RIGHT-ORDERED GROUPS

Paul Conrad

1. INTRODUCTION

In this note, "order" will always mean linear order. A group is *right-ordered* (notation: it is an ro-group) if it is an ordered set and if multiplication on the right preserves this order ($a < b$ inplies $ac < bc$). In the process of investigating the group $\mathscr{A}$ of order-preserving automorphisms of an ordered group ([3] and [4]), it became apparent that $\mathscr{A}$ could always be right-ordered. In Section 5 it is shown that every group of order-preserving permutations of an ordered set can be right-ordered. Also, every ro-group G is o-isomorphic to a subgroup of the ro-group of all o-permutations of the set G. In Section 4 (Theorem 4.1), we prove that the following two properties of an ro-group G are equivalent: (a) for each pair of positive elements a, b in G, there exists a positive integer n such that $(ab)^n > ba$; (b) if C and C' are convex subgroups of G, and C' covers C, then C is normal in C' and there exists an order-preserving isomorphism of C'/C into the additive group of real numbers. In Section 2 it is shown (Theorem 2.1) that a right-ordering of G is an ordering if and only if $a < b$ implies $b^{-1} < a^{-1}$ for all a, b in G. We also derive four properties, each of which is a necessary and sufficient condition for the right-ordering of a group. In Section 3, some well-known properties of ordered groups are shown to hold for ro-groups.

2. NECESSARY AND SUFFICIENT CONDITIONS FOR A GROUP TO ADMIT A RIGHT-ORDERING

Let G be a group with identity e. Then G is an ro-*group* provided

(1) $(G, <)$ is an ordered set;

(2) if $a < b$, then $ac < bc$ for all a, b, c in G.

LEMMA 1.1. *A group* G *admits a right-order if and only if there exists a subsemigroup* P *of* G *that satisfies*

$$(*) \qquad e \notin P; \quad \textit{if } g \neq e \textit{ and } g \in G, \textit{then } g \in P \textit{ or } g^{-1} \in P.$$

The proof is entirely similar to the one for o-groups. For if G is right-ordered, let $P = \{g \in G: g > e\}$, and if P is a subsemigroup of G that satisfies (*), then define $a < b$ if $ba^{-1} \in P$. P is the *semigroup of positive elements of* G.

COROLLARY I. *If* G *is an* ro-*group, then* G *is torsion-free.*

For consider $e \neq g \in G$. If $e < g$, then $e < g < g^2 < \cdots$, and if $g < e$, then $\cdots < g^2 < g < e$.

COROLLARY II. *If* G *is abelian, then every right-ordering of* G *is an ordering. Every abelian subgroup of an* ro-*group is an* o-*group.*

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For an ordering of G is determined by a normal subsemigroup P of G that satisfies (*), and every subsemigroup of an abelian group is normal.

Let G be an ro-group with the semigroup P of positive elements. If we define $a \dashv b$ if $a^{-1}b \in P$, then it is easy to show that:

(i) $(G, \dashv)$ is an ordered set.

(ii) If $a \dashv b$, then $ca \dashv cb$ for all a, b, c in G.

Conversely, if G satisfies (i) and (ii), and if we define $Q = \{g \in G: g \vdash e\}$, then Q is a subsemigroup of G that satisfies condition (*). Thus a subsemigroup of G that satisfies (*) determines two orderings of G. We shall usually ignore the left-ordering $\dashv$.

THEOREM 2.1. *Suppose that* G *is an* ro-*group. Let* P *be its semigroup of positive elements, and let* a, b $\in$ G. *Then the following are equivalent.*

(1) G *is an* o-*group; that is,* P *is normal in* G.

(2) *If* $ba^{-1} \in P$, *then* $a^{-1}b \in P$.

(3) *If* $a < b$, *then* $b^{-1} < a^{-1}$.

(4) *If* $a < b$, *then* $a \dashv b$.

(5) *If* $ab > b$, *then* $ba > b$.

Proof. If P is normal and $ba^{-1} \in P$, then $b^{-1}ba^{-1}b = a^{-1}b \in P$. Thus (1) implies (2). If (2) is true and $a < b$, then $ba^{-1} \in P$, and thus $a^{-1}b \in P$. Therefore $b^{-1} < a^{-1}$, and (2) implies (3). If (3) is true and $a < b$, then $b^{-1} < a^{-1}$, and thus $a^{-1}b \in P$. Therefore $a \dashv b$, and (3) implies (4). If (4) is true and $ab > b$, then

$$a = abb^{-1} > bb^{-1} = e.$$

Thus $a \vdash e$ and $ba \vdash b$. If $ba < b$, then $ba \dashv b$, a contradiction. Therefore $ba > b$, and (4) implies (5). Finally, suppose that (5) is true. If $a \in P$, then $a > e$ and $ab > b$. Thus $ba > b$, and $bab^{-1} > e$. Therefore P is normal, and (5) implies (1)

Computation without the use of rule (3) is difficult, and the fact that $a^2 = b^2$ does not imply that $a = b$ is annoying (see Example I in Section 5).

For each finite subset $\{x_1, \cdots, x_n\}$ of a group G, let $(x_1, \cdots, x_n)$ denote the subsemigroup of G that is generated by $\{e, x_1, \cdots, x_n\}$. Let $I(x_1, \cdots, x_n)$ denote the intersection of the 2^n semigroups $(x_1^{e_1}, \cdots, x_n^{e_n})$, where the e_i are ± 1.

THEOREM 2.2. *For any group* G *the following are equivalent.*

(1) G *can be right-ordered.*

(2) *For each finite subset* $\{x_1, \cdots, x_n\}$ *of* G, $I(x_1, \cdots, x_n) = e$.

(3) *For each element* $a \neq e$ *in* G *there exists a subsemigroup* G_a *of* G *that contains* a *but not* e, *and such that* $xy \in G_a$ *implies* $x \in G_a$ *or* $y \in G_a$ (*or, equivalently,* $G \setminus G_a$ *is a semigroup*).

(4) *If* $\{x_1, \cdots, x_n\}$ *is a finite subset of* G *that does not contain* e, *then there exist* $e_i = \pm 1$ *for* $i = 1, \cdots, n$ *such that* e *does not belong to the subsemigroup of* G *that is generated by* $\{x_1^{e_1}, \cdots, x_n^{e_n}\}$.

(5) *There exists a set* S *of subsemigroups of* G *such that* $e = \bigcap_{\mathcal{G} \in S} \mathcal{G}$ *and for each* $g \in G$ *and* $\mathcal{G} \in S$ *either* $g \in \mathcal{G}$ *or* $g^{-1} \in \mathcal{G}$.

Proof. The equivalence of (1), (2) and (3) follows immediately from Ohnishi's proof of the corresponding theorem for o-groups [7] (simply omit the word "invariant" wherever it occurs). Suppose that G is an ro-group, and let $\{x_i, \cdots, x_n\}$ be a finite subset of G that does not contain e. Pick $e_i = \pm 1$ for $i = 1, \cdots, n$, so that $x_i^{e_i} > e$. Then all of the elements of the semigroup generated by $\{x_1^{e_1}, \cdots, x_n^{e_n}\}$ are positive. Therefore (1) implies (4). Suppose that G satisfies (4) but not (2). Then there exists a finite subset $\{x_1, \cdots, x_n\}$ of G such that $e \neq a \in I(x_1, \cdots, x_n)$. Without loss of generality, $x_i \neq e$ for $i = 1, \cdots, n$. For each choice of the e_i,

$$a = \text{some product of the } x_i^{e_i} \text{ and } a = \text{some product of the } x_i^{-e_i}.$$

Thus a^{-1} = some product of the $x_i^{e_i}$. Therefore $\{a, x_1, \cdots, x_n\}$ does not satisfy (4). Thus (4) implies (2).

Suppose that G is an ro-group, and let $A = \{g \in G: g \geq e\}$ and $B = \{g \in G: g \leq e\}$. Clearly $S = \{A, B\}$ satisfies (5). Therefore (1) implies (5). Finally suppose that G satisfies (5). Well-order the elements in S, $\mathscr{G}_1 \dashv \mathscr{G}_2 \dashv \cdots$. For each $e \neq g$ in G, let $\alpha(g)$ be the first $\mathscr{G}_i$ in this well-ordering such that $g \notin \mathscr{G}_i$. If $\alpha(g) = \alpha(g^{-1})$, then there exists a $\mathscr{G}_i$ such that $g, g^{-1} \notin \mathscr{G}_i$, a contradiction. Therefore $\alpha(g) \neq \alpha(g^{-1})$ for all $e \neq g \in G$. Let

$$P = \{g \in G: g \neq e \text{ and } \alpha(g) \dashv \alpha(g^{-1})\}.$$

Clearly P satisfies condition (*) of Lemma 1.1, and by a straightforward computation it follows that P is a subsemigroup of G. Therefore (5) implies (1).

Let Q be a subsemigroup of G that does not contain e. As before, define that $a < b$ in G if $ba^{-1} \in Q$. Then $<$ is a partial ordering of G that satisfies condition (2) of our definition of an ro-group. Thus Q determines a partial right-ordering of G. Everett [5] has shown that if G is an abelian group that can be ordered (that is, is torsion-free), then any partial ordering of G can be extended to an ordering. Ohnishi [6] derived conditions that are necessary and sufficient for the extension of any partial ordering of a non-abelian group to an ordering. Ohnishi [7] essentially shows that the partial right-ordering of G determined by each of the subsemigroups G_a defined in (3) of Theorem 2.2 can be extended to a right-ordering of G, provided that G can be right-ordered. Thus we have

THEOREM 2.3. *If* G *is an* ro-*group*, Q *is a subsemigroup of* G *that does not contain* e, *and* $G \setminus Q$ *is a semigroup, then the partial right-order of* G *that is determined by* Q *can be extended to a right-ordering of* G.

Proof. Let $\mathscr{S}$ be the set of all subsemigroups T of G such that $e \notin T$, $Q \subseteq T$, and $G \setminus T$ is a semigroup. By Zorn's lemma there exists a maximal element P in $\mathscr{S}$. Suppose (by way of contradiction) that there exists $e \neq a \in G$ such that $a, a^{-1} \notin P$. Pick a semigroup G_a that satisfies (3) of Theorem 1.2. Let

$$T = P \cup \{x \in G_a: x, x^{-1} \notin P\}.$$

It follows by a routine computation that $T \in \mathscr{S}$. Thus, since $a \in T \setminus P$ and P is maximal in $\mathscr{S}$, we have a contradiction. Therefore P determines an extension of the partial right-order of G to a right-order of G.

3. SOME PROPERTIES OF ro-GROUPS

Throughout this section, we assume that G is an ro-group and that P is its semigroup of positive elements.

3.1. *If* $x, y \in P$ *and* $x < y^n$ *for some positive integer* n, *then* $xyx^{-1} \in P$.

Proof. Let n be the least positive integer such that $x < y^n$. If $xyx^{-1} < e$, then $xy < x$ and $x < y^n$, hence $xy < y^n$. But then $x < y^{n-1}$, a contradiction.

3.2. *If* $x, y \in G$, $b \in P$, $x < y$ *and there exists a positive integer* n *such that* $b < (yx^{-1})^n$, *then* $bx < by$.

Proof. By 3.1, $e < b(yx^{-1})b^{-1} = by(bx)^{-1}$. Therefore $bx < by$.

A subgroup C of G is *convex* if $x \in G$ and $e < x < c \in C$ imply that $x \in C$. If G is an o-group, then this definition is equivalent to the usual one (if $g \in G$ and $c_1, c_2 \in C$, then $c_1 < g < c_2$ implies that $g \in C$).

3.3. *The set of all convex subgroups of* G *is ordered by inclusion, and closed with respect to intersections and joins.*

Proof. Let C and C' be convex subgroups of G and suppose that $c' \in C' \setminus C$. Without loss of generality, $c' \in P$. Pick $c \in C \cap P$. Then $e < c < c'$, because C is convex and $c' \notin C$. Thus $c \in C'$ and $C \cap P \subset C'$. But since C' is a group, $C \subset C'$.

3.4. *If* C *is a convex subgroup of* G *and* $a \in (G \setminus C) \cap P$, *then* $aC \subset (G \setminus C) \cap P$.

Proof. Pick $c \in C$. Then $c^{-1} < a$, for otherwise $e < a \leq c^{-1}$, hence $a \in C$. Thus $e = c^{-1}c < ac$. In general, $Ca \not\subset (G \setminus C) \cap P$.

3.5. *Suppose that* C *is a normal convex subgroup of* G. *Define that* $X \in G/C$ *is positive if* $X = aC$, *where* $a \in (G \setminus C) \cap P$. *This definition right-orders* G/C.

Proof. If $X = aC = bC$, where $a \in (G \setminus C) \cap P$, then $b = ac$, where $c \in C$. Thus by 3.4, $b \in (G \setminus C) \cap P$. Therefore this definition is independent of the particular choice of the representative a. If $C \neq X \in G/C$, then either X contains a positive element, or X contains a negative element. Therefore either X is positive or X^{-1} is positive. If the elements X and Y of G/C are positive, then $X = aC$ and $Y = bC$, where $a, b \in (G \setminus C) \cap P$. Thus $ab \in P$, and if $ab \in C$, then $X = Y^{-1} = b^{-1}C$. But then X is not positive, a contradiction. Thus $ab \in (G \setminus C) \cap P$, and $XY = abC$ is positive. Therefore G/C is right-ordered. In particular, if C is a normal convex subgroup of G and $a \in (G \setminus C) \cap P$, then $Ca = aC \subset (G \setminus C) \cap P$.

Let G' be another ro-group with identity e' and positive semigroup P'. A homomorphism π of G onto G' is an *o-homomorphism* if $P\pi \subset P' \cup \{e'\}$. Thus π is an o-homomorphism if and only if $a < b$ in G implies $a\pi \leq b\pi$ in G' for all a, b in G. If C is a normal convex subgroup of G, and G/C is right-ordered as in 3.5, then the natural homomorphism of G onto G/C is an o-homomorphism.

3.6. *If* π *is an o-homomorphism of* G *onto* G' *and* C *is a convex subgroup of* G, *then* $C\pi$ *and the kernel* $K(\pi)$ *of* π *are convex subgroups.*

Proof. If $e < a < b \in K(\pi)$, then $e' \leq a\pi \leq b\pi = e'$. Thus $a \in K(\pi)$ and hence $K(\pi)$ is convex. Suppose that $e' < a' < b' \in C\pi$. Then $a' = a\pi$ and $b' = b\pi$, where $a \in G \cap P$ and $b \in C \cap P$. If $a \notin C$, then $b < a$, and hence $b' = b\pi \leq a\pi = a'$, a contradiction. Thus $a \in C$ and $a' = a\pi \in C\pi$. Therefore $C\pi$ is convex.

If C is a convex normal subgroup of G, then it follows from 3.5 and 3.6 that there exists a one-to-one order-preserving correspondence between the convex subgroups of G/C and the convex subgroups of G that contain C.

3.7. *Suppose that* N *is a normal subgroup of a group* H, *and that* N *and* H/N *are* ro-*groups. Define that* $e \neq h \in H$ *is positive if either* $h \in N$ *and* h *is positive in* N *or else* $h \in H \setminus N$ *and* hN *is positive in* H/N. *This definition right-orders* H *so that* N *is a convex subgroup and the natural homomorphism* η *of* H *onto* H/N *is an* o-*homomorphism.*

For it follows by straightforward grinding that the definition right-orders H. Clearly η is an o-homomorphism. Thus by 3.6, $N = K(\eta)$ is convex.

G is *archimedean* if for every pair $a, b \in P$ there exists a positive integer n such that $a < b^n$.

3.8. *If* G *is archimedean, then* G *is an* o-*group. Thus* G *is* o-*isomorphic to a subgroup of the additive group* $\mathfrak{R}$ *of real numbers.*

Proof. If $x, y \in P$, then there exists an n such that $x < y^n$. Thus by 3.1, $xyx^{-1} \in P$. Therefore $P\tau \subset P$, where τ is the inner automorphism of G that is induced by x ($g\tau = xgx^{-1}$ for all $g \in G$). Suppose (by way of contradiction) that $x^{-1}ax < e$ for some $a \in P$. Then $e < (x^{-1}ax)^{-1} = x^{-1}a^{-1}x$. But then

$$e < x(x^{-1}a^{-1}x)x^{-1} = a^{-1},$$

a contradiction. Thus $P\tau^{-1} \subset P$. Hence $P\tau = P$, and P is normal in G. Therefore G is an o-group. The last part of this proposition is the well-known result of Hölder (see [8, p. 6] for a proof).

In this section we have shown that an ro-group has many of the properties of an o-group. In fact, in some ways ro-groups are easier to deal with. In particular, note 3.7 and the examples in Section 5. But some of the fundamental properties of o-groups are not possessed by ro-groups. For example, if C and C' are convex subgroups of G and C' covers C, then C is not necessarily normal in C'. In the next section we deal with this and related problems.

4. A STRUCTURE THEOREM FOR ro-GROUPS

Throughout this section we assume that G is an ro-group, and that P is its semigroup of positive elements.

LEMMA 4.1. *Let* a *and* b *belong to* P. *Then the following three properties of* G *are equivalent:*

(i) *there exists a positive integer* n *such that* $(ab)^n > ba$;

(ii) *if* $a < b$, *then there exists a positive integer such that* $ab^na^{-1} > b$;

(iii) *there exists a positive integer* n *such that* $a^nb > a$.

Proof. If $a < b$, then $a, ba^{-1} \in P$. Thus by (i) there exists an n such that

$$ab^na^{-1} = (aba^{-1})^n > ba^{-1}a = b.$$

Therefore (i) implies (ii). Since $e < a$, $b < ab$. Thus by (ii) there exists an n such that $(ba)^n = b(ab)^nb^{-1} > ab$. Therefore (ii) implies (i). If (i) is false, then

$$(ba)^n b < a(ba)^n b = (ab)^{n+1} < ba$$

for all n. Thus (iii) is false, and hence (iii) implies (i). If $a \leq b$, then $a \leq b < ab$.

If $a > b$, then $a = cb$, where $c, b \in P$. Thus by (i), $(cb)^n > bc$ for some n. Therefore

$$a^n b = (cb)^n b > bcb = ba > a.$$

Therefore (i) implies (iii).

LEMMA 4.2. *Suppose that* G *has the properties in Lemma* 4.1. *Let* $x \in G$ *and* $a, y \in P$. *If* $x < a^m$ *and* $y < a^n$ *for some positive integers* m *and* n, *then there exists a positive integer* q *such that* $xy < a^q$.

Proof. Without loss of generality, let $m = n$. Then $xy < a^m y$. If there exists an integer r such that $a^m \leq a^r y^{-1}$, then $a^m y \leq a^r$. Thus $xy < a^r$. Suppose (by way of contradiction) that $a^r y^{-1} < a^m$ for all r. Then $a^{m+q} y^{-1} < a^m$ for all $q \geq 0$. But $y < a^m$, hence $ya^q < a^{m+q}$, and $ya^q y^{-1} < a^{m+q} y^{-1}$. Thus $ya^q y^{-1} < a^m$ for all $q \geq 0$. Therefore $y(a^m)^q y^{-1} < a^m$ for all $q \geq 0$. Now let $z = a^m$. Then $e < y < z$ and $yz^q y^{-1} < z$ for all $q \geq 0$. This contradicts property (ii) of Lemma 4.1.

LEMMA 4.3. *Suppose that* G *satisfies the conditions in Lemma* 4.1. *If* C *and* C' *are convex subgroups of* G *such that* C' *covers* C, *and if* $a, b \in (C' \setminus C) \cap P$, *then there exists a positive integer* n *such that* $a^n > b$. *In particular, if* G *contains no proper convex subgroup, then* G *is* o-*isomorphic to a subgroup of* $\mathfrak{R}$.

Proof. Suppose (by way of contradiction) that $a^n < b$ for all positive integers n. Let $S = \{x \in G: e < x < a^n \text{ for some } n\}$. Clearly S is a convex set, and by Lemma 4.2, S is a semigroup. Let $T = S \cup S^{-1} \cup \{e\}$, where $S^{-1} = \{s^{-1}: s \in S\}$. We next show that T is a subgroup of G.

If $x \in T$, then $x = e$ or $x \in S$ or $x \in S^{-1}$, thus $x^{-1} = e$ or $x^{-1} \in S^{-1}$ or $x^{-1} \in S$. Therefore $x^{-1} \in T$. Consider $x, y \in T$. If $x = e$ or $y = e$ or $xy = e$, then $xy \in T$. If $x, y \in S$, then $xy \in S \subset T$. If $x, y \in S^{-1}$, then $y^{-1}, x^{-1} \in S$. Thus $y^{-1}x^{-1} = (xy)^{-1} \in S$, and $xy \in S^{-1} \subset T$. Next suppose that $x \in S^{-1}$ and $y \in S$. Then $x < e$ and $xy < y$. If $e < xy$, then $e < xy < y \in S$, and $xy \in S \subset T$. If $xy < e$, then $e < (xy)^{-1} = y^{-1}x^{-1}$ and $y^{-1} < e$. Thus

$$e < (xy)^{-1} = y^{-1}x^{-1} < x^{-1} \in S.$$

Therefore $(xy)^{-1} \in S$ and $xy \in S^{-1} \subset T$. Finally, suppose that $x \in S$ and $y \in S^{-1}$. If $xy, yx \in P$, then by (i) of Lemma 4.1,

$$x = y^{-1} yx < (yxy^{-1})^n = yx^n y^{-1}$$

for some n. Thus $e < xy < yx^n < x^n \in S$, and $xy \in S \subset T$. If xy and yx are negative, then $y^{-1}x^{-1}, x^{-1}y^{-1} \in P$, $y^{-1} \in S$ and $x^{-1} \in S^{-1}$. Thus, by the last argument, $(xy)^{-1} = y^{-1}x^{-1} \in S$. Therefore $xy \in S^{-1} \subset T$. If $yx < e < xy$, then x and $x^{-1}y^{-1}$ belong to P. Thus by (iii) of Lemma 4.1, $x < x^n(x^{-1}y^{-1})$ for some n. Therefore $e < xy < x^{n-1} \in S$, and $xy \in S \subset T$. If $xy < e < yx$, then $x^{-1}y^{-1} < e < y^{-1}x^{-1}$, $y^{-1} \in S$, and $x^{-1} \in S^{-1}$. Thus, by the last argument, $(xy)^{-1} = y^{-1}x^{-1} \in S$.

Therefore T is a convex subgroup of G. If $c \in C \cap P$, then $c < a$, for otherwise $e < a < c \in C$, hence $a \in C$. Therefore $C \cap P \subset T$, and since T is a group, $C \subset T$. $C \neq T$ because $a \in T \setminus C$. If $t \in T \cap P$, then $e < t < a^n \in C'$, hence $t \in C'$. Therefore $T \subset C'$. $T \neq C'$ because $b \in C' \setminus T$. Therefore $C \subset T \subset C'$ and $C \neq T \neq C'$; but this contradicts the fact that C' covers C. This completes the proof of the first assertion of the lemma. If G has no proper convex subgroup, then clearly G is archimedean. Thus, by 3.8, G is o-isomorphic to a subgroup of $\mathfrak{R}$.

COROLLARY. *Suppose that* G *satisfies the conditions in Lemma* 4.1. *Then* G *is archimedean if and only if* G *contains no proper convex subgroup.*

LEMMA 4.4. *Suppose that* G *satisfies the conditions in Lemma* 4.1. *If* C *and* C' *are convex subgroups of* G *such that* C' *covers* C, *then* C *is normal in* C'.

Proof. Pick $b \in (C' \setminus C) \cap P$. Define $x\pi = bxb^{-1}$ for all $x \in G$. We first show that $y\pi < b$ for all $y \in C$. For by 3.4, $by^{-1} \in (C' \setminus C) \cap P$. Thus by Lemma 4.3, $(by^{-1})^n > b$ for some n. Hence by 3.1, $b(by^{-1})b^{-1} > e$, and $b(yb^{-1})b^{-1} < e$. Therefore $y\pi = byb^{-1} < b$. Consider $c \in C$ and suppose that $c\pi \in C' \setminus C$. If $c\pi \in (C' \setminus C) \cap P$, then by Lemma 4.3, $c^n\pi = (c\pi)^n > b$ for some n. But $c^n \in C$, hence $c^n\pi < b$. If $(c\pi)^{-1} = c^{-1}\pi \in (C' \setminus C) \cap P$, then $c^{-n}\pi = (c^{-1}\pi)^n > b$. But $c^{-n} \in C$, hence $c^{-n}\pi < b$. Therefore $C\pi \subset C$.

By Lemma 4.3, for each $a \in (C' \setminus C) \cap P$ there exists a positive integer n such that $a^n > b$. By (ii) of Lemma 4.1, there exists a positive integer q such that $(ba^n b^{-1})^q > a^n$. Therefore

$$(bab^{-1})^{nq} \in (C' \setminus C) \cap P, \qquad bab^{-1} \in (C' \setminus C) \cap P,$$

$$[(C' \setminus C) \cap P]\pi \subset (C' \setminus C) \cap P, \qquad (C' \setminus C)\pi \subset C' \setminus C,$$

and $C \subset C\pi$. Therefore $C\pi = C$ and C is normal in C'.

THEOREM 4.1. *The following properties of an* ro-*group* G *are equivalent.*

(i) *For each pair* $a, b \in P$ *there exists a positive integer* n *such that* $(ab)^n > ba$.

(ii) *If* C *and* C' *are convex subgroups of* G *and* C' *covers* C, *then* C *is normal in* C' *and* C'/C *is* o-*isomorphic to a subgroup of* $\mathfrak{R}$.

Proof. The fact that (i) implies (ii) follows from Lemmas 4.1, 4.3 and 4.4. Conversely, suppose that G satisfies (ii), and consider $a, b \in P$. First assume that $b \leq a$. Let G^a be the intersection of all convex subgroups of G that contain a, and let G_a be the join of all convex subgroups of G that do not contain a. Then G^a and G_a are convex subgroups of G and G^a covers G_a (see 3.3). Thus G^a/G_a is o-isomorphic to a subgroup of $\mathfrak{R}$. By the properties of $\mathfrak{R}$ it follows that there exists a positive integer n such that

$$G_a(ab)^n = [(G_a a)(G_a b)]^n > (G_a b)(G_a a) = G_a ba.$$

Thus $G_a(ab)^n(ba)^{-1}$ is positive in G^a/G_a. Therefore $(ab)^n(ba)^{-1} > e$ and $(ab)^n > ba$. An entirely similar proof takes care of the case where $a < b$.

Remark. We show in Example III that not all ro-groups have property (i). In Example I we construct an ro-group which has property (ii) but cannot be ordered.

Let $\mathscr{C}$ be the class of all ro-groups that have property (i) in Theorem 4.1, and let $G \in \mathscr{C}$. The set Γ of all pairs of convex subgroups G^γ, G_γ of G such that G^γ covers G_γ is ordered by inclusion. Moreover, by Theorem 4.1, each G_γ is normal in G^γ, and each G^γ/G_γ is o-isomorphic to a subgroup of $\mathfrak{R}$. Let S be a subgroup of G. Then G is an a-*extension* of S if for every $e < g$ in G there exists an $e < s$ in S and a positive integer n such that $g^n \leq s \leq g^{n+1}$. G is a c-*extension* of S if for each $\gamma \in \Gamma$, $G^\gamma = G_\gamma(S \cap G^\gamma)$. G is a-*closed* (c-*closed*) if it does not admit any proper a-extension (c-extension) in $\mathscr{C}$. The proof of Lemma 1.1 [1, p. 323] is valid for any group in $\mathscr{C}$. Thus, corresponding to Corollary II [1, p. 324], we have

THEOREM 4.2. *If* $G \in \mathscr{C}$, *then there exists an* a-*closed* a-*extension* (c-*closed* c-*extension*) *of* G *in* $\mathscr{C}$.

Once again, let G be an arbitrary ro-group. An element $a \in G$ is a *left-keeper* if $x < y$ implies $ax < ay$ for all x, y in G. Clearly the set K of all left-keepers in G contains the center of G. Let H be the greatest subgroup of G that is convex and ordered.

THEOREM 4.3. *K is an ordered subgroup of G and* $K \supset H$.

Proof. It is easy to show that $K = \{a \in G: aPa^{-1} = P\}$. Thus it follows at once that K is a group, and hence an o-group. To prove that $H \subset K$ it suffices to show that $H \cap P \subset K$. Consider $q \in H \cap P$ and $x, y \in G$ such that $x < y$. Then $y = zx$, where $z \in P$, and $qy = qzx = qzq^{-1}qx$. Thus it suffices to show that $qzq^{-1} \in P$. If $q < z$, then by 3.1, $qzq^{-1} \in P$. If $z \leq q$, then since H is convex and ordered, $qzq^{-1} \in P$.

Let S be the set of all normal abelian convex subgroups of G. Let $M = \bigcup_{A \in S} A$. Then M is the greatest normal abelian convex subgroup of G, and M is an o-group. There exists a unique (to within an isomorphism) rational vector space D that contains M, such that for any d in D, $nd \in M$ for some positive integer n. This is a straightforward generalization of the usual construction of the rationals. D is called the d-*closure* of M, and the order of M can be extended to an order of D in one and only one way.

THEOREM 4.4. *There exists an* a-*extension* H *of* G *such that* (i) H *contains the* d-*closure* D *of* M; (ii) H *is generated by* D *and* G; *and* (iii) *if* K *is any* a-*extension of* G *that satisfies* (i) *and* (ii), *then* K *is equivalent to* H.

The proof is the same as the proof of Theorem 3.1 [2, p. 518] except that "o-automorphism" is replaced by "automorphism" whenever it occurs.

5. EXAMPLES OF ro-GROUPS

I. *Every normal extension of an* ro-*group by an* ro-*group can be right-ordered.* This is merely a restatement of 3.7. Thus it is easy to construct ro-groups that cannot be ordered. For example, let $G = I \times I$, where I is the group of integers. Define

$$(a', a) + (b', b) = (a' + b', a(-1)^{b'} + b),$$

and define that (a', a) is positive if $a' > 0$ or $a' = 0$ and $a > 0$. Then G is an ro-group, but G cannot be ordered because $-(1, 0) + (0, 1) + (1, 0) = -(0, 1)$. Thus $(0, 1)$ cannot be positive or negative. Moreover, $(1, 1) + (1, 1) = (2, 0) = (1, 0) + (1, 0)$.

II. *The unrestricted direct product of a set of* ro-*groups can be right-ordered.* For let S be a set, and for each element $s \in S$, let G_s be an ro-group. Let G be the set of all mappings α of S into $\bigcup_{s \in S} G_s$ such that $s\alpha \in G_s$ for all $s \in S$. For $s \in S$ and $\alpha, \beta \in G$, define $s(\alpha\beta) = s\alpha\, s\beta$. Let θ be the identity of G. Well-order S: $s_1 \dashv s_2 \dashv \cdots$. For each $\theta \neq \alpha \in G$, let $L(\alpha)$ be the first element in the well-ordering such that $L(\alpha)\alpha \neq e$, where e is the identity of $G_{L(\alpha)}$. Define α to be positive if $L(\alpha)\alpha > e$. Then G is an ro-group, and G is an o-group if and only if every G_s is an o-group.

III. *Every group of order-preserving permutations of an ordered set can be right-ordered.* For let S be an ordered set, let G be a group of o-permutations of S, and let ϕ be the identity permutation of S. Well-order S: $s_1 \dashv s_2 \dashv \cdots$. For each $\phi \neq \pi \in G$, let $L(\pi)$ be the first element in this well-ordering such that $L(\pi)\pi \neq L(\pi)$. Define π to be positive if $L(\pi)\pi > L(\pi)$. This definition right-orders G (see the proof of Theorem 3 [3, p. 388] for the details). In particular, *the group of all* o-*automorphisms of an* ro-*group can be right-ordered.*

Suppose that S itself is an ro-group, and let the first element s_1 in the above well-ordering be the identity e of S. Then the mapping of $s \in S$ upon the right multiplication s' of S ($xs' = xs$ for all $x \in S$) is an o-isomorphism of S into G. Therefore, *every* ro-*group* S *is* o-*isomorphic to a subgroup of the* ro-*group of all* o-*permutations of the set* S.

Finally we use this method to construct an ro-group that does not have property (iii) in Lemma 4.1 and hence does not have properties (i) or (ii) of Theorem 4.1. Let S be the set of real numbers with their natural order, and let G be the group of all o-permutations of S. We distinguish two particular elements α and β of G:

$$x\alpha = \begin{cases} x + 1/2 & \text{for all } x \leq 0, \\ (x+1)/2 & \text{for all } 0 \leq x \leq 1, \\ x & \text{for all } 1 \leq x, \end{cases}$$

$$x\beta = \begin{cases} x/2 & \text{for all } x \leq 1, \\ x - 1/2 & \text{for all } 1 \leq x. \end{cases}$$

Well-order S so that 0 is the first element and -1 is the second. Then $L(\alpha) = 0 \dashv -1 = L(\beta)$.

$$L(\beta)\beta = (-1)\beta = -1/2 > -1 = L(\beta).$$

Thus $\beta > \phi$. $L(\alpha)\alpha = 0\alpha = 1/2 > 0 = L(\alpha)$. Thus $\alpha > \phi$.

$$0\alpha^n\beta\alpha^{-1} = [(2^n - 1)2^{-n}]\beta\alpha^{-1} = [(2^n - 1)2^{-n-1}]\alpha^{-1} = (2^n - 1)2^{-n-1} - 2^{-1} = -2^{-n-1}.$$

This means that $L(\alpha^n\beta\alpha^{-1}) = 0$ and $\alpha^n\beta\alpha^{-1} < \phi$ for all positive integers n. Thus $\alpha^n\beta < \alpha$, and G does not have property (iii).

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Tulane University