ON MAPS WITH NONNEGATIVE JACOBIAN

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The purpose of this note is to prove the following theorem.

THEOREM 1. *Let* M *and* N *be two oriented* n*-dimensional differentiable manifolds with* M *compact and* N *connected. Let* f *be a differentiable map of* M *into* N *whose Jacobian* J(f) *is nonnegative. Then either* $J(f) \equiv 0$, *or* N *is compact,* f *is onto, and* f *has positive degree on each component of* M *on which* $J(f) \not\equiv 0$.

Remark. Since the two manifolds are oriented, that is, since we have chosen a fixed orientation for both, the sign of the Jacobian at any point of M is well defined. The special case of this theorem where M and N are surfaces is treated in [1].

In the course of the proof of this theorem we shall give a proof in modern terminology, and without the use of triangulations, of some classical results about degrees of maps. The results below can also be stated and proved for relative manifolds without any serious modification.

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1. ORIENTATION AND DEGREE

Let M be a connected n-manifold (not necessarily differentiable). If $x \in M$ and if $U \subset M$ is a cell containing x, then $H_n(M, M - x) \approx H_n(U, U - x)$ by excision, and $H_n(U, U - x) \approx Z$. A generator of $H_n(M, M - x)$ is called a local orientation at x. If x and y are two points contained in the same open n-cell V, there is a canonical isomorphism $\phi^V_{x,y}$, depending only on V, from $H_n(M, M - x)$ to $H_n(M, M - y)$, defined as follows. By excision, the inclusion maps $(M, M - V) \subset (M, M - x)$ and $(M, M - V) \subset (M, M - y)$ induce isomorphisms on the corresponding H_n. Taking the composition of these in the obvious way gives $\phi^V_{x,y}$. A manifold M is called orientable if we can choose a generator μ_x for every $H_n(M, M - x)$ in a consistent manner; that is, if for any V, x, and y,

$$\phi^V_{x,y}\,\mu_x = \mu_y . \tag{1}$$

LEMMA 1. *Let* $\mathfrak{B}$ *be a covering of* M *by open cells. A necessary and sufficient condition for* M *to be orientable is that* (1) *hold for all* $V \in \mathfrak{B}$.

The necessity of the condition is trivial. As to its sufficiency, it is clear that if $x, y \in V' \subset V$, then

$$\phi^{V'}_{x,y} = \phi^V_{x,y} . \tag{2}$$

Thus if (1) holds for V, it holds for V'. In particular, (1) holds for all sufficiently small V. Now let U be any cell containing x and y. Choose an arc γ containing x

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and y and lying in U. Cover γ by a finite number of cells $V_i \subset U$ such that (1) holds. Then, choosing a sequence of points $x = x_1, x_2, \cdots, x_n = y$ such that x_i and x_{i+1} lie in V_i, we find by (2) that

$$\phi^U_{x,y} = \phi^{V_1}_{x_1,x_2} \cdots \phi^{V_{n-1}}_{x_{n-1},x_n}.$$

Since each of the factors satisfies (1), so does $\phi^U_{x,y}$. (In terms of the language of sheaves, the groups $H_n(M; M - x)$ form a sheaf, and M is orientable if and only if this sheaf is a product sheaf.)

For any compact orientable manifold M, $H_n(M) \approx Z$, and the inclusion $M \subset (M, M - x)$ induces an isomorphism $H_n(M) \approx H\ (M, M - x)$ for all $x \in M$. This follows from the Alexander-Lefschetz duality theorem, which says that

$$H_i(M, U) \approx H^{n-i}(M - U);$$

see [3]. We have only to apply this theorem with $U = \emptyset$ and $U = M - x$ to obtain the diagram

$$\begin{array}{ccc} H_n(M) & \xrightarrow{\approx} & H^0(M) \\ \downarrow & & \downarrow \approx \\ H_n(M, M - x) & \xrightarrow{\approx} & H^0(x). \end{array}$$

In particular, a choice of μ_x determines a generator μ of $H_n(M)$. A manifold M together with such a μ is called an oriented manifold.

Let M and N be two oriented n-manifolds, and let f be a continuous map $M \to N$. If x is an isolated point of $f^{-1}(y)$ $(y \in N)$, we choose a neighborhood U of x containing no other point of $f^{-1}(y)$. Then the map $f: (U, U - x) \to (N, N - y)$ is well defined. We thus get a map $f_*: H_n(U, U - x) \to H_n(N, N - y)$. Since $H_n(M, M - x)$ is canonically isomorphic to $H_n(U, U - x)$, f_* induces a map $H_n(M, M - x) \to H_n(N, N - y)$, which we shall denote by f_*^x. If μ_x and ν_y denote the local orientations at x and y, respectively, then $f_*^x\, \mu_x = d_x(f)\, \nu_y$. The integer $d_x(f)$ is called the local degree of f at x. Furthermore the map f induces a map $f_*: H_n(M) \to H_n(N)$. If M and N are compact and if μ and ν are the orientations of $H_n(M)$ and $H_n(N)$, then $f_*(\mu) = d(f)\nu$. The integer d(f) is called the degree of f. If M is compact and N is not, we define d(f) to be 0.

THEOREM 2. *Let* M *and* N *be two oriented* n*-manifolds with* M *compact. If there is a* $y \in N$ *such that* $f^{-1}(y)$ *is a finite set of points* $x_1, \cdots, x_k$, *then*

$$d(f) = \sum_k d_{x_k}(f). \tag{3}$$

Proof. Choose disjoint open sets U_i such that $x_i \in U_i$. Consider the diagram

$$\begin{array}{ccc} H_n(M) & \xrightarrow{f_*} & H_n(N) \\ \iota_* \downarrow & & \downarrow \iota_* \\ H_n(M, M - f^{-1}(y) & \xrightarrow{f_*} & H_n(N, N - y), \end{array}$$

where the vertical maps are induced by inclusion. By excision,

$$H_n(M, M - f^{-1}(y)) \approx \sum_i H_n(U_i, U_i - x_i),$$

which is canonically isomorphic to $\Sigma_i H_n(M, M - x_i)$. Under the composite isomorphism i_*, μ goes into $\Sigma \mu_{x_i}$. It is then easy to see that $f_* i_* \mu = \Sigma f_* \mu_{x_i} = (\Sigma d_{x_i}(f))\nu_y$. If N is compact, we are done, because $H_n(N) \approx H_n(N, N - y)$. If N is not compact, $H_n(N) = 0$ since by duality, it is isomorphic to the 0-dimensional cohomology with compact supports. This proves the theorem.

It should be remarked that if $d(f) \neq 0$, then $f(M) = N$. For if there were a $y \in N$, $y \notin f(M)$, then f would map M into $N - y$; and thus f_* would factor through $H_n(N - y)$, which is 0 since $N - y$ is not compact.

2. ORIENTED DIFFERENTIABLE MANIFOLDS

In this section we discuss orientation of differentiable manifolds, and we conclude the proof of Theorem 1. If we choose an orientation for Euclidean n-space, this gives an orientation for any open subset.

LEMMA 2. *Let* U *and* V *be open subsets of* E^n, *and let* f *be a differentiable map of* U *into* V. *If* $J(f) > 0$ *at* x, *then* $d_x(f) = 1$.

We first prove that any affine transformation with positive determinant preserves local orientation. Compactify E^n by adding a point ∞. Then $E^n \cup \infty$ is the sphere S^n, and by excision $H_n(E^n, E^n - x) \approx H_n(E^n \cup \infty, \infty)$. If f is an affine map with $f(x) = y$, then the map $f_*: H_n(E^n, E^n - x) \to H_n(E^n, E^n - y)$ can be factored through the map $\hat{f}_*: H_n(E^n \cup \infty, \infty) \to H_n(E^n \cup \infty, \infty)$, where $\hat{f}$ is the natural extension of f to $E^n \cup \infty$. If f has positive determinant, $\hat{f}$ is homotopic to the identity (keeping ∞ fixed). Thus $\hat{f}_*$ is the identity, and f preserves orientation.

Now let f be a differentiable map with $f(x) = y$ and having positive Jacobian at x. Let g denote the affine transformation sending x into y and having the same Jacobian matrix at x as f. For any t, the map $tf + (1 - t)g$ has the same Jacobian at x as does f. By the compactness of $[0, 1]$, we can find a neighborhood W of x such that $tf + (1 - t)g$ maps $W - x$ into $V - y$ for all $t \in [0, 1]$. Thus

$$f_* = g_*: H_n(W, W - x) \to H_n(V, V - y).$$

Since g_* preserves local orientation, so does f_*.

LEMMA 3. *Let* M *be a differentiable manifold.* M *is orientable if and only if there is a covering of* M *by coordinate neighborhoods whose transition functions have positive Jacobians.*

Proof. Each coordinate map induces a local orientation at every point contained in it. If in the overlap of two neighborhoods the transition functions have positive Jacobian, the local orientation is the same, by the previous lemma. Thus if we can cover M by coordinate neighborhoods whose transition functions are all of positive Jacobian, we can make a consistent choice of local orientation; that is, M is orientable. Conversely, suppose M is orientable. Choose an orientation μ for M, and cover M by coordinate neighborhoods, choosing coordinate maps which induce the same local orientation as that given by μ. (This can be done by choosing any coordinate map and then composing with a reflection if necessary.) Then the transition functions have positive Jacobian.

Proof of Theorem 1. If $J(f) \not\equiv 0$, then f(M) contains some open set. By Sard's theorem, the image of those points where $J(f) = 0$ has measure 0. Thus there is a point $y \in f(M)$ with $J(f) \neq 0$ at all points of $f^{-1}(y)$. Since M is compact, this implies that $f^{-1}(y)$ is finite, otherwise $J(f)$ would be 0 at any limit point of $f^{-1}(y)$. Since $J(f) > 0$ at $x_i \in f^{-1}(y)$, $d_{x_i}(f) = 1$. By (3), $d(f) > 0$, which proves Theorem 1.

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