ON A THEOREM OF LEFSCHETZ

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1. INTRODUCTION

This note concerns the Lefschetz hyperplane theorem in both homology and homotopy. (See Theorem I and its corollary, for the statement of the result.) We shall deduce our refinement of this oft-proved theorem (see [2], [3], [4], [6]) as an immediate consequence of what in [2] I called the main theorem of the nondegenerate Morse theory.

Morse and Lefschetz lived within a few steps of one another for over twenty years. It is therefore amusing that the idea of applying the former's theory in this connection is quite recent. I first saw this approach taken in a lecture by R. Thom in 1957, and this note is no more than a technical elaboration of his idea. My main observation is that the notion of a nondegenerate critical manifold, when properly applied, eliminates all the troubles with infinity which Thom encountered in his original account.

The proper dual of the Lefschetz theorem states that the homology of a Stein manifold vanishes above its middle dimension. In a forthcoming paper [1], Andreotti and Frankel use the Morse theory in its most elementary form to prove this dual statement. Poincaré duality then completes their proof of the classical version of the Lefschetz theorem. This approach is in a sense the simplest. But it does not yield the homotopy statement.

2. STATEMENT OF THE LEFSCHETZ THEOREM

Throughout this note, X will denote a compact, complex, analytic manifold of complex dimension n. Let E be an analytic line bundle over X. A global holomorphic section s of E will be called *nonsingular* if the following condition is satisfied:

CONDITION T. *For each* $x \in X$ *with* $s(x) = 0$, *there exist* (a) *a holomorphic section* s_* *of* E *over some neighborhood of* x *with* $s_*(x) \neq 0$, and (b) *a local analytic coordinate system* $(z_1, z_2, \cdots, z_n)$, *centered at* x, *such that near* x *the section* s *is represented by*

$$s = z_1 s_* . \tag{2.1}$$

Suppose now that s is a nonsingular section of E, and denote the set of zeros of s by S. By (2.1), this set is a closed complex analytic submanifold of X. The Lefschetz theorem compares S with X, under certain conditions on E which will be formulated next.

The fiber of E over $x \in X$ is denoted by E_x. A smooth (that is, C^∞) function $\mathfrak{h}$ which assigns to each $x \in X$ a positive definite hermitian form $\mathfrak{h}_x$ on E_x is called a hermitian structure on E. That such structures exist is easily verified. Given

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a hermitian structure $\mathfrak{h}$ and a global section s of E, we denote by (s, s) the function on X which assigns to x the value of the hermitian form on E_x at the point s(x). Thus $(s, s)(x) = \mathfrak{h}_x(s(x), s(x))$, represents the square of the "height" of s(x) over the zero section.

A hermitian structure $\mathfrak{h}$ on E defines a form $c(E, \mathfrak{h})$ on X. This form is characterized by the following condition: *If* U *is an open set of* X *which admits a nonvanishing holomorphic section* s_U *of* E *restricted to* U, *then, over* U,

$$c(E, \mathfrak{h}) = \frac{i}{2\pi} \bar{\partial}\partial \log (s_U, s_U). \tag{2.2}$$

The form is clearly closed, and it is known to represent the first Chern class of E. In terms of a local analytic coordinate system $(z_1, z_2, \cdots, z_n)$, it is of the type $i\, g_{\alpha\beta}\, dz_\alpha \wedge d\bar{z}_\beta$, where the matrix $g_{\alpha\beta}$ is hermitian. We say that $\mathfrak{h}$ is a positive structure if at each point of X the matrix just defined is positive definite. This condition is also expressed by the statement $c(E, \mathfrak{h}) > 0$. Finally, the bundle E is called positive if E admits a positive hermitian structure. All these definitions are of course standard; see [5], for instance.

The version of the Lefschetz theorem which we are after now takes the following form.

THEOREM I. *Let* E *be a positive line bundle over* X, *let* s *be a nonsingular holomorphic section of* E, *and let* S *denote the null set of* s. *Then* X *is obtained from* S *by successively attaching cells of dimension* $\geq$ n. *Symbolically,*

$$X = S \cup e_1 \cup e_2 \cup \cdots \cup e_r \qquad (\dim e_k \geq \dim_C X = n). \tag{2.3}$$

Here we use the standard notion of attaching a cell to a space: Given S, a cell e, and a map α of the boundary, $\dot{e}$, of e into S, we form the space $S \cup e$ (S with e attached) by identifying the points $x \in \dot{e}$ with $\alpha(x) \in S$. The formula (2.3) then just means that, up to homotopy type, X is obtained by a finite series of such steps, the important part of the theorem being that at each step the dimension of the cell to be attached is no less than the complex dimension of X.

COROLLARY. *Let* $j: S \subset X$ *be the inclusion of* S *in* X. *Then under the conditions of Theorem* I, *the homomorphism induced by* j *in both homotopy and integral homology is onto in dimensions* $< n$, *and is one-to-one in dimensions* $< n - 1$.

This is a standard consequence of (2.3); see [7]. In homotopy it is established by freeing a point in the attaching cell. In homology, excision and exactness yield the corollary.

Remarks. In the situation envisaged by Lefschetz, X was an algebraic manifold imbedded regularly in a complex projective space $P_m(C)$, and S was the intersection of X with a hyperplane of $P_m(C)$ which cut X transversally. His conclusion was then the homology statement of our corollary. It is well known that $P_m(C)$ admits a positive line bundle $\tilde{E}$ whose global holomorphic sections vanish precisely on the hyperplanes of $P_m(C)$. Hence the restriction of $\tilde{E}$ to X can play the role of E in our version. Condition T then expresses the transversality to X of the hyperplane determined by s. Hence (2.3) at any rate contains the classical result. The main generalization expressed by (2.3) is that one obtains the homotopy analogue of the Lefschetz result. Even if the theorem were assumed to be true in homology and for the fundamental group, the general homotopy statement would not be immediate; one encounters the usual road block that $\pi_1(S)$ may operate nontrivially on $\pi_k(X; S)$.

In [5], K. Kodaira proved the beautiful theorem that if X admits a positive line bundle, then X is an algebraic manifold. Thus our statement is no improvement over the classical one in this direction. Nevertheless a slight step forward is involved here, because not every positive line bundle is induced by an imbedding from the hyperplane bundle.

3. A THEOREM IN THE NONDEGENERATE MORSE THEORY

In the next section, Theorem I will be derived from Theorem II, which is formulated below. In [2] this result appears as the third corollary of Theorem III. The reader is referred to [2] for details; here we shall only review the pertinent definitions.

Throughout, we use the term "smooth" to denote C^∞, and we assume all manifolds to be smooth. Our functions will always be real-valued.

Let ϕ be a smooth function on the manifold M. The differential of ϕ is denoted by $d\phi$. The points where this form vanishes are called the critical points of ϕ. In terms of local coordinates, these are precisely the points where all the first partial derivatives of ϕ vanish.

Let $m \in M$ be a critical point of ϕ, and let M_m be the tangent space to M at m. The *Hessian* of ϕ, denoted by $H_m\phi$, is the symmetric quadratic form defined on M_m which in terms of local coordinates near m is defined by

$$H_m \phi \left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta}\right) = \frac{\partial^2}{\partial x_\alpha \, \partial x_\beta}\phi\bigg|_m .$$

The dimension of a maximal subspace on which the Hessian $H_m\phi$ is negative definite is called the *index* of the critical point.

DEFINITION 3.1. *A connected smooth submanifold* V *of* M *is called a nondegenerate critical manifold of* ϕ *provided*

(3.1) $$d\phi = 0 \text{ on } V,$$

(3.2) *if* $v \in V$, *then the nullspace of* $H_v\phi$ *is precisely the tangent space to* V *at* v.

Finally, we agree to call a function on M *nondegenerate* if it is smooth and its critical set consists entirely of nondegenerate critical manifolds.

THEOREM II. *Let* ϕ *be a nondegenerate function on the compact manifold* M. *Let* M_* *be the set on which* ϕ *takes on its absolute minimum. Also, let* $|\phi|$ *be the lowest index which occurs among the indices of critical points on* $M - M_*$. *Then* M *is obtained from* M_* *by successively attaching a finite number of cells of dimension* $\geq |\phi|$. *Thus*

(3.3) $$M = M_* \cup e_1 \cup e_2 \cup \cdots \cup e_r \qquad (\dim e_k \geq |\phi|).$$

Comparing (3.3) with (2.3), we see that in order to deduce (2.3) from Theorem II it will be sufficient to construct a nondegenerate function ϕ on X with $X_* = S$ and $|\phi| = \dim_C X$. As we shall see, a slight perturbation of the function (s, s) will have these properties.

4. THE PROOF OF THEOREM I

In this section we assume the conditions and the notation used in Theorem I.

Let $\mathfrak{h}$ be a positive hermitian structure on E, and consider the corresponding function (s, s).

(4.1) *Each component of* S *is a nondegenerate critical manifold of the function* (s, s).

Proof. Let p be a point of S, and let $\{(z_1, z_2, \cdots, z_n); s_*\}$ have the same meaning as in condition T, so that near p the section s has the representation $s = z_1 s_*$. We set (s_*, s_*) equal to a. Then a is a smooth positive function near p, and in terms of it we have $(s, s) = a z_1 \bar{z}_1$. Hence condition (3.1) is certainly satisfied on S. By Condition T, the set S is also a smooth submanifold of M. Hence there remains only the verification of (3.2). This amounts to checking that $H_p(s, s)$ is nondegenerate on the transversal quotient space M_p/S_p. Let $\{x_\alpha\}$ be functions such that $z_1 = x_1 + i x_2$. The x_α ($\alpha = 1, 2$) form a part of a real coordinate system near p, and we clearly have $H_p(s, s)(\partial/\partial x_\alpha, \partial/\partial x_\beta) = 2a(p)\delta_{\alpha\beta}$. Hence (3.2) is verified.

(4.2) *Let* p *be a critical point of* (s, s) *on* X - S. *Then the index of* p *is no less than* $\dim_C X = n$.

Proof. On X - S, the function (s, s) is positive. Hence the function $f = \log(s, s)$ is smooth and well-defined near p, and we have $(s, s) = e^f$. From the Taylor series of the exponential function, we see immediately that p is a critical point of f, and that the index of p, as a critical point of (s, s), is equal to its index as a critical point of f. Let then H be the Hessian of f at p. We extend H to a hermitian form on the complexification of M_p. This form, denoted by $\tilde{H}$, will have the same index as H. Now, by hypothesis, the hermitian structure on E is positive. Hence the differential form $(i/2\pi)\bar{\partial}\partial f$ is positive near p. In terms of local analytic coordinates, this is expressed by

(4.3) $$\textit{the form } \frac{\partial^2 f}{\partial z_\alpha \partial \bar{z}_\beta} u_\alpha \bar{u}_\beta \textit{ is negative definite.}$$

Here we have written

$$\frac{\partial}{\partial z_\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} - i\frac{\partial}{\partial y_\alpha}\right), \quad \frac{\partial}{\partial \bar{z}_\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} + i\frac{\partial}{\partial y_\alpha}\right),$$

the x_α and y_α being the real and imaginary parts of z_α. It follows that $\tilde{H}$ is negative definite on a complex subspace of dimension n, whence the index of p is at least equal to n.

Combining (4.1) and (4.2), we see that the function (s, s) has all the properties of the function ϕ of the last section, except possibly the nondegeneracy on X - S. We shall next show that this drawback can be eliminated by a suitable small perturbation of (s, s).

For this purpose, recall the generic character of nondegenerate functions, as expressed by the following approximation theorem. Let U be a finite covering of X with smooth coordinate functions. If then k is a positive integer and ε is a positive number, one has the notion of an (ε, k)-small function on X (with respect to U). Namely, the function f is called (ε, k)-small if the absolute value of f and of all its partial derivatives (with respect to the coordinates in U) up to order k are less than ε at all points of X.

APPROXIMATION THEOREM. *Let* F *be a smooth function on* X. *Then, corresponding to each pair* (ε, k) *as above, there exists an* (ε, k)-*small function* η *such that* $F + \eta$ *is nondegenerate.*

(This is by now a well-known theorem. For a completely selfcontained account, see [8].)

We apply this theorem in our situation in the following manner. Let A and B be two open, smooth, normal neighborhoods of S in X such that $\bar{A} \subset B$ and such that (s, s) has no critical values in $\bar{B}$ other than S. Such neighborhoods exist, because S is nondegenerate. Let g be a smooth function on X which is 1 on X - B and vanishes on A.

PROPOSITION 4.1. *There exists a positive number* ε_1 *such that, if* η *is any* $(\varepsilon_1, 2)$-*small function on* X, *then the function* $\phi = (s, s) + g\eta$ *satisfies the conditions*

(4.4) ϕ *is positive on* X - A,

(4.5) ϕ *has no critical points on the closure of* B - A,

(4.6) *the form* $\frac{i}{2\pi}\bar{\partial}\partial \log \phi$ *is positive on* X - A.

Remark first that if η is an (ε, k)-small function, then $g\eta$ is $(M\varepsilon, k)$-small, where M is some fixed positive number depending only on k. Hence the factor g causes no trouble. Next, observe that each of the conditions imposed on ϕ is an "open" condition to be imposed on a compact set, and is furthermore a condition which holds for the function (s, s). Hence the proposition is true.

By the approximation theorem, there exists an $(\varepsilon_1, 2)$-small function η_1 such that $(s, s) + \eta_1$ is nondegenerate. Consider now the function $\phi = (s, s) + g\eta_1$. The factor g affects this function only on the closure of B. Hence, by (4.5), ϕ is also nondegenerate. Because g vanishes on A, the set S is a nondegenerate critical manifold of ϕ. Finally, by (4.4) and (4.6), the argument of (4.2) applies to ϕ, whence $|\phi| \geq \dim_C X = n$. This is therefore the desired function on X, which in view of (3.3) establishes Theorem I.

5. THE GENERALIZED LEFSCHETZ THEOREM

Looking back upon the proof of Theorem I, we see that modulo the Morse theory the heart of the proof lies in (4.1) and (4.2). These propositions in turn have their direct antecedents in condition T and in the condition $c(E, \mathfrak{h}) > 0$, respectively. It is therefore natural to see whether these conditions can be weakened without substantially affecting the remainder of the argument. For instance, the condition T could well be replaced by the condition that the null set of s be a deformation retract of the set $(s, s) \leq \varepsilon$ for $\varepsilon > 0$ small enough. Here we shall record only one such technical generalization, which is obtained by weakening the second condition, in the hope that it will find some application in the future.

DEFINITION 5.1. *The hermitian structure* $\mathfrak{h}$ *on* E *will be called of type* k *if at each point of* X *the form* $c(E, \mathfrak{h})$ *is positive definite on a* k-*dimensional subspace of the tangent space.*

For example, if $\mathfrak{h}$ is of type $n = \dim_C X$, then $\mathfrak{h}$ is positive in the old sense.

THEOREM III. *Let* $\mathfrak{h}$ *be a hermitian structure of type* k *on the line bundle* E *over* X. *Let* s *be a nonsingular holomorphic section of* E *with null set* S. *Then* X *is obtained from* S *by successively attaching cells of dimension* $\geq$ k.

The proof parallels exactly the proof of Theorem I. The only change to be made is that wherever the positiveness of $c(E, \mathfrak{h})$ was used, we now apply the condition that $\mathfrak{h}$ be of type k, and we obtain a corresponding result. Thus (4.1) applies as before, while in (4.2) the n has to be replaced by k. (In the new argument, the form $\tilde{H}$ is negative definite on a k-dimensional subspace.) Finally, in Proposition 4.3 the condition (4.6) is changed in the obvious manner.

REFERENCES

1. A. Andreotti and T. Frankel, *The Lefschetz theorem on hyperplane sections*, Ann. of Math. (to appear).

2. R. Bott, *The stable homotopy of the classical groups*, Ann. of Math. (to appear).

3. I. Fary, *Cohomologie des variétés algébriques*, Ann. of Math. (2) 65 (1957), 21-73.

4. S. Lefschetz, *L'analysis situs et la géométrie algébrique*, Gauthier-Villars, Paris, 1924.

5. K. Kodaira, *On Kähler varieties of restricted type, (an intrinsic characterization of algebraic varieties)*, Ann. of Math. (2) 60 (1954), 28-48.

6. A. H. Wallace, *Homology theory on algebraic varieties*, New York, Pergamon Press, 1958.

7. G. W. Whitehead, *Homotopy theory*, M.I.T. notes (1953).

8. H. Whitney, *On singularities of mappings of Euclidean spaces. I. Mappings of the plane into the plane*, Ann. of Math. (2) 62 (1955), 374-410.

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