ON THE DERIVATIVE OF A POLYNOMIAL

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In this note I deal with a problem suggested to me by Professor P. Erdös.

THEOREM 1. *Let* E *be a connected, closed, bounded set of capacity* κ, *and let* f(z) *be a polynomial of degree* n *such that* $|f(z)| \leq 1$ *on* E. *Then, for all* $z \in E$,

$$|f'(z)| \leq \frac{e}{2}\frac{n^2}{\kappa} < 1.36\frac{n^2}{\kappa}. \tag{1}$$

Proof. Since E is connected, the complement of E consists of one or more simply connected regions. One of them contains the point at infinity. Let

$$z = \phi(w) = \kappa w + c_0 + c_1 w^{-1} + \cdots$$

be the function (with κ real and positive) that maps $|w| > 1$ conformally onto this region. Then κ is the capacity of E (also called the transfinite diameter or the outer radius of E). The function

$$F(w) = w^{-n} f(\phi(w)) = b_0 + b_1 w^{-1} + \cdots$$

is regular in $1 < |w| \leq \infty$. Since

$$\limsup_{|w| \to 1} |F(w)| \leq 1,$$

we have $|F(w)| \leq 1$ in $|w| > 1$, and therefore [3, p. 168]

$$|F'(w)| \leq \frac{1 - |F(w)|^2}{|w|^2 - 1}. \tag{2}$$

The derivative of F(w) is

$$\begin{aligned} F'(w) &= -nw^{-n-1} f(\phi(w)) + w^{-n}\phi'(w) f'(\phi(w)) \\ &= -nw^{-1} F(w) + w^{-n}\phi'(w) f'(\phi(w)). \end{aligned}$$

Together with (2), this implies the inequality

$$|\phi'(w) f'(\phi(w))| \leq n|w|^{n-1}|F(w)| + \frac{|w|^n}{|w|^2 - 1}(1 - |F(w)|^2).$$

If we take $|w| = (n+1)^{1/2}(n-1)^{-1/2}$, we obtain (putting $\alpha = |F(w)|$)

$$|\phi'(w) f'(\phi(w))| \leq n\left(\frac{n+1}{n-1}\right)^{(n-1)/2}\alpha + \frac{1}{2}(n-1)\left(\frac{n+1}{n-1}\right)^{n/2}(1 - \alpha^2).$$

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The derivative with respect to α of the right side is

$$n\left(\frac{n+1}{n-1}\right)^{(n-1)/2}\left[1-\frac{n-1}{n}\left(\frac{n+1}{n-1}\right)^{1/2}\alpha\right],$$

and this quantity is positive because $0 \leq \alpha \leq 1$. Hence we have

$$|\phi'(w)f'(\phi(w))| \leq n\left(\frac{n+1}{n-1}\right)^{(n-1)/2}. \tag{3}$$

Since [2, p. 69]

$$|\phi'(w)| \geq \kappa(1-|w|^{-2}) = 2\kappa(n+1)^{-1},$$

we obtain from (3) (again putting $z = \phi(w)$)

$$|f'(z)| \leq \frac{n(n+1)}{2\kappa}\left(\frac{n+1}{n-1}\right)^{(n-1)/2} < \frac{en^2}{2\kappa}. \tag{4}$$

This holds for all $z \in C_n$, where C_n is the curve onto which the circle

$$|w| = (n+1)^{1/2}(n-1)^{-1/2}$$

is mapped by $z = \phi(w)$. Since C_n contains E in its interior, (4) holds for all z in E.

Remarks. If $T_n(x)$ denotes the Tchebysheff polynomial of degree n, then $T_n'(1) = n^2$. Since the segment $[-2, 2]$ has capacity 1, it follows that the constant $e/2$ in Theorem 1 can not be replaced by any constant less than 1/2, for any n. It is easy to see that (1) need not hold if E is not connected.

COROLLARY. *If* $f(z) = z^n + \cdots$ *is a polynomial of degree* n *and if the set* C *on which* $|f(z)| = 1$ *is connected, then*

$$\max_{z \in C} |f'(z)| < \frac{en^2}{2}.$$

This follows from Theorem 1 and the fact that C has capacity 1 [1, p. 216].

THEOREM 2. *Let* $f(z) = z^n + \cdots$ *be a polynomial of degree* n, *let* C *be the curve on which* $|f(z)| = 1$, *and let* L *be the length of* C. *If* C *is connected, the mean value of* $f'(z)$ *on* C *is at most* n; *that is,*

$$\frac{1}{L}\int_C |f'(z)|\,|dz| \leq n,$$

with equality only for $f(z) \equiv z^n$.

Proof. For $f(z) \equiv z^n$ the theorem is trivial. If $f(z) \not\equiv z^n$, the connectedness of C implies that $L > 2\pi$ [4, Theorem 2]. Since $w = f(z)$ maps C onto the unit circle $|w| = 1$ described n times, we have

$$\frac{1}{L}\int_C |f'(z)|\,|dz| = \frac{2\pi n}{L} < n.$$

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