ON NORMAL AND EPr MATRICES

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1. INTRODUCTION

H. Schwerdtfeger [2] has called a matrix A of order n and rank r with elements from the complex field C an EPr matrix if it satisfies the condition

$$(1) \qquad \sum_{i=1}^{n} \alpha_i A_i = 0 \quad \text{if and only if} \quad \sum_{i=1}^{n} \bar{\alpha}_i A^i = 0 \qquad (\alpha_i \in C),$$

where A_i is the ith row of A and A^i is the ith column of A. Another formulation of this definition is that

$$(1') \qquad A\xi = 0 \quad \text{if and only if} \quad A^*\xi = 0,$$

where ξ is contained in the complex n-dimensional Euclidean space C_n. The class of EPr matrices ($r = 0, 1, \cdots, n$) contains the normal matrices and the nonsingular matrices as subclasses.

In Section 2, other characterizations of complex EPr matrices are given, and in Section 3, EPr matrices are used to develop characterizations for normal matrices with elements from an arbitrary field.

2. COMPLEX EPr MATRICES

THEOREM 0 (Schwerdtfeger [2, p. 131]). *A necessary and sufficient condition that* A *be an* EPr *matrix is that there exist a nonsingular matrix* Q *such that* QAQ* *is the direct sum of a nonsingular matrix* D *of order* r *and a zero matrix.*

The following theorem gives other necessary and sufficient conditions that A be an EPr matrix.

THEOREM 1. *The following statements are equivalent:*

(i) A *is an* EPr *matrix;*

(ii) A *is unitarily similar to the direct sum of a nonsingular matrix* D *of order* r *and a zero matrix;*

(iii) A *is congruent to the direct sum of a nonsingular matrix* D *of order* r *and a zero matrix;*

(iv) A *is the matrix of a linear transformation* T *acting on* C_n *and having the property that* C_n *can be expressed as the direct sum of two mutually orthogonal* T-*spaces* V_1 *and* V_2 *such that* $T(V_1) = V_1$, $T(V_2) = 0$ *and* V_1 *has dimension* r;

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(v) A *has rank* r, *and there exists a nonsingular matrix* N *such that* $A^* = NA$;

(vi) A *can be represented as*

$$P\begin{pmatrix} D & DX^* \\ XD & XDX^* \end{pmatrix} P^* = P\begin{pmatrix} I & 0 \\ X & I \end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I & X^* \\ 0 & I \end{pmatrix} P^*, \tag{2}$$

where P *is a permutation matrix and* D *is nonsingular and of order* r.

It is clear that (ii) implies (i), (iii), (iv) and (v), since each statement is invariant under unitary transformations and is immediate when A is in the form $D \dotplus 0$.

(i) → (ii). Let A be an EPr matrix of order n. There is a unitary matrix which transforms A into $B = [B_{ij}]$ (i, j = 1, 2) such that B_{11} is nonsingular, B_{12} is zero and B_{22} is a triangular matrix with zeros on and above the main diagonal. If $n > r$, the last column of B is zero, and hence the last row of B is also zero. If $n > r + 1$, the last two columns of B are zero and hence the last two rows of B are zero. A repetition of this argument yields the result $B_{21} = B_{22} = 0$.

(iii) → (i). The equivalence of (i) and (iii) is contained in Theorem 0.

(iv) → (ii). After a suitable unitary transformation on C_n (under which the EPr property is invariant) we can select the unit vectors $e_1, e_2, \cdots, e_r$ and $e_{r+1}, \cdots, e_n$ as bases for V_1 and V_2, respectively. We partition A into matrices $[A_{ij}]$ (i, j = 1, 2). Relative to this basis, A_{11} is nonsingular and $A_{12} = A_{21} = A_{22} = 0$.

(v) → (i). Since N is nonsingular, $A^*\xi = 0$ implies $A\xi = 0$.

(i) → (vi). Let A be an EPr matrix. It is known [2] that A has a nonsingular principal submatrix D of order r. Therefore, for a proper choice of P, the matrix B_{11} in the upper left-hand corner of $P^*AP = B = [B_{ij}]$ (i, j = 1, 2) is D. If we set $X = B_{12}D^{-1}$ and $Y = D^{-1}B_{21}$, then

$$B = \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & B_{22} - XDY \end{pmatrix}\begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}.$$

Since the rank of A is r, it follows that $B_{22} = XDY$, and hence $B = \begin{pmatrix} D & DY \\ XD & XDY \end{pmatrix}$.

Let

$$C = \begin{pmatrix} I & 0 \\ -X & I \end{pmatrix}.$$

Then CP^*APC^* is an EPr matrix and is in the form $\begin{pmatrix} D & DY - DX^* \\ 0 & 0 \end{pmatrix}$. Hence $DY = DX^*$, and therefore $Y = X^*$.

(vi) → (iii) is immediate, and the proof is complete.

3. NORMAL AND EPr MATRICES OVER AN ARBITRARY FIELD

Let $\lambda: a \rightleftharpoons \bar{a}$ be an involutory automorphism of a field F. For a matrix $A = [a_{ij}]$ contained in the algebra of n-by-n matrices with elements in F, the conjugate transpose A^* of A is defined by

$$A^* = [b_{ij}] \qquad (b_{ij} = \bar{a}_{ji}),$$

and A is an EPr matrix if it satisfies the condition

$$(1'') \qquad \sum_{i=1}^{n} \alpha_i A_i = 0 \quad \text{if and only if} \quad \sum_{i=1}^{n} \bar{\alpha}_i A^i = 0 \qquad (\alpha_i \in F).$$

By these definitions, (i), (iii), (v) and (vi) of Theorem 1 are equivalent.

A simple example demonstrates that a normal matrix (that is, a matrix A such that $AA^* = A^*A$) over F need not be an EPr matrix. In particular, let F be GF(5), $a = \bar{a}$, and consider

$$A_0 = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

However, the following lemma holds.

LEMMA. *If* A *is normal and has the same rank as* AA^*, *then* A *is an* EPr *matrix.*

If $A^*\xi = 0$, then $A^*\xi = AA^*\xi = A^*A\xi = 0$. Since A and A^*A have the same rank, their null spaces, $\eta(A)$ and $\eta(A^*A)$, have the same dimension, n - r. But $\eta(A) \subset \eta(A^*A)$, and therefore $\eta(A) = \eta(A^*A)$. Thus $A\xi = 0$. Similarly, $A\xi = 0$ implies that $A^*\xi = 0$.

It is known [1] that if A is a complex normal matrix, then A^* can be expressed as the product of A and a unitary matrix which commutes with A. Conversely, a matrix having this property is normal. However, A_0 is not an EPr matrix. For $A_0^* = NA_0$ would imply that $\eta(A_0) \subset \eta(A_0^*)$. Clearly $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \eta(A_0)$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \notin \eta(A_0^*)$. However, we can prove

THEOREM 2. *Let* A *have the same rank as* AA^*. *Then a necessary and sufficient condition that* A *be normal is that* $A^* = NA = AN$ *for some nonsingular* N. *The matrix* N *may always be chosen to be unitary* (that is, so that $NN^* = N^*N = I$).

That the condition is sufficient is obvious.

Let A be normal and have the same rank as AA^*. By the lemma, A is an EPr matrix. We shall use the representation (2) of an EPr matrix for A. It follows from the normality of A that $D(I + X^*X)D^* = D^*(I + X^*X)D$, and since A and AA^* have the same rank, $I + X^*X$ is nonsingular. The general solution of the equation $D^* \dot{+} 0 = N(D \dot{+} 0)$, for nonsingular N, is

$$N = \begin{pmatrix} D^*D^{-1} & N_1 \\ 0 & N_2 \end{pmatrix},$$

where N_1 is arbitrary, and where the only condition on N_2 is that it be nonsingular. Hence the general solution of $A^* = NA$ for nonsingular N is given by

$$N = P\begin{pmatrix} I & 0 \\ X & I \end{pmatrix}\begin{pmatrix} D^*D^{-1} & N_1 \\ 0 & N_2 \end{pmatrix}\begin{pmatrix} I & 0 \\ -X & I \end{pmatrix}P^*,$$

Finding N such that $A^* = AN = NA$ is equivalent to solving the equation

$$D(I + X^*X)N_1 + DX^*N_2 = D^*X^*.$$

Since D and $I + X^*X$ are nonsingular, a solution always exists. If N_2 is selected to be I, then $N^*N = I$.

THEOREM 3. *Let* K *be a field containing* F *and the characteristic roots of* A. *For each* β *in* K, *let* $A - \beta I$ *and* $(A - \beta I)(A - \beta I)^*$ *have the same rank* r_β, *and let* $A - \beta I$ *be an* EPr_β *matrix. Then* A *is normal, and there exists a scalar polynomial* f *such that* $A^* = f(A)$.

Since $A - \beta I$ is an EPr_β matrix, $(A - \beta I)^* = (A - \beta I)N_\beta$ for some nonsingular N_β, and $A - \beta I$ has the same rank as $(A - \beta I)^2 N_\beta$. Thus all roots of the minimal equation of A are simple, and to each root of multiplicity e_β of A, there correspond e_β linearly independent characteristic vectors $\xi_1, \xi_2, \cdots, \xi_{e_\beta}$. The totality of these vectors for all roots of A constitutes a set of n linearly independent characteristic vectors of A. For any characteristic vector ξ of A, with root β, we have, using (1'), $A^*\xi = \bar{\beta}\xi$ and $AA^*\xi = \bar{\beta}A\xi = \bar{\beta}\beta\xi = A^*A\xi$. Thus $(A^*A - AA^*)\xi = 0$, and therefore $A^*A - AA^* = 0$.

Let $B = A - \beta I$. We have shown that the roots of the minimal equation of B are simple, and hence that B is similar to a diagonal matrix. Let

$$SBS^{-1} = T = T_1 \dotplus T_2 \dotplus \cdots \dotplus T_k,$$

where $T_i = \beta_i I$ and $\beta_i \neq \beta_j$ for $i \neq j$. Since B is normal,

$$S^{-1}TSS^*T^*(S^*)^{-1} = S^*T^*(S^*)^{-1}S^{-1}TS. \tag{3}$$

Set $SS^* = U$. Then (3) is equivalent to $T(UT^*U^{-1}) = (UT^*U^{-1})T$, and it follows that $UT^*U^{-1} = SB^*S^{-1}$ is of the form $X_1 \dotplus X_2 \dotplus \cdots \dotplus X_k$, and that $T_iX_i = X_iT_i$. However, B^* is similar to a diagonal matrix, and hence each of the X_i is similar to a diagonal matrix. Let $U_iX_iU_i^{-1} = Z_i$ be diagonal. Set $U = U_1 \dotplus U_2 \dotplus \cdots \dotplus U_k$ and $W = US$. Then

$$WBW^{-1} = T_1 \dotplus T_2 \dotplus \cdots \dotplus T_k \quad \text{and} \quad WB^*W^{-1} = Z_1 \dotplus Z_2 \dotplus \cdots \dotplus Z_k.$$

Since B, B^* and BB^* have the same rank, $T_i = 0$ if and only if $Z_i = 0$. Thus WAW^{-1} has β as its ith diagonal element if and only if WA^*W^{-1} has $\bar{\beta}$ as its ith diagonal element. The Lagrange interpolation formula yields a polynomial f such that $WA^*A^{-1} = f(WAW^{-1})$ and hence $A^* = f(A)$.

Although the hypothesis that $A - \beta I$ has the same rank as $(A - \beta I)(A - \beta I)^*$ is not needed to prove that A is normal when $n = 2$, it cannot be dropped in general. For let F be the field $GF(11)$ $(\bar{a} = a)$, and consider the matrix

$$A_1 = \begin{pmatrix} 2 & 3 & 7 \\ 1 & 3 & 8 \\ 1 & 6 & 6 \end{pmatrix}.$$

The characteristic polynomial of A_1 is x^3, and therefore its only root is 0. Since A_1 is an EP2 matrix (with $\xi = (1, -3, 1)$), $A_1 - \beta I$ is an EPr_β matrix for all β in any extension field of F. However, A_1 is not normal.

For a complex normal matrix A, it is well known [3] that there exists a scalar polynomial f such that $A^* = f(A)$. Theorem 3 extends this result to a class of normal matrices over an arbitrary field. However, the result does not hold in general. In GF(7), with $\bar{a} = a$, the matrix

$$A_2 = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 2 & -1 & 0 & 4 \\ 3 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is normal, and $A_2^2 = 0$. Each polynomial in A_2 is of the form $pI + qA_2$. Suppose there exist p and q such that $A_2^* = pI + qA_2$. Then, by considering the (1, 1) and (4, 4) positions of each side of this equation, we have $p = 0$ and $q = 1$, which implies that $A_2 = A_2^*$. This is a contradiction.

Over the complex field, A and AA^* always have the same rank, and we can therefore characterize complex normal matrices as follows.

THEOREM 4. *A necessary and sufficient condition that the matrix* A *be normal is that* $A - \beta I$ *be an* EPr_β *matrix for all* β *in* C.

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