ON MODULAR FORMS OF NEGATIVE DIMENSION

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1. Modular forms of arbitrary real negative dimension $r < -2$ may be constructed by means of the generalized Poincaré series of Petersson [3]:

$$F_\mu(\tau) = \frac{1}{2}\sum_V \frac{\exp\{-2\pi i(\mu - \alpha)V\tau\}}{\varepsilon(V)(-i(c\tau + d))^s} \qquad (\mu = 1, 2, \cdots; s > 2). \tag{1.1}$$

The series is extended over all matrices $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of the modular group with different lower row; $\varepsilon(V)$ is a multiplier system for the dimension $r = -s$, and $0 \leq \alpha < 1$. (Precise definitions are given in Section 2.) It can be seen directly from (1.1) that when $s > 2$, the series converges absolutely and uniformly in every region $\Im\, \tau \geq y_0 > 0$, which implies that $F_\mu(\tau)$ is regular for $\Im\, \tau > 0$. The absolute convergence of the series enables us to rearrange the terms and thus to establish without difficulty the transformation property

$$F_\mu(W\tau) = \varepsilon(W)(-i(\gamma\tau + \delta))^s F_\mu(\tau), \tag{1.2}$$

for every $W = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ belonging to the modular group. $F_\mu(\tau)$ is not identically zero; for when it is expanded in a Fourier series (Laurent series in $\exp 2\pi i\tau$), it has a single term with negative exponent, namely $\exp(-2\pi i(\mu - \alpha)\tau)$.

When $s = 2$ (that is, $r = -2$), absolute convergence fails, and we cannot obtain (1.2) and the other facts mentioned above so readily. One method of overcoming the difficulty was suggested by Hecke [2], who introduced the convergence factor $|c\tau + d|^\sigma$ in the denominator. This method has been successfully exploited by Petersson [4].

In the following sections, we present an alternative approach based on the conditional convergence of (1.1). We show that the series (1.1) for $s = 2$, when summed in a certain order, does in fact converge uniformly to a function $F_\mu(\tau)$ which is regular in $\Im\, \tau > 0$ and satisfies (1.2) there. That is, $F_\mu(\tau)$ is a modular form of dimension -2, with multipliers $\varepsilon(V)$. Moreover, the method enables us to represent the Fourier coefficients of F_μ as series of Bessel functions, similar to the ones obtained by Petersson [3] and by Rademacher and Zuckerman [8].

However, these results are obtained on the basis of a certain Assumption A, namely, that the exponential sums

$$A_{k,\mu}(m) = \sum_{\substack{h=0 \\ (h,k)=1}}^{k-1} \varepsilon^{-1}(V_{k,-h}) \exp\{-2\pi i((\mu - \alpha)h' + (m + \alpha)h)/k\} \tag{1.3}$$

(see Section 2 for definitions) can be estimated as $O((m,k)^{\frac{1}{2}}k^{\frac{1}{2}+\varepsilon})$. In Section 7 we show, by reducing the sums in question to classical Kloosterman sums, that this

Received January 28, 1958.

estimate is valid for all $F_\mu(\tau)$ of dimension -2. Thus, the linear combinations of the series (1.1) are sufficient to represent the modular forms of dimension -2 having a polar singularity (in $\exp 2\pi i\tau$) at infinity, at least up to cusp forms (see Theorem 2, Section 7).

Even more interesting is the situation when $-2 < r < -3/2$. Our development will show that the series (1.1) still converges ($3/2 < s$) if Assumption A is valid; and $F_\mu(\tau)$ would then be a modular form of dimension r. This would yield convergent series representations of the Fourier coefficients of modular forms of dimension r for $-2 < r < -3/2$, representations not known at present.

However, the verification of the required estimate for the exponential sums (1.3) in this case is not trivial. We shall return to this question in a later publication.

I owe the idea for this investigation to Rademacher's paper [5], in which forms of dimension 0 rather than -2 are treated.

2. *Preliminaries.* The modular group $\Gamma(1)$ is the set of all 2-by-2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with rational integral entries and with determinant one. A modular substitution is a nonhomogeneous linear transformation

$$V\tau = \frac{a\tau + b}{c\tau + d};$$

we see that V and $-V = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ correspond to the same substitution. $\Gamma(1)$ is known to be generated by the matrices

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

with the relations

$$(-I)^2 = U^3 = I, \qquad T_2 = -I.$$

We shall refer to both V and $V\tau$ as a modular substitution.

An entire modular form of dimension r is a function $F(\tau)$, regular in the upper half-plane and having at most a polar singularity at $\tau = i\infty$, which satisfies the transformation equation

$$F(V\tau) = \varepsilon(V)\,(-i(c\tau + d))^{-r}\,F(\tau) \qquad (|\varepsilon(V)| = 1) \tag{2.1}$$

for every modular substitution $V\tau$. Since $V\tau = (-V)\tau$, we may always assume that $c \geq 0$, and fix $|\arg(-i(c\tau + d))| \leq \pi/2$. In particular, (2.1) implies

$$\varepsilon(I) = e(-r/4), \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.2}$$

where, throughout this paper,

$$e(z) = e^{2\pi iz}.$$

When $c = 0$, V is of the form S^m, with m an integer and $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We write

$$F(S\tau) = \varepsilon(S)\, e(r/4)\, F(\tau) = e(\alpha)\, F(\tau)\,, \tag{2.3}$$

where we can choose α in the range

$$0 \leq \alpha < 1\,. \tag{2.4}$$

We can extend the multipliers $\varepsilon(V)$ to matrices V for which $c < 0$, as follows: apply (2.1) with V replaced by $-V$. Since $V\tau = (-V)\tau$, we get

$$\varepsilon(-V)\,(-i(-c\tau - d))^{-r} = \varepsilon(V)\,(-i(c\tau + d))^{-r}\,. \tag{2.5}$$

In particular, with $V = I$, (2.2) gives

$$\varepsilon(-I) = e(r/4)\,. \tag{2.6}$$

For any two substitutions V_1 and V_2, we can evaluate $F(V_1 V_2 \tau)$ in two ways; comparison then yields a "consistency condition":

$$\varepsilon(V_1 V_2)\,(-i(c_{12}\,\tau + d_{12}))^{-r} = \varepsilon(V_1)\,\varepsilon(V_2)\,(-i(c_1 V_2\tau + d_1))^{-r} \cdot (-i(c_2\tau + d_2))^{-r}\,, \tag{2.7}$$

where $V_1 V_2 = (\dot{c}_{12}\;\; \dot{d}_{12})$.

We apply this principle to $F(VS^m\tau)$ and $F(S^m V\tau)$, and recall (2.3). This gives

$$\varepsilon(S^m V) = \varepsilon(VS^m) = e(m\alpha)\,\varepsilon(V) \qquad (m \text{ an integer})\,. \tag{2.8}$$

The multipliers $\varepsilon(V)$ are said to form a *multiplier system for the dimension* r if $\varepsilon(V)$ is a complex-valued function of $V \in \Gamma(1)$, $|\varepsilon(V)| = 1$ for all V, and $\varepsilon(V)$ satisfies (2.7). It is easily seen that the relation (2.7) is independent of τ, so that $\varepsilon(V)$ can be calculated for any V if $\varepsilon(V)$ is known for the generators U, T, $-I$ of $\Gamma(1)$. ($\varepsilon(-I)$ is already determined by r, as we saw in (2.6).) It is known, moreover, that $\varepsilon(V)$ is determined *uniquely* by the values $\varepsilon(U)$, $\varepsilon(T)$ ([3], 393-401). Since $U = TS$, this amounts to saying that $\varepsilon(V)$, *a multiplier system for the dimension* r, *is uniquely determined by* α and $\varepsilon(T)$. Thus, in order to show that $F(\tau)$ is a modular form, it is only necessary to prove that

$$F(T\tau) = \varepsilon(T)\,(-i\tau)^{-r}\, F(\tau)\,,$$

$$F(S\tau) = e(\alpha)\, F(\tau)\,.$$

In the following sections we shall encounter certain exponential sums:

$$A_{k,\mu}(m) = \sum_{\substack{h=0 \\ (h,k)=1}}^{k-1} \varepsilon^{-1}(V_{k,-h})\, e(-[(\mu - \alpha)h' + (m + \alpha)h]/k) \qquad (k \geq 1)\,, \tag{2.9}$$

where m, μ are integers, and $V_{k,-h}$ is the modular substitution

$$V_{k,-h} = \begin{pmatrix} h' & k' \\ k & -h \end{pmatrix}. \tag{2.10}$$

Here h' is defined by

$$hh' \equiv -1 \pmod{k} \qquad (0 \le h' < k) \tag{2.11}$$

and

$$-k' = (hh' + 1)/k. \tag{2.12}$$

Concerning these sums we make the following

ASSUMPTION A. *For every* $\varepsilon > 0$,

$$|A_{k,\mu}(m)| \le C_\varepsilon (\rho m + \sigma, k)^{\frac{1}{2}} k^{\frac{1}{2}+\varepsilon} \qquad (m = 0, 1, 2, \cdots) \tag{2.13}$$

unless $\alpha = 0$, $m = 0$, *in which case*

$$|A_{k,\mu}(0)| \le C. \tag{2.14}$$

Here C_ε, C *and the integers* $\rho > 0$ *and* σ *depend on* r, μ *and* α, *and* $\rho m + \sigma \neq 0$.

3. We are going to study the series

$$H(\tau) = \sum_{k=1}^{\infty} \sum_{m=-\infty}^{\infty}{}' \frac{e(-(\mu - \alpha) V_{k,-m}\tau)}{\varepsilon(V_{k,-m}) (-i(k\tau - m))^s}, \tag{3.1}$$

where α satisfies (2.4), $\tau = x + iy$ $(y > 0)$, $\mu = 1, 2, 3, \cdots$, and

$$3/2 < s = -r \le 2, \tag{3.2}$$

and where Σ' means that the summation variable is prime to k. $V_{k,-m}$ is the substitution

$$V_{k,-m} = \begin{pmatrix} m' & k' \\ k & -m \end{pmatrix}; \tag{3.3}$$

since it is unimodular, we have

$$mm' + kk' + 1 = 0. \tag{3.4}$$

Thus

$$m'm \equiv -1 \pmod{k}, \tag{3.5}$$

and m' is determined only modulo k.

Despite this ambiguity, the terms of the series are determined uniquely, for (2.8) shows that $e(-(\mu - \alpha)V_{k,-m}\tau)\varepsilon^{-1}(V_{k,-m})$ is invariant under $m' \to m' + k$. Indeed, under this replacement $V_{k,-m} \to SV_{k,-m}$, and, by (2.8), $\varepsilon \to e(\alpha)\cdot\varepsilon$; on the other hand, $V_{k,-m}\tau = m'/k - 1/k(k\tau - m)$ picks up the added term 1.

Our first problem is to establish the convergence of the series (3.1). Write

(3.6)
$$H(\tau) = G_1 + G_2 = \lim_{K\to\infty} G_1(K) + \lim_{K\to\infty} G_2(K),$$

(3.7)
$$\begin{cases} G_1(K) = \sum_{k=1}^{K} \sum_{m=-\infty}^{\infty}{}' \dfrac{e(-(\mu - \alpha)m'/k)\{e((\mu - \alpha)/k(k\tau - m)) - 1\}}{\varepsilon(V_{k,-m})(-i(k\tau - m))^s}, \\ G_2(K) = \sum_{k=1}^{K} \sum_{m=-\infty}^{\infty}{}' \dfrac{e(-(\mu - \alpha)m'/k)}{\varepsilon(V_{k,-m})(-i(k\tau - m))^s}. \end{cases}$$

We have, on expanding the exponential,

(3.8)
$$G_1(K) = \sum_{k=1}^{K} \sum_{m=-\infty}^{\infty}{}' \frac{e(-(\mu - \alpha)m'/k)}{\varepsilon(V_{k,-m})} \sum_{\ell=1}^{\infty} \frac{(2\pi(\mu - \alpha))^\ell}{\ell!\, k^\ell (-i(k\tau - m))^{\ell+s}}.$$

This triple series converges absolutely; for it is dominated by

$$\sum_{k,m,\ell} \frac{(2\pi(\mu - \alpha))^\ell}{\ell!\, k^\ell [(kx - m)^2 + k^2 y^2]^{(\ell+s)/2}}$$

$$\leq \sum_{\ell=1}^{\infty} \frac{(2\pi\mu)^\ell}{\ell!\, k^\ell} \sum_{k=1}^{\infty} \left\{ \frac{1}{(ky)^{\ell+s}} + 2 \sum_{m=1}^{\infty} \frac{1}{(m^2 + k^2 y^2)^{(\ell+s)/2}} \right\}$$

(3.9)
$$\leq \sum_{\ell=1}^{\infty} \frac{(2\pi\mu)^\ell}{\ell!} \left\{ \sum_{k=1}^{\infty} \frac{1}{y^{\ell+s} k^{2\ell-s}} + 2 \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(2kmy)^{(\ell+s)/2} k^\ell} \right\}$$

$$\leq y^{-s} \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \left(\frac{2\pi\mu}{y} \right)^\ell \sum_{k=1}^{\infty} \frac{1}{k^2} + 2(2y)^{-s/2} \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \left(\frac{2\pi\mu}{\sqrt{2y}} \right)^\ell \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} \cdot \sum_{m=1}^{\infty} \frac{1}{m^{\frac{1+s}{2}}}$$

$$\leq C_1 y^{-s} \exp(2\pi\mu/y) + C_2 y^{-s/2} \exp(2\pi\mu/\sqrt{2y}).$$

Hence, the series converges absolutely and uniformly in $y \geq y_0 > 0$, and $\lim_{K\to\infty} G_1(K) = G_1$ is an analytic function of τ, regular in $\Im\tau > 0$. (In fact, this is true for $s > 1$.)

To handle $G_2(K)$, we recall that the expression $e(-(\mu - \alpha)m'/k)\,\varepsilon^{-1}(V_{k,-m})$ is independent of the choice of m'. Hence, in $G_2(K)$ we may choose m' in the range $0 \leq m' < k$. Now write

$$m = qk + h \quad (0 \leq h < k,\ (h, k) = 1,\ -\infty < q < \infty).$$

Then $V_{k,-m} = V_{k,-h} S^{-q}$. It follows from the remark above that $m' = h'$, where h' is defined by (2.11). Using this and (2.8), we get

$$G_2(K) = \sum_{k=1}^{K} \sum_{h=0}^{k-1}{}' \varepsilon^{-1}(V_{k,-h})\, e(-(\mu - \alpha)h'/k) \sum_{q=-\infty}^{\infty} \frac{e(\alpha q)}{[-i(k\tau - h) + ikq]^s}.$$

We now distinguish two cases, $\alpha > 0$ and $\alpha = 0$, and treat the former.

The inner sum is, by the Lipschitz formula ([1], p. 206),

$$\frac{(2\pi/k)^s}{\Gamma(s)} \sum_{\ell=0}^{\infty} (\ell + \alpha)^{s-1}\, e((\tau - h/k)(\ell + \alpha));$$

hence,

$$\begin{aligned} G_2(K) &= \frac{(2\pi)^s}{\Gamma(s)} \sum_{k=1}^{K} k^{-s} \sum_{\ell=0}^{\infty} (\ell + \alpha)^{s-1} e((\ell + \alpha)\tau) \\ &\quad \cdot \sum_{h=0}^{k-1}{}' \varepsilon^{-1}(V_{k,-h})\, e(-[(\mu - \alpha)h' + (\ell + \alpha)h]/k), \end{aligned} \tag{3.10}$$

where we have interchanged the order of the summations with respect to h and ℓ. The sum on h is an exponential sum which, by Assumption A, is $O((\ell + 1)^{\frac{1}{2}} k^{\frac{1}{2}+\varepsilon})$, for $\rho\ell + \sigma \neq 0$ implies $(\rho\ell + \sigma, k) = O(\ell + 1)$. It follows that

$$|G_2(K)| \leq C_\varepsilon \sum_{k=1}^{\infty} k^{-s+\frac{1}{2}+\varepsilon} \,|1 - e^{-2\pi y}|^{-2}.$$

Since $s > 3/2$, $\lim_{K\to\infty} G_2(K) = G_2$ exists uniformly in $y \geq y_0 > 0$ and G_2 is regular in the upper half-plane.

When $\alpha = 0$, $G_2(K)$ becomes

$$G_2(K) = \sum_{k=1}^{K} \sum_{h=0}^{k-1}{}' \varepsilon^{-1}(V_{k,-h})\, e(-\mu h'/k) \sum_{q=-\infty}^{\infty} [-i(k\tau - h) + ikq]^{-s}.$$

Applying the appropriate Lipschitz formula ([1], p. 206), namely,

$$\{(2\pi)^s/\Gamma(s)\} \sum_{\ell=1}^{\infty} \ell^{s-1}\, e(\ell(\tau - h/k)) = \sum_{q=-\infty}^{\infty} [-i(\tau - h/k) + iq]^{-s},$$

we get

$$G_2(K) = \frac{(2\pi)^s}{\Gamma(s)} \sum_{k=1}^{K} k^{-s} \sum_{\ell=1}^{\infty} \ell^{s-1} e(\ell\tau) \sum_{h=0}^{k-1}{}' \varepsilon^{-1}(V_{k,-h})\, e(-[\mu h' + \ell h]/k).$$

The innermost sum is an exponential sum which, by Assumption A, can be estimated by $O(\ell^{\frac{1}{2}} k^{\frac{1}{2}+\varepsilon})$, since $\ell > 0$ implies $\rho\ell + \sigma \neq 0$. Thus we obtain in this case also the result that $G_2 = \lim_{K\to\infty} G_2(K)$ is regular in $\Im\tau > 0$.

Combining this with the result on G_1 we have the following lemma.

LEMMA 1. *The function* $H(\tau)$, *defined by* (3.1), *is a regular function of* τ *in* $\Im\tau > 0$.

4. To prove the transformation properties of $H(\tau)$ under modular substitutions, we shall need another expression for it. For this purpose, we require a lemma which follows closely one of Rademacher's ([5, p. 238], [6]). Since H is regular in the upper half-plane, we can confine our attention to $\tau = iy$, $y > 0$, and later extend our results by analytic continuation.

LEMMA 2. *Let* $\tau = iy$, $y > 0$, *and let* $s > 3/2$. *Then*

$$\sum_{k=1}^{\infty}\sum_{m=-\infty}^{\infty} \varepsilon^{-1}(V_{k,-m})\, e(-(\mu-\alpha)m'/k)\cdot(-i(k\tau-m))^{-s}$$

$$= \lim_{K\to\infty}\sum_{k=1}^{K}\sum_{m=-K}^{K}{}' \varepsilon^{-1}(V_{k,-m})\, e(-(\mu-\alpha)m'/k)\cdot(-i(k\tau-m))^{-s}.$$

The convergence of the left member has already been demonstrated. Thus the statement of the lemma is equivalent to

$$\lim_{K\to\infty}\sum_{k=1}^{K}\sum_{|m|>K}{}' \varepsilon^{-1}(V_{k,-m})\, e(-(\mu-\alpha)m'/k)\cdot(-i(k\tau-m))^{-s} = 0. \tag{4.1}$$

Let

$$T_k(K) = \sum_{|m|>K}{}' \varepsilon^{-1}(V_{k,-m})\, e(-(\mu-\alpha)m'/k)\cdot(-i(k\tau-m))^{-s}. \tag{4.2}$$

Define the function

$$g(m) = \begin{cases} \varepsilon^{-1}(V_{k,-m})\cdot e(-(\mu-\alpha)m'/k) & ((m,k)=1), \\ 0 & \text{otherwise.} \end{cases}$$

where, as in Section 2, we make m' unique by requiring that $0 \le m' < k$. If we replace m by $m + k$, $V_{k,-m}$ goes into $V_{k,-m}S^{-1}$, and, by (2.8), $\varepsilon^{-1} \to \varepsilon^{-1}\cdot e(\alpha)$. Hence, $g(m)\,e(-\alpha m/k)$ is periodic in m with period k, so that we have the finite Fourier series

$$g(m) = \sum_{j=0}^{k-1} B_j\, e((j+\alpha)m/k),$$

with

$$B_j = k^{-1}\sum_{\ell=0}^{k-1}{}' \varepsilon^{-1}(V_{k,-\ell})\, e(-[(\mu-\alpha)\ell' + (j+\alpha)\ell]/k) = k^{-1}A_{k,\mu}(j),$$

where $A_{k,\mu}(j)$ is defined by (2.9).

Then

(4.3)
$$
\begin{aligned}
T_k(K) &= \sum_{j=1}^{k-1} B_j \sum_{m=K+1}^{\infty} \frac{e((j+\alpha)m/k)}{(-i(k\tau - m))^s} \\
&+ \sum_{j=1}^{k-1} B_j \sum_{m=K+1}^{\infty} \frac{e(-(j+\alpha)m/k)}{(-i(k\tau + m))^s} + B_0 \sum_{|m|>K} (-i(k\tau - m))^{-s} e(\alpha m/k) \\
&= V_1 + V_2 + V_3 .
\end{aligned}
$$

Now when $\alpha > 0$ we have, by Assumption A,

$$|B_0| \le C_\varepsilon\, (\sigma, k)^{\frac{1}{2}} k^{-\frac{1}{2}+\varepsilon} \le C_\varepsilon\, k^{-\frac{1}{2}+\varepsilon},$$

while $\alpha = 0$ implies

$$B_0 \le C k^{-1} < C k^{-\frac{1}{2}+\varepsilon};$$

thus

(4.4)
$$
\begin{aligned}
|V_3| &\le 2 C_\varepsilon\, k^{-\frac{1}{2}+\varepsilon} \sum_{m=K+1}^{\infty} (k^2 y^2 + m^2)^{-s/2} \\
&< C_\varepsilon\, k^{-\frac{1}{2}+\varepsilon} \sum_{m=K+1}^{\infty} m^{-s} < C_\varepsilon\, k^{-\frac{1}{2}+\varepsilon} K^{1-s},
\end{aligned}
$$

the C_ε being not necessarily the same at each appearance.

To study V_1 we proceed, as in Rademacher's proof, by examining the finite sum

$$\sum_{m=K+1}^{N} \frac{e((j+\alpha)m/k)}{(-i(k\tau - m))^s} = \left(\int_{N+\frac{1}{2}-i\infty}^{N+\frac{1}{2}+i\infty} - \int_{K+\frac{1}{2}-i\infty}^{K+\frac{1}{2}+i\infty} \right) \frac{e((j+\alpha)z/k)}{(-i(k\tau - z))^s} \frac{dz}{e(z) - 1},$$

where we set $0 < \arg(-i(k\tau - z)) < \pi$, since $\tau = iy$. We then find, as in [5, p. 243], [6], that

$$\sum_{m=K+1}^{N} \frac{e((j+\alpha)m/k)}{(-i(k\tau - m))^s} \le c_j\, (N^{-s} + K^{-s}), \qquad c_j = k\left(\frac{1}{j} + \frac{1}{k - j - \alpha}\right),$$

so that

$$\left| \sum_{m=K+1}^{\infty} \frac{e((j+\alpha)m/k)}{(-i(k\tau - m))^s} \right| \le c_j\, K^{-s} .$$

Hence, by Assumption A,

$$|V_1| \leq K^{-s} \sum_{j=1}^{k-1} c_j |B_j| \leq C_\varepsilon \; K^{-s} k^{\frac{1}{2}+\varepsilon} \sum_{j=1}^{k-1} \frac{(\rho j + \sigma, k)^{\frac{1}{2}}}{j}.$$

Setting $\rho j + \sigma = \ell$, we find for the inner sum S the inequality

$$S \leq \rho \sum_{d|k} d^{\frac{1}{2}} \sum_{(\ell,k)=d} \frac{1}{\ell - \sigma} \qquad (\rho + \sigma \leq \ell < \rho k + \sigma).$$

It is readily verified that $\ell - \sigma \geq c_1 |\ell|$, where $c_1, c_2, \cdots$ denote constants depending on ρ and σ. Hence,

$$S \leq c_2 \sum_{d|k} d^{\frac{1}{2}} \sum_{(\ell,k)=d} \frac{1}{|\ell|} \qquad (\rho + \sigma \leq \ell < \rho k + \sigma).$$

Now $(\ell, k) = d$ is equivalent to $\ell = \ell_1 d$, $k = k_1 d$, $(\ell_1, k_1) = 1$. Therefore

$$S \leq c_3 \sum_{d|k} d^{-\frac{1}{2}} \sum_{\ell_1=1}^{\rho k+\sigma} \frac{1}{\ell_1} \leq c_4 \log c_5 k \sum_{d|k} d^{-\frac{1}{2}}.$$

Let $k = \prod_i p_i^{e_i}$; then

$$\sum_{d|k} d^{-\frac{1}{2}} = \prod_i \frac{1 - p_i^{-(e_i+1)/2}}{1 - p_i^{-1/2}}$$

$$< \prod_i{}' (1 - p_i^{-1/2})^{-1} < 4 \cdot 2^{\omega(k)},$$

$\omega(k)$ denoting the number of distinct prime factors of k. Since, for every $\varepsilon > 0$, $2^{\omega(k)} = O(k^\varepsilon)$, we have, finally,

$$S \leq c_1 \log c_2 k \cdot C_\varepsilon k^\varepsilon \leq C_\varepsilon k^\varepsilon.$$

With this estimate for S, we obtain

$$(4.5) \qquad |V_1| \leq C_\varepsilon K^{-s} k^{\frac{1}{2}+\varepsilon}.$$

Obviously, V_2 has the same estimate.

Combining (4.3) to (4.5), we have

$$|T_k(K)| \leq C_\varepsilon k^{-\frac{1}{2}+\varepsilon} K^{1-s} + C_\varepsilon k^{\frac{1}{2}+\varepsilon} K^{-s}.$$

It follows that

$$\left|\sum_{k=1}^{K} T_k(K)\right| \le \sum_{k=1}^{K} |T_k(K)|$$

$$\le C_\varepsilon K^{\frac{3}{2}+\varepsilon-s} + C_\varepsilon K^{\frac{3}{2}+\varepsilon-s} = O(K^{\frac{3}{2}+\varepsilon-s}).$$

Since $s > 3/2$, we have $\lim_{K\to\infty}\Sigma_{k=1}^{K} T_k(K) = 0$. In view of (4.1) and the definition (4.2) of $T_k(K)$, we see that this completes the proof of the lemma.

We now go back to the sum $G_1(K)$ of (3.7). Considered as the triple series in (3.8), $G_1(K)$ is absolutely convergent and may be rearranged in the manner of Lemma 2. Adding $G_1(K)$ and $G_2(K)$, we get

$$H(\tau) = \lim_{K\to\infty} \sum_{k=1}^{K} \sum_{m=-K}^{K}{}' \frac{e(-(\mu - \alpha) V_{k,-m}\tau)}{\varepsilon(V_{k,-m}) (-i(k\tau - m))^s}.$$

This formula is valid only for $\tau = iy$, $y > 0$, since that condition is a hypothesis of the lemma.

It is from this formula (4.6) that we shall prove the transformation property of $H(\tau)$ under the substitution $\tau \to -1/\tau$.

5. We first wish to extend the series in (4.6) over negative values of k. Now $V \to -V$ implies $k \to -k$, $-m \to m$; but $(-V)\tau = V\tau$. By (2.5) we see that the term in (k, m) goes over unchanged into the term in (-k, -m). Hence, we can write

$$H(\tau) = \lim_{K\to\infty} \frac{1}{2} \sum_{\substack{k=-K \\ k\neq 0}}^{K} \sum_{m=-K}^{K}{}' \frac{e(-(\mu - \alpha) V_{k,-m}\tau)}{\varepsilon(V_{k,-m}) (-i(k\tau - m))^s}.$$

The terms with $k = 0$ are missing. Since $k = 0$ implies $m = \pm 1$ (because of $(m, k) = 1$), these terms are

$$e(-(\mu - \alpha)\tau)\{\varepsilon^{-1}(I)\, e(s/4) + \varepsilon^{-1}(-I)\, e(-s/4)\} = 2e(-(\mu - \alpha)\tau),$$

by (2.2) and (2.6). We can write the substitution I as $V_{0,1}$. Then $m' = 1$ is not determined by (3.5), since $k = 0$. But (3.4) is still satisfied. Likewise, we write $-I = V_{0,-1}$. Thus

$$mm' + kk' + 1 = 0 \tag{5.1}$$

for all integral values of k and m.

We are therefore led to define a new function

$$F_\mu(\tau) = e(-(\mu - \alpha)\tau) + H(\tau) = \lim_{K\to\infty} \frac{1}{2} \sum_{k=-K}^{K} \sum_{m=-K}^{K}{}' \frac{e(-(\mu - \alpha) V_{k,-m}\tau)}{\varepsilon(V_{k,-m}) (-i(k\tau - m))^s}. \tag{5.2}$$

In (5.2), replace τ by $T\tau = -1/\tau$, where $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Note that

$$V_{k,-m}\,T = V_{-m,-k} = \begin{pmatrix} k' & -m' \\ -m & -k \end{pmatrix}$$

for all values of k and m. From (2.7), with $V_1 = V_{k,-m}$, $V_2 = T$, we find $(s = -r)$

$$\varepsilon(V_{-m,-k})(-i(-m\tau - k))^s = \varepsilon(V_{k,-m})\varepsilon(T)(-i(-k/\tau - m))^s(-i\tau)^s .$$

Hence, under the substitution $\tau \to -1/\tau$, (5.2) goes into

$$F_\mu(-1/\tau) = \varepsilon(T)(-i\tau)^{-r} \lim_{K\to\infty} \frac{1}{2} \sum_{k=-K}^{K} \sum_{m=-K}^{K}{}' \frac{e(-(\mu - \alpha) V_{-m,-k}\tau)}{\varepsilon(V_{-m,-k})(-i(-m\tau - k))^s} .$$

In the finite sums we replace -m by k, and k by m. The ranges of summation are symmetric in m and k, since (m, k) = 1 is equivalent to (-k, m) = 1. Also

$$V_{-m,-k} \to V_{k,-m} = \begin{pmatrix} m' & k' \\ k & -m \end{pmatrix},$$

for $mm' + kk' + 1 = 0$ for all m and k, by (5.1). Interchanging the order of summation in the finite sums, we get

$$F_\mu(-1/\tau) = \varepsilon(T)(-i\tau)^{-r} \lim_{K\to\infty} \frac{1}{2} \sum_{k=-K}^{K} \sum_{m=-K}^{K}{}' \frac{e(-(\mu - \alpha) V_{k,-m}\tau)}{\varepsilon(V_{k,-m})(-i(k\tau - m))^s}$$

$$= \varepsilon(T)(-i\tau)^{-r} F_\mu(\tau) ,$$

the desired transformation formula.

This formula has been proved only for $\tau = iy$, $y > 0$. But $F_\mu(\tau)$ is regular in the whole upper half-plane, since $H(\tau)$ is regular, by Lemma 1; therefore, by the principle of analytic continuation, we have

(5.3) $$F_\mu(-1/\tau) = \varepsilon(T)(-i\tau)^{-r} F_\mu(\tau) \qquad (\Im\tau > 0) .$$

6. We still have to show that

(6.1) $$F_\mu(\tau + 1) = e(\alpha) F_\mu(\tau) .$$

We shall do this by expanding F_μ in a Fourier series, from which (6.1) will follow at once.

We start with $G_2(K)$, which is already in the right form (3.10). Let $\alpha > 0$. If we interchange the order of the summations on ℓ and k, replace ℓ by m, and introduce $A_{k,\mu}(m)$ from (2.9) we get

(6.2) $$G_2 = \sum_{m=0}^{\infty} e((m + \alpha)\tau) \sum_{k=1}^{\infty} A_{k,\mu}(m) \frac{(2\pi)^s}{\Gamma(s)} k^{-s} (m + \alpha)^{s-1} .$$

The sum $G_1(K)$ of (3.8) can be rearranged, by virtue of its absolute convergence. The procedure is the same as for G_2, and it yields the following result

$$G_1 = \sum_{m=0}^{\infty} e((m+\alpha)\tau) \sum_{k=1}^{\infty} k^{-1} A_{k,\mu}(m) \sum_{\ell=1}^{\infty} \frac{(2\pi)^{2\ell+s} (m+\alpha)^{\ell+s-1} (\mu-\alpha)^{\ell}}{\ell!\,\Gamma(\ell+s)\,k^{2\ell+s-1}}.$$

We see that (6.2) is just the missing term $\ell = 0$ in this series. Therefore, when $\alpha > 0$,

$$H(\tau) = G_1 + G_2$$

$$(6.3) \qquad = 2\pi \sum_{m=0}^{\infty} e((m+\alpha)\tau) \sum_{k=1}^{\infty} k^{-1} A_{k,\mu}(m) \left(\frac{\mu-\alpha}{m+\alpha}\right)^{\frac{r+1}{2}} I_{-r-1}\left(\frac{4\pi}{k}(\mu-\alpha)^{\frac{1}{2}}(m+\alpha)^{\frac{1}{2}}\right)$$

where I_r is the Bessel function of the first kind with purely imaginary argument.

The parallel calculation for $\alpha = 0$ yields

$$(6.4) \qquad H(\tau) = \sum_{m=1}^{\infty} e(m\tau) \sum_{k=1}^{\infty} k^{-1} A_{k,\mu}(m) \left(\frac{\mu}{m}\right)^{\frac{r+1}{2}} I_{-r-1}\left(\frac{4\pi}{k}\mu^{\frac{1}{2}} m^{\frac{1}{2}}\right).$$

Thus $H(\tau + 1) = e(\alpha) H(\tau)$. Going back to the definition (5.2) of F_μ, we see at once that (6.1) is proved. Moreover, we have obtained a representation of the Fourier coefficients of $F_\mu(\tau)$ as convergent infinite series.

Since $F_\mu(\tau)$ satisfies the transformation equation (2.1) on $V = S, T$, it is a consequence of the remark following (2.8) that F_μ is a modular form of dimension r.

7. All the work up to now made essential use of the estimate (2.13) (Assumption A). In this section, we shall show this estimate to be justified when $r = -2$.

The set of entire modular forms of real dimension has been parametrized by Rademacher and Zuckerman [8, Thm. 2, p. 453]. The parameters, besides the dimension r, are certain integers β, γ, κ with the restrictions $0 \leq \beta \leq 1$, $0 \leq \gamma \leq 2$, $0 \leq \kappa$. The quantity α is not arbitrary, but is determined by the relation [8, (8.95)]

$$(7.1) \qquad \alpha = -\frac{r}{12} - \frac{\beta}{2} - \frac{\gamma}{3} - \left[-\frac{r}{12} - \frac{\beta}{2} - \frac{\gamma}{3}\right],$$

and we have, moreover,

$$\mu - \kappa = -\left[-\frac{r}{12} - \frac{\beta}{2} - \frac{\gamma}{3}\right].$$

We see that for each dimension r, there are exactly 6 permissible values of α.

Our present interest is the sum $A_{k,\mu}(m)$, which is expressed in terms of the above parameters in [8, (9.53), (9.54)]. (Formula (9.54) holds for arbitrary real r, though it is claimed only for $r > 0$.) Thus we can write

(7.2) $$A_{k,\mu}(m) = \sum_{h=0}^{k-1}{}' e(-2s(h', k))\, \xi_1^{\beta} \xi_2^{\gamma}\, e\{-(\kappa h' + (m + \mu - \kappa)h)/k\},$$

where

$$\xi_1 = e\{[-h'(h^2 + 1) + h'k^2 + (hh' + 1)k]/2k\},$$

$$\xi_2 = e\{(h - h')[(hh' + 1 + k^2)(2hh' + 1) + 1]/3k\},$$

and h' is defined in (2.11).

By a straightforward calculation we find

(7.3) $$\begin{aligned} \xi_1 &= -e(\theta_1(h - h^*)/k) && (k \text{ odd},\ 2\theta_1 \equiv 1 \pmod k),\\ \xi_1 &= e((h - h^*)/2k) && (k \text{ even}),\\ \xi_2 &= e(\theta_2(h - h^*)/k) && (3\nmid k,\ 3\theta_2 \equiv 1 \pmod k),\\ \xi_2 &= e((h - h^*)/3k) && (3\mid k), \end{aligned}$$

where $hh^* \equiv -1 \pmod{Dk}$, Dk being the denominator of the expression involved.

For e(-2s(h, k)), we use the equivalent expression $\omega^{-4}(h, k)$ given in [7, (16), (20)]. This yields, after some computation,

(7.4) $$e(-2s(h, k)) = \begin{cases} e(\theta_3(h^* - h)/k) & \text{if } (k, 6) = 1 \text{ and } 6\theta_3 \equiv 1 \pmod k,\\ -e(\theta_2(h^* - h)/2k) & \text{if } (k, 6) = 2,\\ e(\theta_4(h^* - h)/3k) & \text{if } (k, 6) = 3 \text{ and } 2\theta_4 \equiv 1 \pmod k,\\ -e((h^* - h)/6k) & \text{if } (k, 6) = 6. \end{cases}$$

Now, combining formulas (7.2) to (7.4), we get

(7.5) $$A_{k,\mu}(m) = \pm \sum_{h=0}^{k-1}{}' e((ah^* + bh)/Dk),$$

where D = (k, 6), $b = -a - D(m + \mu)$, and a is the integer given by the following table.

D	a
1	$\theta_3 - \beta\theta_1 - \gamma\theta_2 - \kappa$
2	$\theta_2 - \beta - 2\gamma\theta_2 - 2\kappa$
3	$\theta_4 - 3\beta\theta_1 - \gamma - 3\kappa$
6	$1 - 3\beta - 2\gamma - 6\kappa$

When $D = 1$, (7.5) is a Kloosterman sum [9]. This is not yet true for $D > 1$. Suppose, for example, that $D = 3$. Define

$$B_{k,\mu}(m) = \sum_{h=0}^{3k-1}{}' \; e((ah^* + bh)/3k).$$

Set $h = qk + j$ $(0 \leq j < k,\ (j, k) = 1,\ q = 0, 1, 2)$. Then $h^* = j^*(1 + qkj^*)$, where $jj^* \equiv -1 \pmod{3k}$, and

$$B_{k,\mu}(m) = \pm \sum_{j=0}^{k-1}{}' \; e((aj^* + bj)/3k) \sum_{q=0}^{2} e(q(aj^{*2} + b)/3).$$

The sum on q equals 3, for $a \equiv -b \pmod 3$, and $3 \mid k$ implies $3 \nmid h^*$, that is, $3 \nmid j^*$; hence $j^{*2} \equiv 1 \pmod 3$. We then get

$$A_{k,\mu}(m) = \pm B_{k,\mu}(m)/3,$$

and $B_{k,\mu}(m)$ is a Kloosterman sum. The other values of $D > 1$ are handled similarly.

In every case, therefore, $A_{k,\mu}(m)$ is a sum of the form

$$A_{k,\mu}(m) = \pm D^{-1} \sum_{h=0}^{Dk-1}{}' \; e((ah^* + bh)/Dk). \tag{7.6}$$

Note that $(a, Dk) \leq a^*$, where a^* is independent of m and k. For $6/D$ is prime to Dk; hence,

$$(a, Dk) = (6a/D, Dk) = (1 - 3\beta - 2\gamma - 6\kappa, Dk),$$

and $1 - 3\beta - 2\gamma - 6\kappa \neq 0$, as we see by examining all possible cases. Let $(a, b, Dk) = d > 1$. Clearly

$$A_{k,\mu}(m) = \pm D^{-1} d \sum_{h=0}^{k_1-1}{}' \; e([a_1 h^* + b_1 h]/k_1), \tag{7.7}$$

where $a = a_1 d$, $b = b_1 d$, $Dk = k_1 d$; hence, $(a_1, b_1, k_1) = 1$. Since $d \leq (a, Dk) \leq a^*$, we see that $A_{k,\mu}(m)$ has the order of magnitude of the sum in (7.6). That is, we may assume in (7.6) that $(a, b, Dk) = 1$.

Hence, by theorems of Salié and Weil ([9, pp. 266-267], [10]), we deduce that

$$|A_{k,\mu}(m)| \leq C_\varepsilon (b, Dk)^{\frac{1}{2}} (Dk)^{\frac{1}{2}+\varepsilon}.$$

But, as before,

$$(b, Dk) \leq D(b, k) = D(6b/D, k) = D(-1 + 3\beta + 2\gamma + 6\kappa - 6\mu - 6m, k).$$

If we set $\rho = 6$, $\sigma = 1 - 3\beta - 2\gamma - 6\kappa + 6\mu$, then

$$(b, Dk) \leq D(\rho m + \sigma, k),$$

and the required estimate (2.13) follows immediately, unless $\rho m + \sigma = 0$.

But $\rho m + \sigma = 0$ implies $\rho m + \sigma \equiv 0 \pmod 6$, or $1 - 3\beta - 2\gamma \equiv 0 \pmod 6$. This can happen only if $\beta = 1, \gamma = 2$. By reference to (7.1) we see that this implies $\alpha = 0$ (since $r = -2$). We now have $0 = \rho m + \sigma = 6m$, where we have used the line following (7.1); therefore, $m = 0$. And of course $b = 0$, for $-6b/D = \rho m + \sigma$.

The exponential sum (7.6) then has $b = - a - D\mu$. Since $b = 0$, we have $a = -D\mu$ and (7.6) becomes

$$A_{k,\mu}(0) = \pm D^{-1} \sum_{h=0}^{Dk-1}{}' e(-D\mu h^*/Dk) = \pm \sum_{h=0}^{k-1}{}' e(-\mu h^*/k) = \pm \sum_{d} d\mu\left(\frac{k}{d}\right).$$

with the Möbius μ-function, where d runs over the common divisors of μ and k. Hence,

$$|A_{k,\mu}(0)| \leq \sum_{d|\mu} d = O(1),$$

as $k \to \infty$. This completes the proof of Assumption A when $r = -2$.

We summarize our results in

THEOREM 1. *For* $\mu = 1, 2, 3, \cdots$, *let*

$$F_\mu(\tau) = e(-(\mu - \alpha)\tau) + \sum_{k=1}^{\infty} \sum_{m=-\infty}^{\infty}{}' \frac{e(-(\mu - \alpha) V_{k,-m}\tau)}{\varepsilon(V_{k,-m}) (-i(k\tau - m))^s}, \tag{7.8}$$

where $s > 3/2$, α *is given by* (7.1), $\varepsilon(V_{k,-m})$ *is a multiplier system for the dimension*

$$r = -s,$$

and the summation is understood in the sense

$$\lim_{K\to\infty} \sum_{k=1}^{K} \sum_{m=-\infty}^{\infty}{}'.$$

Then, if the estimates (2.13) *and* (2.14) *for the exponential sum* $A_{k,\mu}(m)$ *(defined in* (2.9)) *are correct, the function* $F_\mu(\tau)$ *is a modular form of dimension* r, *with multipliers* $\varepsilon(V_{k,-m})$. $F_\mu(\tau)$ *has the Fourier series*

$$F_\mu(\tau) = e(-(\mu - \alpha)\tau) + \sum_{m=\delta}^{\infty} a_m\, e((m + \alpha)\tau),$$

with

$$a_m = 2\pi \sum_{k=1}^{\infty} k^{-1} A_{k,\mu}(m) \left(\frac{\mu - \alpha}{m + \alpha}\right)^{\frac{r+1}{2}} I_{-r-1}\left(\frac{4\pi}{k}(\mu - \alpha)^{\frac{1}{2}}(m + \alpha)^{\frac{1}{2}}\right),$$

where

$$I_\ell(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{2n+\ell}}{n!\,\Gamma(n + \ell + 1)},$$

and $\delta = 0$ *for* $\alpha > 0$, $\delta = 1$ *for* $\alpha = 0$.

In particular, when $r = -2$, *the estimates* (2.13) *and* (2.14) *are correct and the conclusions above are valid.*

Any linear combination of the $F_\mu(\tau)$ of dimension r is obviously a modular form of dimension r. We confine ourselves to $r = -2$. Given a form $G(\tau)$ of dimension -2 and having an $\alpha > 0$, we construct a linear combination $F(\tau) = \Sigma_{\nu=1}^{\mu} b_\nu F_\nu(\tau)$ such that the expansion of $G(\tau) - F(\tau)$ at $\tau = i\infty$ has no terms with negative exponents. Then

$$\lim_{\tau\to\infty} \{G(\tau) - F(\tau)\} = \lim_{\tau\to\infty} O(e(\alpha\tau)) = 0.$$

Hence $G - F$ is a modular form which vanishes at $\tau = \infty$, that is, a cusp form (which may be identically zero).

If $G(\tau)$ has $\alpha = 0$, then, as we have seen, we must have $\beta = 1$, $\gamma = 2$. Now, using the parametrization [8, (9.41)], we find that G has the expansion

$$G(\tau) = e(-(\kappa + 1)\tau) + \cdots.$$

Since $\kappa \geq 0$, G has a pole at $\tau = \infty$. Thus there are no cusp forms of dimension -2 with $\alpha = 0$, nor are there any forms whose expansions at ∞ begin with a constant term. Hence, if we choose $F(\tau) = \Sigma_{\nu=1}^{\mu} b_\nu F_\nu(\tau)$ so that the principal parts of $F(\tau)$ and $G(\tau)$ agree at ∞, it follows that $F(\tau) \equiv G(\tau)$. We have, then,

THEOREM 2. *Let* $G(\tau)$ *be a modular form of dimension* -2 *with a value of* $\alpha > 0$. *Then there exist constants* $b_1, b_2, \cdots, b_\mu$ *such that*

$$G(\tau) = \sum_{\nu=1}^{\mu} b_\nu F_\nu(\tau) + K(\tau),$$

where $K(\tau)$ *is a cusp form.* If $\alpha = 0$, *constants* $b_1, b_2, \cdots, b_\mu$ *can be found such that*

$$G(\tau) = \sum_{\nu=1}^{\mu} b_\nu F_\nu(\tau).$$

The theorem shows that there is no modular form of dimension -2 and $\alpha = 0$ whose Fourier expansion contains a constant term. For every such form is a linear combination of the $F_\nu(\tau)$, and F_ν has no constant term, as we saw in Theorem 1.

8. Among the modular forms of dimension -2, $J'(\tau)$ is of particular interest. Here $J(\tau)$ is the absolute modular invariant,

$$J(\tau) = e(-\tau) + \cdots, \tag{8.1}$$

and satisfies

$$J(V\tau) = J(\tau) \qquad (V \in \Gamma(1)). \tag{8.2}$$

Hence,

$$\begin{cases} J'(\tau) = -2\pi i\, e(-\tau) + \cdots, \\ J'(V\tau) = (c\tau + d)^2 J'(\tau) = -(-i(c\tau + d))^2 J'(\tau) \qquad (c > 0), \end{cases} \tag{8.3}$$

so that

$$\varepsilon(V_{k,-h}) = -1 \qquad (k > 0). \tag{8.4}$$

From (8.3) we get $\alpha = 0$, $\mu = 1$.

Theorems 1 and 2 then show that

$$-J'(\tau)/2\pi i = e(-\tau) + \sum_{k=1}^{\infty} \sum_{m=-\infty}^{\infty}{}' \frac{e(-V_{k,-m}\tau)}{(k\tau - m)^2}. \tag{8.5}$$

The Fourier coefficients can be read off from Theorem 1. If we set

$$-J'(\tau)/2\pi i = e(-\tau) + \sum_{m=1}^{\infty} a_m e(m\tau), \tag{8.6}$$

then

$$a_m = -2\pi\sqrt{m} \sum_{k=1}^{\infty} k^{-1} A_k(m)\, I_1\left(\frac{4\pi}{k}\sqrt{m}\right), \tag{8.7}$$

where

$$A_k(m) = \sum_{h=0}^{k-1}{}' e(-(h' + mh)/k). \tag{8.8}$$

By integration we can recover the known coefficients of $J(\tau)$.

Note Added in Proof. The form of Assumption A (see (2.13), (2.14)) is unnecessarily complicated. It can be replaced by the following:

ASSUMPTION A'. *For every* $\varepsilon > 0$,

$$|A_{k,\mu}(m)| \leq C_\varepsilon k^{\frac{1}{2}+\varepsilon} \qquad (m = 0, 1, 2;\ \mu = 1, 2, 3, \cdots), \tag{*}$$

where C_ε *does not depend on* m *or* k.

The proof of (*) is the same as in the text, up to and including the paragraph which contains (7.7). We then notice that we may write (7.5) in the form

$$A_{k,\mu}(m) = \pm \sum_{h=0}^{k-1}{}' \, e((bh^* + ah)/Dk).$$

Hence, by the theorems of Salié and Weil quoted in the text,

$$|A_{k,\mu}(m)| \leq C_\varepsilon (a, Dk)^{\frac{1}{2}} (Dk)^{\frac{1}{2}+\varepsilon}.$$

But since, as noted in the lines following (7.6), $(a, Dk) \leq a^*$, with a^* independent of m and k, we obtain (*) immediately.

The use of Assumption A' simplifies the foregoing developments, particularly in the estimate of the sum S (lines preceding (4.5)), which is now simply

$$S = \sum_{j=1}^{k-1} j^{-1} < C \log k < Ck^\varepsilon.$$

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