T_0$ is any maximal split torus of $G_0$. Let $\Sigma'$ be the subsystem of $E_6$ generated by the roots $\{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$. We then have that $G_{0,\Sigma'} =: H$ is a split subgroup of type $^1A_5$; that is, $H \simeq \mathrm{SL}_6$.

Let $P = F(\sqrt{-1})$ and $L$ be as above; then $L \cdot P$ is a Galois extension of degree 6 over $P$. Consider the regular representation $R^{(1)}_{L\cdot P/F}(\mathrm{G_m}) \hookrightarrow H$, and let $T_1$ be the image of this representation so that $T_1$ is an anisotropic torus in $H_0$ of absolute rank 5. Let $T_2 \le G_{0,\tilde{\alpha}}$ be $R^{(1)}_{P/F}(\mathrm{G_m})$ and define $T = T_1 \cdot T_2 \le G_0$.

Lemma 5.3. *There exists a $[\mu] \in H^1(F, \bar{T})$ such that $[\mu] \mapsto [D]$ under $H^1(F, \bar{T}) \to H^2(F, Z(G_0))$.*

*Proof.* Consider the diagram

$$
\begin{array}{ccccc}
\mu_2 & \longrightarrow & \mu_2 & & \\
\downarrow & & \downarrow & & \\
\mu_6 & \longrightarrow & T_1 \times T_2 & \longrightarrow & \bar{T} \\
\downarrow & & \downarrow & & \downarrow \\
Z(G_0) & \longrightarrow & T & \longrightarrow & \bar{T}
\end{array}
$$

with exact columns and rows. This gives a diagram of interconnected long exact sequences with segment

$$
\begin{array}{ccccccc}
H^1(F, T_1 \times T_2) & \longrightarrow & H^1(F, \bar{T}) & \xrightarrow{\phi_1} & H^2(F, \mu_6) & \xrightarrow{\phi_2} & H^2(F, T_1 \times T_2) \\
\downarrow & & \downarrow & & \downarrow \phi_3 & & \\
H^1(F, T) & \longrightarrow & H^1(F, \bar{T}) & \xrightarrow{\phi_4} & H^2(F, \mu_3) & &
\end{array}
$$

By commutativity, $\mathrm{Im}(\phi_4) = \mathrm{Im}(\phi_3 \circ \phi_1) = \phi_3(\ker(\phi_2))$. Then using Shapiro's lemma, we have that

$$H^2(F, T_1 \times T_2) = \ker(\mathrm{Br}(L \cdot P) \xrightarrow{\text{Norm}} \mathrm{Br}(F)) \times \ker(\mathrm{Br}(P) \xrightarrow{\text{Norm}} \mathrm{Br}(F)).$$

Recall that elements of ${}_6\mathrm{Br}(F)$ can be written in the form $[D_1 \otimes D_2]$, with $D_1$ cubic cyclic and $D_2$ a quaternion algebra, because $F$ is a number field. The map $\mu_6 \to T_1 \times T_2$ takes $\xi_6 \mapsto (\xi_6, \xi_6^3)$, so

$$\begin{aligned}\phi_2([D_1 \otimes D_2]) &= ([D_1 \otimes_F D_2 \otimes_F L \cdot P], [D_1 \otimes_F D_2 \otimes F]^3) \\ &= ([D_1 \otimes_F D_2 \otimes_F L \cdot P], [D_2 \otimes_F P]).\end{aligned}$$

If $[D_1 \otimes_F D_2]$ is in the kernel of this map, then $D_2$ is split by $P$ and $D_1 \otimes_F D_2$ is split by $L \cdot P$. The first condition gives that $D_1$ is split by $L \cdot P$ and so, because the degree of $D_1$ is relatively prime to the degree of $P$ over $F$, we have that $D_1$ is split over $L$. This means that the kernel of $\phi_2$ is given by $\{[D_1 \otimes D_2] \in {}_6\mathrm{Br}(F) \mid [D_1 \otimes L] = 1 = [D_2 \otimes P]\}$. The map $\mu_6 \to \mu_3$ is given by squaring, so $\phi_3([D_1 \otimes_F D_2]) = [D_1 \otimes_F D_2]^2 = [D_1]^{-1}$. Combining these results gives that $[D]$ is in the image of $\phi_4$ if and only if $[D]^{-1}$ contains $L$ as a maximal subfield, which is true because $[D]$ is assumed to contain $L$ and $[D]^{-1} = [D^{op}]$. Thus we have proved the existence of $[\mu]$. □

### 5.3. *Concluding Argument*

*Proof of Proposition 5.2.* By Lemma 5.2 we have that the image of $H^1(F_v, T) \to H^1(F_v, G_0)$ is nontrivial for every $v \in V^F_{\infty,\mathbb{R}}$ such that $T$ has rank 2 over $F_v$. Because $|H^1(F_v, G_0)| = 2$, we can apply Lemma 4.2 to $[\mu]$ from Lemma 5.3 to see that there exists a $[\gamma] \in H^1(F, \bar{T})$ such that $[\gamma] \mapsto [\xi]$. Then ${}^\gamma H$ is a simple subgroup of $G = {}^\xi G_0$ containing $T_2$; hence ${}^\gamma H$ has appropriate real rank because $T_2$ does and thus $G$ is not minimal. □

### 5.4. *Anisotropic Tori in $E_6$ over $\mathbb{R}$*

The following was used in the proof of Lemma 5.2.

PROPOSITION 5.3. *Over $\mathbb{R}$, any maximal anisotropic torus of a split group $G_0$ of type $E_6$ has absolute rank 4.*

*Proof.* Because all maximal anisotropic tori are conjugate, it suffices to prove that there exists an anisotropic torus of rank 4 in $G_0$ that is not properly contained in a larger anisotropic torus. Using the numbering found in [B], consider the subgroup $H_0$ of type ${}^1D_4$ generated by the root subgroups $G_{\alpha_2}, G_{\alpha_3}, G_{\alpha_4}, G_{\alpha_5}$. This subgroup is isogenous to the group $\mathrm{SO}_8\big(\sum_{i=1}^4 x_i^2 - \sum_{i=1}^4 y_i^2\big)$ and therefore contains an anisotropic torus of rank 4 (take products of the $\mathrm{SO}(x_i^2 + x_{i+1}^2)$). Call this torus $T$.

*Claim.* $C_{G_0}(T)$ *is a torus.*

Note that this claim holds over $F$ if it holds over $\bar{F}$. To prove this claim, take a maximal torus of $G_0$ that includes $T$ and then consider the root system of $G_0$ with respect to this torus over the closure. Because $T$ is a torus, $C_{G_0}(T)$ is reductive;

hence $C_{G_0}(T)$ is the almost direct product of a central torus and its derived subgroup. The derived subgroup is generated by those root subgroups that commute with $T$, of which I claim there are none. This may be proved by computing

$$h_{\alpha_2}(t_2)h_{\alpha_3}(t_3)h_{\alpha_4}(t_4)h_{\alpha_5}(t_5)X_\alpha(h_{\alpha_2}(t_2)h_{\alpha_3}(t_3)h_{\alpha_4}(t_4)h_{\alpha_5}(t_5))^{-1}$$

and showing that it is not $X_\alpha$ for any $\alpha$. Indeed, if this is true for some $\alpha$ then $\langle\alpha_i,\alpha\rangle = 0$ for $i = 2,3,4,5$. If $\alpha = \sum_{i=1}^{8} c_i\varepsilon_i$ (again, in the notation of [B]), then these equations give

$$c_1 = -c_2, \quad c_1 = c_2, \quad c_2 = c_3, \quad c_3 = c_4;$$

these equalities imply $c_1 = c_2 = c_3 = c_4 = 0$, which is impossible for any root $\alpha \in E_6$. This proves the claim.

Any torus is contained in a maximal torus, so there is a maximal torus (call it $S$) contained in $C_{G_0}(T)$. Because $C_{G_0}(T)$ is also a torus, we must have that $C_{G_0}(T) = S$. Assume that $S$ contains a split torus of rank 2. If there is an anisotropic torus properly containing $T$, say $S'$, then we would have $S' \subset C_{G_0}(T) = S$ and so $S$ could have rank at most 1—a contradiction. Thus, it suffices to prove that $S$ contains a split torus of rank 2.

Note that, if $C_{G_0}(H_0)$ contains a split torus of rank 2, then $C_{G_0}(T)$ does as well. In order for an element $\prod h_{\alpha_i}(t_i)$ (recall that we take roots with respect to an $F$-split torus) to commute with $H_0$, we have the following restrictions on $t_i$:

$$t_2^2t_4 = 1, \quad t_1t_3^2t_4 = 1, \quad t_3t_4^2t_2t_5 = 1, \quad t_6t_4t_5 = 1.$$

Now elements of the form $h_{\alpha_1}(s^2t^2)h_{\alpha_2}(s)h_{\alpha_3}(t)h_{\alpha_4}(s^{-2})h_{\alpha_5}(t^{-1})h_{\alpha_6}(s^2t)$ form a 2-dimensional split torus that commutes with $H_0$ (and thus with $T$). □

## 6. Non–Absolutely Simple Groups

Collecting the results from Sections 2–5 completes the proof of Theorem 1.2. It remains to prove Theorem 1.3. Thus, we consider $G$ that is not absolutely simple. By [BOI, (28.8)], we know that simple algebraic groups over number fields that are not absolutely simple are the restriction of scalars of absolutely simple groups over finite extensions of $F$. Moreover, the following lemma shows that we may restrict ourselves to the case where $G$ is the restriction of a minimal absolutely simple group.

LEMMA 6.1. *If $G = R_{K/F}(H)$, where $H$ is an absolutely simple group over $K$ of absolute rank at least 2 and $H$ is not minimal, then $G$ is not minimal.*

*Proof.* Choose a subgroup $H' \leq H$ that has appropriate real rank over $K$. Consider $G' = R_{K/F}(H') \leq G$. This is proper because $H'$ is. For $v \in V_{\infty,\mathbb{R}}^F$ we have

$$G'_{F_v} = R_{K_{w_1}/F_v}(H'_{K_{w_1}}) \times \cdots \times R_{K_{w_s}/F_v}(H'_{K_{w_s}}),$$

where $w_i$ are the valuations on $K$ that restrict to $v$ on $F$. Assume $v \in S'_G$. If $K_{w_i} \simeq \mathbb{C}$ for some $i$, then $G_{F_v}$ has a factor of the form $R_{K_{w_i}/F_v}(H_{K_{w_i}})$ that has rank at least 2, which contradicts $v \in S'_G$. If $K_{w_i} \simeq \mathbb{R}$ for each $i$, then $H_{K_{w_i}}$ has rank 1 for some $i$ and so $H'_{K_{w_i}}$ has rank 1 as well; thus $G'$ has $F_v$-rank 1.

If $v \in S''_G$ and $w_i \in S''_H$ for some $i$, then $H'_{K_{w_i}}$ has higher rank and thus so does $G'_{F_v}$. Moreover, if $K_{w_i} \simeq \mathbb{C}$ for some $i$, then $G'$ also has $F_v$-rank at least 2 because $R_{K_{w_i}/F_v}(H')$ does. Thus, we may assume that no $w_i$ is in $S''_H$ and no $w_i$ has $K_{w_i} \simeq \mathbb{C}$. This gives that at least two $w_i$ are in $S'_H = S'_{H'}$, so $G'$ has appropriate $F_v$-rank. □

Notice that $\mathrm{SL}_1(D)$ and $\mathrm{SU}(D, \tau)$ are simply connected and have no $F$-defined proper semisimple subgroups for $\deg(D) = p$ prime. The following lemma strongly limits the possible simple subgroups $R_{K/F}(G)$ when $G$ has no semisimple $K$-defined subgroups.

LEMMA 6.2. *Suppose that $G = R_{K/F}(H)$, where $H$ is defined over $K$, is simply connected and has no proper semisimple subgroups defined over $K$. Then every $F$-simple proper subgroup of $G$ is isomorphic to $R_{P/F}(H')$ for $F \subset P \subsetneq K$, where $H'$ is defined over $P$ and $H'_K$ is isomorphic to $H_K$. In particular, if $G$ has proper $F$-simple subgroups, then $H$ admits descent to a subfield $P \subset K$.*

*Proof.* Suppose that $G' \leq G$ is a nontrivial proper semisimple subgroup of $G$ as before. Let $K \otimes_F K \simeq K \times K'$, where $K'$ is an étale extension of $K$ and $G_K \simeq H_K \times R_{K'/K}(H_1)$ for some $H_1$ defined over $K'$. Let $\pi$ be the projection $G_K \twoheadrightarrow H_K$. Then $\pi(G'_K)$ is a semisimple subgroup of $H_K$, so $\pi(G'_K)$ is either trivial or all of $H_K$.

Assume that the image of $G'_K$ under $\pi$ is trivial. Over $\bar{K}$, $G_{\bar{K}}$ becomes

$$H_{\bar{K}} \times \cdots \times H_{\bar{K}}$$

with $\Gamma = \mathrm{Gal}(\bar{K}/K)$ permuting the components of $G_{\bar{K}}$ transitively. Let $1 \neq g = (g_1, \ldots, g_n) \in G'_K(\bar{K})$ and suppose that $g_j \neq 1$. Because $\Gamma$ permutes the components of $G_{\bar{K}}$ transitively, there exists a $\sigma \in \Gamma$ such that the first component of $\sigma(g)$ is $\sigma(g_j)$. Then $\pi(\sigma(g)) = \sigma(g_j) \neq 1$, but $\sigma(g) \in G'_F(K)$ because $G'$ is $F$-defined and so $\pi(\sigma(g)) = 1$, a contradiction.

If $G'$ is absolutely simple then the kernel of $\pi$ is finite; hence, setting $H' = G'$ and $P = F$, we have that $\pi$ is a finite covering of $H_K$ by $H'_K$. By the assumption that $H$ is simply connected, we obtain that $\pi$ is an isomorphism.

If $G'$ is not absolutely simple, then $G' = R_{F'/F}(H')$ for some $H'$ absolutely simple over $F'$. Suppose $F' \otimes_F K \simeq K_1 \times \cdots \times K_\ell$ with $K_i/K$ finite field extensions. Then

$$G'_K \simeq R_{K_1/K}(H'_{K_1}) \times \cdots \times R_{K_\ell/K}(H'_{K_\ell}).$$

Let $\pi_i$ be the composition $R_{K_i/K}(H'_{K_i}) \hookrightarrow G'_K \xrightarrow{\pi} H_K$. If the images of all of the $\pi_i$ are trivial then the image of $\pi$ is trivial, which is impossible. Therefore, since $H_K$ contains no proper semisimple subgroups and since the $R_{K_i/K}(H'_{K_i})$ are $K$-simple, it follows that some $\pi_i$ is an $K$-defined isogeny. By the assumption that $H_K$ is simply connected, we get that $\pi_i$ is an isomorphism. If $K_i/K$ is a nontrivial field extension, then $\pi_i$ is an isomorphism between one group that is absolutely simple and one that is not, which is impossible. Hence $K_i = K$ and $\pi_i$ is an isomorphism $H'_K \twoheadrightarrow H_K$. Identifying $P$ with the image of $F'$ in $K_i = K$, we see that $H'$ is defined over $P$ and $G' = R_{P/K}(H')$, as required. □

This lemma allows us to handle several cases, as follows.

PROPOSITION 6.1. *If $G = R_{K/F}(\mathrm{SL}_1(D))$ for a central division algebra $D/K$ of prime degree $p \geq 3$, then $G$ is minimal if and only if $D$ does not descend to any subfield $F \subset P \subsetneq K$.*

*Proof.* Assume that $D$ does not descend. By Lemma 6.2, $G$ contains no proper $F$-simple subgroups in this case. If $D$ does descend, then $H = R_{P/F}(\mathrm{SL}_1(D'))$ is a proper $F$-simple subgroup of appropriate real rank. Indeed, by the assumption that $D'$ has prime degree $p \geq 3$, we must have that $D'$ is split over $P_w$ for all $w \in V^P_{\infty,\mathbb{R}}$. □

PROPOSITION 6.2. *If $G$ is of the form $R_{K/F}(\mathrm{SL}_1(D))$ for $D$ a quaternion algebra over $K$, then $G$ is minimal if and only if, for every $F \subset P \subsetneq K$ such that $D$ descends to $P$, there exists a $v_0 \in S_G$ such that:*

- *if $v_0 \in S'_G$, then $P_{w_i} \simeq \mathbb{R}$ and $D' \otimes_P P_{w_i} \simeq \mathbb{H}$ for all $w_i$ lying over $v_0$; and*
- *if $v_0 \in S''_G$, then there is at most one $w_i$ lying over $v_0$ such that either $P_{w_i} \simeq \mathbb{C}$ or $D' \otimes_P P_{w_i} \simeq M_2(\mathbb{R})$.*

*Proof.* Using Lemma 6.2, we find that all possible $F$-simple subgroups correspond to $F \subset P \subsetneq K$ such that $D$ descends to $P$. Then the conditions imposed upon such $P$ exactly yield that the corresponding subgroup cannot have appropriate real rank. □

EXAMPLE. Let $K = \mathbb{Q}(\sqrt[3]{2}, \sqrt{3})$, $D = (-1,-1)$, $F = \mathbb{Q}$, and $G = R_{K/\mathbb{Q}}(\mathrm{SL}_1(D))$. Then $K$ has two real and two complex completions, so

$$G_{\mathbb{R}} \simeq \mathrm{SL}_1(D) \times \mathrm{SL}_1(D) \times R_{\mathbb{C}/\mathbb{R}}(\mathrm{SL}_2(\mathbb{C})) \times R_{\mathbb{C}/\mathbb{R}}(\mathrm{SL}_2(\mathbb{C}))$$

has $\mathbb{R}$-rank 2. For any field $\mathbb{Q} \subset P \subsetneq K$ we have that $D$ descends to $P$, but $P$ has at most one complex completion; hence $R_{P/\mathbb{Q}}(\mathrm{SL}_1(D))$ has $\mathbb{R}$-rank at most 1 and so, by Lemma 6.2, $G$ is minimal.

PROPOSITION 6.3. *If $G = R_{K/F}(\mathrm{SU}(D,\tau))$ for $D$ a central division algebra of degree $p \geq 3$ over $K'/K$ quadratic with involution of the second kind $\tau$ such that $K'^{\tau} = K$, then $G$ is minimal if and only if, for all $F \subset P \subsetneq K$ such that $D$ descends to a central simple algebra $(D',\tau')$ over a quadratic extension $P'/P$ with involution of the second kind $\tau'$ with $P'^{\tau'} = P$, there exists a $v_0 \in S_G$ such that $P_{w_i} \simeq \mathbb{R}$ and $P_{w_i} \otimes P' \simeq \mathbb{C}$ for all $w_i$ lying $v_0$ and such that either:*

(1) *if $v_0 \in S'_G$ then $(D' \otimes_P P_{w_i}, \tau' \otimes 1) \simeq (M_n(\mathbb{C}), \pm\langle 1,\dots,1\rangle)$ for all $w_i$ lying over $v_0$; or*
(2) *if $v_0 \in S''_G$ then $(D' \otimes_P P_{w_i}, \tau' \otimes 1) \simeq (M_n(\mathbb{C}), \pm\langle 1,-1,1,\dots,1\rangle)$ for at most one $i$ and $(D' \otimes_P P_{w_i}, \tau' \otimes 1) \simeq (M_n(\mathbb{C}), \pm\langle 1,\dots,1\rangle)$ for all others.*

*Proof.* Using Lemma 6.2, we find that all possible simple subgroups correspond to $F \subset P \subsetneq K$ such that $D'$ exists as before. Once again, the conditions imposed upon such $P$ exactly guarantee that the corresponding subgroup cannot have appropriate real rank. □

It remains to consider the restrictions of absolutely simple groups of the form $\mathrm{SU}_3(K', f)$ for $K'/K$ a quadratic extension and $f$ a 3-dimensional hermitian form over $K'$. Notice that there do exist proper, nontrivial, $K$-simple subgroups $H \le \mathrm{SU}_3(K', f)$; however, because $A_2$ does not contain a root system of type $A_1 \times A_1$, these subgroups can only be of absolute rank 1.

PROPOSITION 6.4. *Let $G$ be of the form $R_{K/F}(\mathrm{SU}_3(K', f))$ for $K'/K$ quadratic, and let $f$ be hermitian over $K'^3$. Then $G$ is minimal if and only if the following statements hold.*

(1) *For any $F \subset P \subsetneq K$ such that $\mathrm{SU}_3(K', f)$ descends to $P$, there exists a $v_0 \in S_G$ such that $P_{w_i} \simeq \mathbb{R}$ for all $w_i$ lying over $v_0$ and*:
 (a) *if $\mathrm{SU}_3(K', f)$ descends to $\mathrm{SU}_3(P', f')$, where $f' = \langle 1, a_2, a_3 \rangle$, then $P_{w_i} \otimes P' \simeq \mathbb{C}$ for every $w_i$ and*
  (i) *if $v_0 \in S'_G$ then the image of $a_j$ in $P_{w_i}$ is positive for all $i$, or*
  (ii) *if $v_0 \in S''_G$ then the image of $a_j$ in $P_{w_i}$ is negative for at most one $i$; and*
 (b) *if $\mathrm{SU}_3(K', f)$ descends to $\mathrm{SU}(D, \tau)$, where $D$ is a central division algebra of degree 3 over $P'/P$ quadratic with involution $\tau$ of the second kind, then $P' \otimes P_{w_i} \simeq \mathbb{C}$ for every $i$ and*
  (i) *if $v_0 \in S'_G$ then $(D \otimes P_{w_i}, \tau \otimes 1) \simeq (M_3(\mathbb{C}), \sigma)$, where $\sigma(X) = \bar{X}^T$, for every $w_i$, or*
  (ii) *if $v_0 \in S''_G$ then $(D \otimes P_{w_i}, \tau \otimes 1) \simeq (M_3(\mathbb{C}), \sigma)$ for all but at most one $w_i$ and, for at most one $w_i$, $(D \otimes P_{w_i}, \tau \otimes 1) \simeq (M_3(\mathbb{C}), \sigma \circ \mathrm{Int}(\mathrm{diag}(1, -1, 1)))$ or $(M_3(\mathbb{C}), \sigma \circ \mathrm{Int}(\mathrm{diag}(1, -1, -1)))$.*

(2) *For any $F \subset P \subseteq K$ such that a subgroup $\mathrm{SL}_1(D') \le \mathrm{SU}_3(K', f)$ descends to $\mathrm{SL}_1(D)$ over $P$, there exists a $v_0 \in S_G$ such that*:
 (a) *if $v_0 \in S'_G$, then $P_{w_i} \simeq \mathbb{R}$ and $D \otimes P_{w_i} \simeq \mathbb{H}$ for all $w_i$ over $v_0$; or*
 (b) *if $v_0 \in S''_G$, then $P_{w_i} \simeq \mathbb{C}$ or $D \otimes P_{w_i} \simeq M_2(\mathbb{R})$ for at most one $w_i$ over $v_0$.*

*Proof.* Arguing as in the proof of Lemma 6.2, let $G' \le G$ be an $F$-defined, $F$-simple subgroup and let $G_K = \mathrm{SU}_3(K', f) \times R_{K'/K}(H_1)$. Let $\pi : G_K \to \mathrm{SU}_3(K', f)$ be projection on the first component. If $\pi(G'_K) = 1$ then, as before, $G' = 1$—a contradiction. This means that $\pi(G'_K)$ is either all of $\mathrm{SU}_3(K', f)$ or isomorphic to $\mathrm{SL}_1(D)$ for a quaternion algebra $D$ defined over $K$. If $\pi(G'_K) \le \mathrm{SL}_1(D) \le \mathrm{SU}_3(K', f)$ and if $g = (g_1, \ldots, g_n) \in G'_K(\bar{K})$ then, for any $g_i$, there exists a $\sigma \in \Gamma$ such that $\sigma(g_i)$ is the first component of $\sigma(g)$. Because $\mathrm{SL}_1(D)$ and $G'_K$ are $K$-defined, we therefore have that $g_i \in \mathrm{SL}_1(D)$. This means that $G' \le R_{K/F}(\mathrm{SL}_1(D))$, so we can apply Lemma 6.2 to find that $G'$ is isomorphic to $R_{P/F}(\mathrm{SL}_1(D'))$ for some $D'$ over $P$. The conditions listed in item (2) are exactly what is necessary to ensure that no subgroup of this form has appropriate real rank.

Assume that $\pi(G'_K) = \mathrm{SU}_3(K', f)$. If $G'_K$ is absolutely simple then $\pi$ is an isomorphism, and by setting $F = P$ we see that the conditions in 1 ensure that any such subgroup does not have appropriate real rank. If $G'$ is not absolutely simple, then $G' \simeq R_{F'/F}(H')$ for some absolutely simple $H'$. Hence

$$G'_K = R_{K_1/K}(H'_{K_1}) \times \cdots \times R_{K_m/K}(H'_{K_m}).$$

Let $\pi_i$ be the restriction of $\pi$ to $R_{K_i/K}(H'_{K_1})$. Because the $R_{K_i/K}(H'_{K_i})$ are $K$-simple, we must have that $\ker(\pi_i)$ is either finite or all of $R_{K_i/K}(H'_{K_i})$. Assume that some $\pi_i$ is surjective. Then $\pi_i$ is an isomorphism because $\mathrm{SU}_3(K', f)$ is simply connected. Arguing as in Lemma 6.2 gives $K_i = K$ and $H'_K \simeq \mathrm{SU}_3(K', f)$, and the conditions listed in item (1) are exactly those required to ensure that $G'$ does not have appropriate real rank.

Assume that $\pi_i$ is not surjective for any $i$. Then the image of $\pi_i$ cannot be trivial for all $i$, else the image of $\pi$ would be trivial; thus there exists some $i$ for which the image of $\pi_i$ is $\mathrm{SL}_1(D)$ for some quaternion algebra $D$ over $K$. This means that $H'_{K_i}$ has type $A_1$, so $\pi_i \colon R_{K_i/K}(\mathrm{SL}_1(D_1)) \to \mathrm{SL}_1(D)$ is a surjection with finite kernel. As a result, $\pi_i$ must be an isomorphism and $G'$ is again of the form $R_{P/F}(\mathrm{SL}_1(D))$ for a quaternion algebra $D$. The conditions listed in item (2) are exactly what is required for such a subgroup not to have appropriate real rank. □

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# Appendix: Isotropy of Hermitian Forms over Finite Field Extensions

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Let $A$ be a central, finite-dimensional division algebra over a field $F$. For any element $a \in A$, we denote by $\Lambda_a$ the set of all elements in $A$ of the form $[a, x] = ax - xa$ where $x \in A$. Clearly, $\Lambda_a$ is an $F$-subspace in $A$.

Lemma A.1. *Let $n \in \mathbb{N}$ and $x \in A$. Then $[a^n, x] \in \Lambda_a$.*

*Proof.* We have

$$\begin{aligned}[a^n, x] &= a^n x - x a^n \\ &= a^n x - a^{n-1} x a + a^{n-1} x a - x a^n \\ &= a(a^{n-1} x) - (a^{n-1} x) a + a^{n-1}(xa) - (xa) a^{n-1}.\end{aligned}$$

By induction, $a^{n-1}(xa) - (xa)a^{n-1} \in \Lambda_a$ and so the result follows. □

Let $F[t]$ and $A[t]$ be the polynomial rings over $F$ and $A$, respectively.

Lemma A.2. *Let $\varphi(t) \in F[t]$ and $\psi(t) \in A[t]$. Then*

$$(\varphi(t)\psi(t))(a) - \varphi(a)\psi(a) \in \Lambda_a.$$

*Proof.* We may assume without loss of generality that $\varphi(t) = bt^n$ and $\psi(t) = ct^m$, where $b \in F$ and $c \in A$. Then

$$\begin{aligned}(\varphi(t)\psi(t))(a) - \varphi(a)\psi(a) &= bca^{n+m} - ba^n ca^m \\ &= b((ca^m)a^n - a^n(ca^m)).\end{aligned}$$

Since $b \in F$ and $(ca^m)a^n - a^n(ca^m) \in \Lambda_a$ by Lemma A.1, we are done. □

Let $\sigma$ be an involution of the first kind on $A$, $V$ a right $A$-module, and $h$ a hermitian form on $V$.

Lemma A.3. *Assume that $a$ is $\sigma$-symmetric; that is, $\sigma(a) = a$. Let $v(t), v'(t) \in V[t]$ and let $\varphi(t) = h(v(t), v'(t))$. Then*

$$\varphi(a) - h(v(a), v'(a)) \in \Lambda_a.$$

*Proof.* We may assume that $v(t) = vt^n$ and $v'(t) = v't^m$, where $v, v' \in V$. Then $\varphi(t) = h(v, v')t^{n+m}$ and so, setting $x = h(v, v')$, by Lemma A.1 we have

$$\varphi(a) - h(v(a), v'(a)) = xa^{n+m} - a^n x a^m = [a^n, -xa^m] \in \Lambda_a$$

as required.

Let $L \subset A$ be a maximal separable subfield.

Lemma A.4. *Let $w_1, \dots, w_n \in L$ be a basis of $L$ over $F$ and let $v_1, \dots, v_n \in V$. If $\sum_{i=1}^n v_i \cdot b \cdot w_i = 0$ for all $b \in A$, then $v_i = 0$ for all $i = 1, \dots, n$.*

*Proof.* Clearly, we may assume without loss of generality that $\dim V = 1$ and so we may identify $V = A$. Assume the contrary. Then the condition $\sum_{i=1}^n v_i \cdot b \cdot w_i = 0$ also holds for all $b \in A_E = A \otimes_F E$, where $E/F$ is an arbitrary field extension. Replacing $F$ by an algebraic closure of $F$, we may assume that $A$ is split (i.e., $A = M_n(F)$). Since $L$ is a split étale subalgebra in $A$ up to conjugation, we may also assume that $L$ consists of all diagonal matrices and that

$$w_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \dots, w_n = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Then the condition $\sum_{i=1}^n v_i \cdot b \cdot w_i = 0$ is equivalent to saying that, for all $i = 1, \dots, n$, the $i$th column of the matrix $v_i \cdot b$ is zero for all $b \in M_n(F)$. This can happen only if $v_i = 0$. □

Theorem A.1. *Let $A$ be a central, finite-dimensional division algebra over a field $F$, $\sigma$ an involution of the first kind on $A$, $V$ a right $A$-module, $h$ a hermitian form on $V$, and $L \subset A$ a maximal separable subfield. If $h$ is isotropic over $L$, then there is a 1-dimensional $A$-subspace $U \subset V$ such that $h|_U$ is isotropic over $L$.*

*Proof.* By the theorem on extensions of involutions, there is an involution of the first kind $\sigma'$ on $A$ that is the identity on $L$. Replacing $\sigma$ by $\sigma'$, we may assume that every element in $L$ is $\sigma$-symmetric. Choose a generator $a$ of $L$ over $F$ and let $f(t) \in F[t]$ be its minimal polynomial. The element $\xi = 1 \otimes a$ is in the center of $A_L = A \otimes_F L$. Since $h$ is isotropic over $L$, there is a polynomial $v(t) \in V[t]$ such that $v(\xi) \neq 0$ and $h(v(\xi), v(\xi)) = 0$. Then $h(v(t), v(t))$ is divisible by $f(t)$; that is,

$$h(v(t), v(t)) = f(t) \cdot g(t) \tag{A.1}$$

for some $g(t) \in A[t]$. Note that we can replace $v(t)$ with $v(t) \cdot b$ for any nonzero $b \in A$. Let $v(t) = v_0 + v_1 t + \cdots + v_{n-1} t^{n-1}$, where $v_i \in V$ and $n = \deg A$. By Lemma A.4, there exists a $b \in A$ such that $\sum v_i \cdot b \cdot a^i \neq 0$. Replacing $v(t)$ with $v(t) \cdot b$, we may assume that $v(a) \neq 0$ in $V$. We shall now show that the 1-dimensional subspace $U$ in $V$ generated by $v(a)$ has the required property. Consider the polynomial

$$g(t) = \frac{f(t)}{t - a} \in L[t] \subset A[t].$$

Clearly, $g(\xi) \cdot (\xi - a) = 0$ and $v(a) \cdot g(\xi)$ is a nonzero vector in $V_L = V \otimes_F L$. Since $g(\xi) \cdot \xi = g(\xi) \cdot a$, in $A_L$ we have

$$g(\xi)[a, x] = g(\xi)ax - g(\xi)xa = g(\xi)\xi x - g(\xi)xa = g(\xi)x(\xi - a). \tag{A.2}$$

By Lemmas A.2 and A.3 applied to $\varphi(t) = f(t)$, we have $\psi(t) = g(t)$ and $v'(t) = v(t)$. Now taking (A.1) into consideration, we find that there is an $x \in A$ such that

$$h(v(a), v(a)) = [a, x].$$

Finally, taking into account (A.2) and that $g(\xi)$ is $\sigma$-symmetric, we obtain

$$\begin{aligned} h(v(a) \cdot g(\xi), v(a) \cdot g(\xi)) &= g(\xi) \cdot h(v(a), v(a)) \cdot g(\xi) \\ &= (g(\xi) \cdot [a, x]) \cdot g(\xi) \\ &= g(\xi) \cdot x \cdot (\xi - a) \cdot g(\xi) \\ &= 0. \end{aligned}$$

Thus, the 1-dimensional subspace $U = \langle v(a) \rangle$ in $V$ is isotropic over $L$. □

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