A CHARACTERIZATION OF GENERALIZED MANIFOLDS

P. E. Conner and E. E. Floyd

1. INTRODUCTION

We are concerned here with generalized manifolds; these are spaces which have certain of the local homological (equivalently: cohomological) properties of manifolds. Such spaces have been of interest in topology since the fundamental work of Wilder [7], as well as that of Lefschetz, Čech, Smith, Begle, and others. Recently, there has been renewed interest in such spaces; see papers of Floyd [5], Yang [9], and Borel [2]. The principal motivation for the present paper is the study of fixed point sets of toral groups of transformations. In a later paper, we shall apply our results.

We consider locally orientable generalized manifolds in the sense of Wilder [7]. In our terminology, a finite-dimensional space X is a *locally orientable generalized n-manifold* if it fulfills two requirements: (a) X must have local cohomology modules $H_x^*(X, K)$ over the coefficient ring K at each point x, and their union must have the structure of a locally constant sheaf; (b) $H_x^i(X, K) = 0$ for $i \neq n$ and $H_x^n(X, K) \approx K$.

In Section 4, we show that X has locally constant local cohomology modules if and only if X has property Q of P. A. Smith [6]; that is, requirement (a) above is equivalent to property Q. Using recent work of Borel [2] and an argument on spectral sequences, we show in Section 6 that condition (a) implies condition (b) when the coefficients are in a field. That is, if X is connected, finite-dimensional and has locally constant local cohomology modules, then X is a locally orientable generalized n-manifold, for some n. Otherwise stated, the properties P and Q of P. A. Smith [6] are equivalent, at least when coefficients are in a field. Yang [9] has already proved, for general K, that P implies Q. In Section 7, we prove slightly weaker theorems for the case where the coefficients are in the ring Z of integers. The problem of determining whether the above results hold in full generality for this case is left unsolved.

Neighborhoods of points will always be open sets in the space. Unless otherwise specified, the coefficient ring K will be assumed to be an arbitrary commutative ring consisting of more than one element. The kernel of a homomorphism $f: F \to G$ is denoted by Ker f, and its image by Im f.

2. INVERSE FAMILIES

Suppose that X is a topological space. Let there be given for each open U in X a K-module F_u, and whenever $U \supset V$, let there be given a homomorphism

$$f_{uv}: F_v \to F_u$$

such that (1) f_{uu} is the identity and (2) $f_{uv} f_{vw} = f_{uw}$ whenever $U \supset V \supset W$. Then we say that $[F_u, f_{uv}]$ is *an inverse family on the space* X. The following concepts

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concerning inverse families are aimed at the discussion of local cohomology modules in Section 4.

We say that a base $\mathfrak{U}$ for the open sets of X is *hereditary* if and only if, whenever $U \in \mathfrak{U}$ and $V \subset U$ is nonempty and open, then $V \in \mathfrak{U}$. We say that the inverse family $[F_u, f_{uv}]$ is *locally constant on the subset* A of X if and only if (i) there exists a hereditary base $\mathfrak{U}$ for the open sets of X, and (ii) for each $U \in \mathfrak{U}$ and each $x \in U \cap A$ there exists a submodule $F_{u,x}$ of F_u such that

(a) if $U, V \in \mathfrak{U}$ and $x \in V \subset U$ and $x \in A$, then f_{uv} maps $F_{v,x}$ isomorphically onto $F_{u,x}$, and $f_{uv}(F_v) = F_{u,x}$;

(b) if $x \in U \cap A$ and $U \in \mathfrak{U}$, then there exists a neighborhood $V \subset U$ of x such that the $F_{u,y}$ are equal for all $y \in V \cap A$. If $A = X$, we say simply that the family is locally constant.

If a base $\mathfrak{U}$ satisfies these conditions, we shall say that it is a *base for the local constancy of* $[F_u, f_{uv}]$.

THEOREM 2.1. *Let* $[F_u, f_{uv}]$ *be an inverse family on the space* X, *and let* A *be a subset of* X. *Then* $[F_u, f_{uv}]$ *is locally constant on* A *if and only if the following conditions are satisfied:*

(a') *given* $x \in A$ *and a neighborhood* U *of* x, *there exists a neighborhood* $V \subset U$ *of* x *such that if* W *is any open subset of* V *which intersects* A, *then* $\operatorname{Im} f_{uw} = \operatorname{Im} f_{uv}$;

(b') *there exists a hereditary base* $\mathfrak{U}$ *for the open sets of* X *such that if* $x \in A$ *and* $x \in V \subset U \in \mathfrak{U}$, *then there exists a neighborhood* $W \subset V$ *of* x *such that* $\operatorname{Ker} f_{vw} = \operatorname{Ker} f_{uw}$.

Proof. Suppose conditions (a') and (b') hold, and let $\mathfrak{U}$ be the base of (b'). For each $x \in U \cap A$ $(U \in \mathfrak{U})$, define $F_{u,x}$ to be the intersection of all $\operatorname{Im} f_{uv}$ for which $x \in V$. It is then seen that $[F_u, f_{uv}]$ is locally constant.

Suppose next that $[F_u, f_{uv}]$ is locally constant on A. We show that (a') is satisfied for all sufficiently small neighborhoods U of x; it is then satisfied for all neighborhoods U of x. Suppose $\mathfrak{U}$ is a base for the local constancy of $[F_u, f_{uv}]$. Suppose that $x \in U \in \mathfrak{U}$, and that V is as in (b) and also so small that $\operatorname{Im} f_{uv} = F_{u,x}$. Let W be an open subset of V, and let $y \in W \cap A$. Then

$$\operatorname{Im} f_{uv} \supset \operatorname{Im} f_{uw} \supset F_{u,y} = F_{u,x} = \operatorname{Im} f_{uv}.$$

Hence $\operatorname{Im} f_{uv} = \operatorname{Im} f_{uw}$, and (a') follows.

We show now that (b') is satisfied. Let $\mathfrak{U}$ be a base for the local constancy of $[F_u, f_{uv}]$. Let V be an open subset of U, and let $x \in V \cap A$. There exists a neighborhood $W \subset V$ of x, with $F_{v,x} = \operatorname{Im} f_{vw}$. Then f_{uv} maps $F_{v,x}$ isomorphically onto $F_{u,x}$. Hence f_{uv} maps $\operatorname{Im} f_{vw}$ isomorphically into F_u. Hence $\operatorname{Ker} f_{uw} = \operatorname{Ker} f_{vw}$, as was to be proved.

For an inverse family $[F_u, f_{uv}]$ and a fixed $x \in X$, we consider the inverse system $\{F_u, f_{uv}; x \in U\}$ of modules and homomorphisms, indexed by the collection of neighborhoods U of x. Define $F_x = \lim_{\leftarrow} \{F_u, f_{uv}; x \in U\}$. There is also the natural homomorphism $f_{u,x}: F_x \to F_u$. If the family is locally constant, then for U sufficiently small, $f_{u,x}$ is an isomorphism into, and its image is $\operatorname{Im} f_{uv}$ for appropriate V.

3. DIRECT FAMILIES AND DUAL FAMILIES

Suppose that for each open U in X there is given a K-module G_u, and that whenever $U \supset V$ there is given a homomorphism $g_{vu}: G_u \to G_v$ such that (1) g_{uu} is the identity and (2) $g_{wv} g_{vu} = g_{wu}$ whenever $U \supset V \supset W$. Call such a $[G_u, g_{vu}]$ a *direct family* on X. It is well known [3, XIV] that each direct family leads to a sheaf G on X with

$$G_x = \varinjlim \{G_u, g_{vu}; x \in U\},$$

We define the direct family $[G_u, f_{vu}]$ to be a *locally constant* family if and only if it satisfies the conditions dual to those in Theorem 2.1. That is, it is locally constant if and only if it satisfies

(a") corresponding to each $x \in X$ and each neighborhood U of x, there exists a neighborhood $V \subset U$ of x such that if W is any nonempty open subset of V, then $\operatorname{Ker} g_{wu} = \operatorname{Ker} g_{vu}$;

(b") there exists a hereditary base $\mathfrak{U}$ for the open sets of X such that if $x \in V \subset U \in \mathfrak{U}$, then there exists a neighborhood $W \subset V$ of x with $\operatorname{Im} g_{wv} = \operatorname{Im} g_{wu}$.

It may be seen that if $[G_u, g_{vu}]$ is a locally constant direct family, then its sheaf G is a locally constant sheaf.

In the remainder of this section, the basic ring K is assumed to be a field. Each G_u is then a vector space over the field K. We say that the direct family $[G_u, g_{vu}]$ is *dual* to the inverse family $[F_u, f_{uv}]$ if and only if each G_u is the dual space of F_u (that is, G_u is the space $\operatorname{Hom}[F_u, K]$ of linear maps from F_u to K) and each $g_{vu}: G_u \to G_v$ is the dual (or transpose) of $f_{uv}: F_v \to F_u$.

THEOREM 3.1. *Suppose that the direct family* $[G_u, g_{vu}]$ *is the dual of the inverse family* $[F_u, f_{uv}]$, *where coefficients are in a field* K. *Then the family* $[G_u, g_{vu}]$ *is locally constant if and only if the family* $[F_u, f_{uv}]$ *is locally constant. In case the families are locally constant, then* G_x *is isomorphic to the dual space of* F_x, *for* $x \in X$.

Proof. Consider $U \supset V \supset W$. In the diagrams

$$\begin{array}{ccc} F_w & \overset{f_{uw}}{\longrightarrow} & F_u \\ & f_{vw}\searrow \quad \nearrow f_{uv} & \\ & F_v & \end{array} \qquad \begin{array}{ccc} G_w & \overset{g_{wu}}{\longleftarrow} & G_u, \\ & g_{wv}\nwarrow \quad \nearrow g_{vu} & \\ & G_v & \end{array}$$

we have $\operatorname{Im} f_{uw} = \operatorname{Im} f_{uv}$ if and only if $\operatorname{Ker} g_{wu} = \operatorname{Ker} g_{vu}$, and $\operatorname{Ker} f_{vw} = \operatorname{Ker} f_{uw}$ if and only if $\operatorname{Im} g_{wv} = \operatorname{Im} g_{wu}$. Hence (a') and (a") are equivalent, as are (b') and (b"). Hence the first assertion follows. For appropriately chosen neighborhoods U and V of x, G_x is isomorphic to $G_u/\operatorname{Ker} g_{vu}$, and F_x to $\operatorname{Im} f_{uv}$, if the families are locally constant. But $G_u/\operatorname{Ker} g_{vu}$ is isomorphic to the dual space of $\operatorname{Im} f_{uv}$. Hence the remark follows.

4. LOCAL COHOMOLOGY MODULES

In this section, X will denote a locally compact Hausdorff space. Denote by $C_c^*(X) = C_c^*(X, K) = \Sigma C_c^i(X)$ the Alexander-Spanier grating of X with compact supports, and by $H_c^*(X) = \Sigma H_c^i(X)$ the corresponding cohomology module. For each open

U, let $C_c^*(U)$ denote the submodule of $C_c^*(X)$ consisting of all elements whose supports are in U, and let $H_c^*(U)$ denote its cohomology module. If $V \subset U$, then $C_c^*(V) \subset C_c^*(U)$. Let j_{uv}: $C_c^*(V) \to C_c^*(U)$ denote the inclusion, and j_{uv}^*: $H_c^*(V) \to H_c^*(U)$ the induced homomorphism. Then we have an inverse family $[H_c^*(U), j_{uv}^*]$. We say that X *has locally constant local cohomology modules on the subset* A if and only if the inverse family $[H_c^*(U), j_{uv}^*]$ is locally constant on A. In that case, define the local cohomology module $H_x^*(X, K)$ to be

$$H_x^*(X, K) = \lim_{\leftarrow}\{H_c^*(U), j_{uv}^*; x \in U\}.$$

These are the duals for cohomology of the local homology groups introduced by Alexandroff [1], [8]. If A = X, we say simply that X has locally constant local cohomology modules.

If (X, B) is a locally compact pair consisting of a locally compact Hausdorff space and a closed subset B, then we denote by $H^n(X, B; K)$ and $H_n(X, B; K)$ the Čech n-cohomology and n-homology modules of the compact pair $(X \cup \infty, B \cup \infty)$, where $X \cup \infty$ denotes a one-point compactification of X. We use the well-known identification $H^n(X, B; K) \approx H_c^n(X - B, K)$. If (X, B') is another such pair with $B' \subset B$, then $(X, B') \subset (X, B)$ induces the homomorphism

$$k_{X-B',X-B}: H^n(X, B; K) \to H^n(X, B'; K).$$

Under the identification, $k_{X-B',X-B}$ is identified with

$$j_{X-B',X-B}^*: H_c^n(X - B, K) \to H_c^n(X - B', K).$$

Hence the inverse family $[H^n(X, X - U), k_{uv}]$ is identified with $[H_c^n(U), j_{uv}^*]$. We also have the direct family $[H_n(X, X - U), \overline{k}_{vu}]$ of homology groups. When coefficients are in a field, the latter family is dual to the family $[H^n(X, X - U), k_{uv}]$ and hence to $[H_c^n(U), j_{uv}^*]$. According to (3.1), when one of these is locally constant, so is the other.

We characterize spaces with locally constant local cohomology modules in terms of property Q, introduced by P. A. Smith [6]. We say that X *has property* Q *on the subset* A *of* X (and over K) if and only if corresponding to each neighborhood U of $x \in A$, there exists a neighborhood $V \subset U$ of x such that if U' is any open subset of V and $y \in U' \cap A$, there exists an open V' with $y \in V' \subset U'$ for which the composition

$$H_c^*(V - V') \to H_c^*(V - U') \to H_c^*(U - U')$$

is trivial; or, in terms of the relative groups, for which

$$H^n(X - V', X - V) \to H^n(X - U', X - U)$$

is trivial for all n. If A = X, we say simply that X has property Q. For fields as coefficients, X has the cohomological property Q described here if and only if X has property Q in the sense of homology as used by Smith [6] and Yang [9]. For abelian groups K, X has our property Q over K if and only if X has property Q in the sense of homology over the character group K*.

THEOREM 4.1. *Suppose* A *is a subset of the locally compact space* X. *Then* X *has locally constant local cohomology modules on* A *if and only if* X *has property* Q *on* A.

Proof. Suppose first that X has property Q on A. Corresponding to each neighborhood U of a point x, there exists a neighborhood V of x such that if U is any open subset of V and $y \in U' \cap A$, there exists an open V', with $y \in V' \subset U'$, for which the composition

$$H^*(V - V') \rightarrow H^*(V - U') \rightarrow H^*(U - U')$$

is trivial. Consider the diagram

$$\begin{array}{ccccccccc}
 & & H^n_c(V) & \rightarrow & H^n_c(V - V') & \rightarrow & H^{n+1}_c(V') & \xrightarrow{j^*_{vv'}} & H^{n+1}(V), \\
 & & \downarrow & & \downarrow & & \downarrow j^*_{u'v'} & & \\
 & & H^n_c(V) & \rightarrow & H^n_c(V - U') & \rightarrow & H^{n+1}_c(U') & & \\
 & & \downarrow j^*_{uv} & & \downarrow & & \downarrow & & \\
H^n_c(U') & \underset{j^*_{uu'}}{\rightarrow} & H^n_c(U) & \rightarrow & H^n_c(U - U') & \rightarrow & H^{n+1}_c(U') & &
\end{array}$$

where each row is the exact sequence of a pair, and the vertical homomorphisms are induced by inclusion.

Diagram-tracing establishes that Ker $j^*_{vv'} \subset$ Ker $j^*_{v'v'}$. The opposite inclusion follows from $V' \subset U' \subset V$.

Diagram-tracing also shows that Im $j^*_{uv} \subset$ Im $j^*_{uu'}$. Hence the images are equal, and (a') of (2.1) follows. Hence property Q implies that X has locally constant local cohomology modules on A.

Suppose now that X has locally constant local cohomology modules on A. Corresponding to each $x \in X$ and each neighborhood U of x, there exists by (a') a neighborhood $V \subset U$ of x such that if W is an open subset of V which intersects A, then Im j^*_{uw} = Im j^*_{uv}. If this condition holds for V, it holds also for each open subset of V. Hence we may suppose V to be so small that it belongs to the $\mathfrak{U}$ of (b'). Then, corresponding to each $U' \subset V$ and each $y \in U'$, there exists a neighborhood $V' \subset U'$ of y such that Ker $j^*_{u'v'}$ = Ker $j^*_{vv'}$. Using the same diagram as before, we may then verify that the composition

$$H^n_c(V - V') \rightarrow H^n_c(V - U') \rightarrow H^n_c(U - U')$$

is trivial. Hence X has property Q on A, and the theorem follows. If we let U be the collection of all such V arising from all possible choices of x and U, then (b') is satisfied.

For the remainder of this section, we suppose that K is a field, and we summarize some results of Borel [2], who has studied the dual family of $[H^n_c(U), j^*_{uv}]$, for this case. Borel defines $C_i(X) = \mathrm{Hom}\,[C^i_c(X, K), K]$, so that $C_i(X)$ is the dual vector space of $C^i_c(X)$. Similarly, $\partial\colon C_i(X) \rightarrow C_{i-1}(X)$ is the dual of the coboundary. A point x is defined not to belong to the support S(a) of $a \in C_i(X)$ if and only if it has a neighborhood U such that $a(c) = 0$ whenever $S(c) \subset U$. Then $C_*(X)$ is a fine, complete grating without torsion. (Borel uses paracompactness of X to show that $C_*(X)$ is fine. However, a very slight modification of his argument enables one to drop the paracompactness hypothesis.) $\mathfrak{F}(X)$ represents the sheaf of the grating $C_*(X)$, whose stalk $\mathfrak{F}_x$ above X is $C_*/C_*(X - x)$, where $C_*(U)$ denotes the set of elements in C_* whose supports are in U.

For $V \subset U$, let $j^{vu}: C_*(U) \to C_*(V)$ denote the dual of the inclusion

$$j_{uv}: C^*(V) \to C^*(U),$$

and let $j_*^{vu}: H_*(C_*(U)) \to H_*(C_*(V))$ denote the induced homomorphism of homology modules. Continuing to follow Borel, we see from the universal coefficient theorem that, for each n, $H_n(C_*(U))$ is the dual space of $H_c^n(U)$. Hence the direct family $[H_n(C_*(U)), j_*^{vu}]$ is the dual family to $[H_c^n(U), j_{uv}^*]$.

THEOREM 4.2. *Suppose that* X *has locally constant local cohomology modules over the field* K. *Then the sheaf* $H(\mathfrak{F})$ *of homology of the sheaf* $\mathfrak{F}$ *of Borel is a locally constant sheaf. Moreover,* $H_n(\mathfrak{F}_x)$ *is the dual space of* $H_x^n(X)$ *for each* x *and* n.

The proof follows from Theorem 3.1 and the equality

$$H(\mathfrak{F}_x) = \varinjlim \{H_*(C_*(U)), j_*^{vu}: x \in U\}$$

of Borel [3, Section 2].

5. SOME ELEMENTARY FACTS CONCERNING THE LOCAL GROUPS

In this section we give some elementary properties that are needed in Section 6. Our first statement is of a well-known type, the fundamental contribution being due to Alexandroff [1]. An outline of a proof may be found in [5, (2.1)].

THEOREM 5.1. *Suppose that* X *is a nonempty locally compact Hausdorff space with* $\dim_{c,K} X < \infty$ (the dimension notation is that of [3, XVI]). *There exists an* $x \in X$ *with* $\varprojlim\{H_c^*(U), j_{uv}^*; x \in U\} \neq 0$. *In particular, if* X *has locally constant local cohomology groups, then each* $H_x^*(X) \neq 0$.

Proof. The second assertion follows from the first. For each $x \in X$ has a neighborhood U such that all the $H_y^*(X)$ are isomorphic for $y \in V$. But $H_y^*(X) = H_y^*(U)$ if $y \in U$. Applying the first assertion to U, we see that $H_y^*(X) \neq 0$ for some $y \in X$. Hence $H_x^*(X) \neq 0$.

THEOREM 5.2. *Suppose* X *is a locally compact Hausdorff space which has locally constant local cohomology modules over* K. *Then* X *is locally connected.*

Proof. Suppose to the contrary that X is not locally connected at $x \in X$. There exists a neighborhood U' of x such that if $x \in V \subset U'$, then V is not contained in a single component of U'.

For each neighborhood U of x, there exists a neighborhood $V \subset U$ of x such that if W is any nonempty open subset of V, then $\text{Im } j_{uw}^* = \text{Im } j_{uv}^*$. Moreover, for U and V sufficiently small, $\text{Im } j_{uv}^*$ is isomorphic to $H_x^*(X)$. By Theorem 5.1, this is nontrivial. Suppose also that $\bar{U} \subset U'$.

Let C_x be the component of $\bar{U}$ which contains x. Then V is not contained in C_x. Since C_x is a component of the compact Hausdorff space $\bar{U}$, there exist in $\bar{U}$ arbitrarily small sets C that are open and closed and contain C_x. In particular, there exists such a C which does not contain V. Let $U_1 = U \cap C$; then U_1 is open and closed in U, and does not contain V. Let $U_2 = U - U_1$; U_2 is also open and closed in U. Let $W_1 = U_1 \cap V$ and $W_2 + U_2 \cap V$. Then by the construction of U and V, we have $\text{Im } j_{uw_1}^* = \text{Im } j_{uw_2}^* = \text{Im } j_{uv}^* = H_x^*(X) \neq 0$. But if $U = U_1 \cup U_2$ is a separation, then

$\operatorname{Im} j^*_{uu_1} \cap \operatorname{Im} j^*_{uu_2} = 0$. Hence $\operatorname{Im} j^*_{uw_1} \cap \operatorname{Im} j^*_{uw_2} = 0$, and we have a contradiction. Hence X is locally connected.

In preparation for our next result, we denote by $C^c_*(X, K)$ the subgrating of $C_*(X, K)$ consisting of all elements of $C_*(X)$ which have compact support.

THEOREM 5.3. *Suppose that* X *is a nonempty, connected, locally connected, locally compact Hausdorff space, and that* K *is a field. Then* $H_0(C^c_*(X, K)) \approx K$.

Proof. Denote by C^*_c the grating $C^*_c(X, K)$, and by $C^*_{c,u}$ the set of elements of C^*_c whose supports are in U. Then, for each compact subset N,

$$H^0(N) = H^0(C^*_c/C^*_{c,X-N}).$$

Also, $\operatorname{Hom}[C^*_c/C^*_{c,X-N}, K]$ is C_{*N}, the set of elements of C_* whose supports are in N. According to the universal coefficient theorem, $H_0(C_*(N)) = \operatorname{Hom}[H^0(N), K]$.

A space satisfying our conditions is a union of compact, connected subsets. Let $\{N\}$ be the family of compact connected subsets of X. For each N, $H^0(N, K) \approx K$; and if $M \subset N$, then the natural homomorphism k^*_{MN} induced by

$$k_{MN}: C^*_c/C^*_{c,X-N} \to C^*_c/C^*_{c,X-M}$$

is an isomorphism onto. Let k^{MN} be the inclusion of M in N. Then

$$k^{NM}_*: H_*(C_{*M}) \to H_*(C_{*N})$$

is dual to k^*_{MN}. Therefore $H_*(C_{*N}) \approx K$ and k^{NM}_* is an isomorphism onto. Finally

$$H_0(C^c_*(X, K)) = \lim_{\to} \{H_0(C_{*N}), k^{NM}_*\},$$

so that $H_0(C^c_*(X, K)) \approx K$.

6. THE FUNDAMENTAL THEOREMS

In this section we prove our main theorems.

THEOREM 6.1. *Suppose that* X *is a locally compact Hausdorff space and that* K *is a field with* $\dim_{c,K} X < \infty$. *Suppose also that* X *has locally constant local cohomology modules over* K. *Then for each* $x \in X$ *there exists a neighborhood* U *of* y *and a nonnegative integer* n *such that* $H^i_y(X; K) = 0$ *if* $i \neq n$ *and* $H^n_y(X; K) \approx K$ *for all* $y \in U$.

Proof. Suppose that n is the first nonnegative integer with $H^n_x(X) \neq 0$. Suppose that there exists an $m > n$ with $H^m_x(X) \neq 0$. Considering the definition of the local groups, we see that, for all sufficiently small neighborhoods U of x, $H^m_c(U) \neq 0$. Moreover, by Theorem 5.2 there exist arbitrarily small connected neighborhoods U of x. Also, by Theorem 4.2, the sheaf $H(\mathfrak{F})$ is locally constant. Hence, if U is sufficiently small, the sheaf induced on U by the sheaf $H(\mathfrak{F})$ is constant on U. We replace X by U. That is, we assume hereafter that X is connected, that $H^m_c(X) \neq 0$ for some $m > n$, and that $H(\mathfrak{F})$ is constant.

Let $D = \Sigma D_i$, where $D_i = C_{-i}(X, K)$. Then D is a fine, complete grating on X, and has coboundary. Also, the grating D_c of elements with compact support is $\Sigma C^c_{-i}(X, K)$. According to the fundamental theorem concerning spectral sequences

[3, XIX Th. 4], there exists a spectral sequence for which $E_2^{p,-q} = H_c^p(X, H_q(\mathfrak{F}))$ and for which $\Sigma_{p-q=-k} E_\infty^{p,-q}$ is the graded module associated with a filtration of $H^{-k}(D_c) = H_k(C_*^c(X))$.

Since $\dim_{c,K} X$ is finite, $H_c^i(X) = 0$ for all i sufficiently large. Suppose $H_c^i(X) = 0$ for all $i > k$, where $k > n$. Since $H(\mathfrak{F})$ is constant,

$$E_2^{p,-q} = H_c^p(X) \otimes H_q(\mathfrak{F}_x).$$

Hence $E_2^{p,-q} = 0$ for $p > k$. Since n is the first integer with $H_n(\mathfrak{F}_x) \neq 0$, we have $E_2^{p,-q} = 0$ when $q < n$ or equivalently $-q > -n$. Hence $E_2^{r,s} = 0$ when $r > k$ or $s > -n$. According to a well-known argument in spectral sequences,

$$E_2^{k,-n} = E_\infty^{k,-n} = H^{k-n}(D_c) = H_{n-k}(C_*^c(X)) = 0$$

since $n - k < 0$. Then

$$E_2^{k,-n} = H_c^k(X) \otimes H_n(\mathfrak{F}_x) = 0.$$

Since $H_n(\mathfrak{F}_x) \neq 0$, it follows that $H_c^k(X) = 0$. By induction, $H_c^k(X) = 0$ for all $k > n$. This contradicts the fact that $H_c^m(X) \neq 0$. Hence there exists an n with $H_i(\mathfrak{F}_x) = 0$ for $i \neq n$. Since $H_i(\mathfrak{F}_x)$ is dual to $H_x^i(X)$ by Theorem 4.2, it follows that $H_x^i(X) = 0$ for $i \neq n$.

Now, with the notation as before, $H_c^k(X) = 0$ for $k > n$, so that $E_2^{r,s} = 0$ for $r > n$ or $s \neq -n$. Hence

$$E_2^{n,-n} \approx E_\infty^{n,-n} \approx H^0(D_c) = H_0(C_*^c(X)).$$

Hence $H_0(C_*^c(X)) \approx K$, by Theorem 5.3. Then $H_c^n(X) \times H_n(\mathfrak{F}_x) \approx K$. Since the tensor product is one-dimensional, each of $H_c^n(X)$ and $H_n(\mathfrak{F}_x)$ is one-dimensional. But $H_n(\mathfrak{F}_x)$ is the dual space of $H_x^n(X)$, so that $H_x^n(X)$ is one-dimensional. Hence $H_x^n(X) \approx K$. The theorem follows.

We now draw some corollaries. Following Wilder [7], we call X a locally orientable generalized n-*manifold* over K if and only if (1) X is finite-dimensional, (2) X has locally constant local cohomology groups over K, and (3) $H_x^i(X, K) = 0$ for $i \neq n$ and $H_x^n(X, K) \approx K$.

COROLLARY 6.2. *Suppose* X *is a connected, locally compact Hausdorff space with* $\dim X < \infty$ *and with locally constant local cohomology modules over a field* K. *Then* X *is a locally orientable generalized* n-*manifold for some integer* n.

Along with property Q for homology, already referred to, P. A. Smith has introduced a fundamental property called property P (this also for homology). Its dual for cohomology is, in our language, that X has property P if and only if X has locally constant local cohomology groups, and that for each $x \in X$ there exists a neighborhood U of x and a nonnegative integer n with $H_x^i(X; K) = 0$ for $i \neq n$ and $H_x^n(X; K) \approx K$ for all $x \in U$. By Theorem 3.1 and the fact that $\{H_n(X, X - U), \bar{k}_{vu}\}$ is the dual of $\{H^n(U), j_{uv}^*\}$, property P for homology is equivalent to property P for cohomology, in the case of a field.

COROLLARY 6.3. *Suppose that* X *is a locally compact Hausdorff space and that* K *is a field such that* $\dim_{c,K} X < \infty$. *Then* X *has property* Q *over* K *if and only if* X *has property* P *over* K.

7. THE FUNDAMENTAL THEOREM FOR Z

In this section we turn to the ring Z of integers as coefficients.

THEOREM 7.1. *If the locally compact Hausdorff space* X *has locally constant local cohomology groups over* Z, *then* X *has locally constant local cohomology groups over* Z_p *for each prime* p.

Proof. By Theorem 4.1, X has property Q over Z. Let $x \in X$, and let U be a neighborhood of x. Then x has a neighborhood $V \subset U$ as described in the definition of property Q. Similarly, for each V, x has a neighborhood $W \subset V$ such that if V' is open in W and $y \in V'$, then there exists an open W', with $y \in W' \subset V'$, such that the composition

$$(*) \qquad H^*(X - W', X - W; Z) \to H^*(X - V', X - V; Z)$$

is trivial. Now let U' be an open subset of W, and let $y \in U'$. There exists an open V', with $y \in V' \subset U'$, such that the composition

$$H^*(X - V', X - V; Z) \to H^*(X - U', X - U; Z)$$

is trivial. Similarly, there is an open W' with $y \in W' \subset V'$ such that (*) is trivial.

The exact sequence $0 \to Z \xrightarrow{p} Z \to Z_p \to 0$ induces exact sequences of cohomology groups. Each row of the following diagram is from such a sequence.

$$\begin{array}{ccccc} & & H^n(X - W', X - W; Z_p) & \to & H^{n+1}(X - W', X - W; Z) \\ & & \downarrow \alpha & & \downarrow \delta \\ H^n(X - V', X - V; Z) & \to & H^n(X - V', X - V; Z_p) & \to & H^{n+1}(X - V', X - V; Z) \\ \downarrow \gamma & & \downarrow \beta & & \\ H^n(X - U', X - U; Z) & \to & H^n(X - U', X - U; Z_p). & & \end{array}$$

Since δ and γ are trivial, $\beta\alpha$ is trivial. Hence X has property Q over Z_p, and the remark follows.

THEOREM 7.2. *If* X *has locally constant local cohomology groups over* Z, *and* p *is a prime, then there is the exact sequence*

$$\cdots \to H_x^n(X, Z) \xrightarrow{p} H_x^n(X, Z) \to H_x^n(X, Z_p) \to \cdots$$

induced by the exact sequence $0 \to Z \xrightarrow{p} Z \to Z_p \to 0$.

Proof. By Theorem 7.1, X has locally constant local cohomology groups over Z_p. Consider neighborhoods $U \supset V \supset W$ of x, and the following diagram:

$$\begin{array}{ccccccccc} \cdots \to & H_c^n(W, Z) & \xrightarrow{p} & H_c^n(W, Z) & \to & H_c^n(W, Z_p) & \to & H_c^{n+1}(W, Z) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & H_c^n(V, Z) & \xrightarrow{p} & H_c^n(V, Z) & \to & H_c^n(V, Z_p) & \to & H_c^{n+1}(V, Z) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & H_c^n(U, Z) & \xrightarrow{p} & H_c^n(U, Z) & \to & H_c^n(U, Z_p) & \to & H_c^{n+1}(U, Z) & \to \cdots . \end{array}$$

We may suppose U, V, W to be chosen so that

$$H_x^*(X, Z) = \mathrm{Im}[H_c^*(W, Z) \to H_c^*(U, Z)] = \mathrm{Im}[H_c^*(V, Z) \to H_c^*(U, Z)],$$

$$H_x^*(X, Z_p) = \mathrm{Im}[H_x^*(W, Z_p) \to H_x^*(U, Z_p)] = \mathrm{Im}[H_c^*(V, Z_p) \to H_c^*(U, Z_p)],$$

$$\mathrm{Ker}[H_c^*(W, Z) \to H_c^*(V, Z)] = \mathrm{Ker}[H_c^*(W, Z) \to H_c^*(U, Z)],$$

$$\mathrm{Ker}[H_c^*(W, Z_p) \to H_c^*(V, Z_p)] = \mathrm{Ker}[H_c^*(W, Z_p) \to H_c^*(U, Z_p)].$$

Diagram-tracing establishes the exactness of the sequence

$$\cdots \to H_x^n(X, Z) \xrightarrow{p} H_x^n(X, Z) \to H_x^n(X, Z_p) \to \cdots$$

of subgroups of

$$H_c^n(U, Z) \xrightarrow{p} H_c^n(U, Z) \to H_c^n(U, Z_p) \to \cdots.$$

The theorem follows.

THEOREM 7.3. *Suppose that* X *is a locally compact Hausdorff space with* $\dim_{c,Z}$ X *finite. Suppose also that* X *has finitely generated, locally constant local cohomology groups over* Z. *Then for each* $x \in X$ *there exists a neighborhood* U *of* x *and a nonnegative integer* n *with* $H_x^i(X, Z) = 0$ *for* $i \neq n$ *and* $H_x^n(X, Z) \approx Z$ *for all* $x \in X$.

Proof. Fix $x \in X$. By Theorem 7.1, X has locally constant local groups over Z_p, for each prime p. Hence, Theorem 6.1, for each prime p there is an integer n_p with

$$H_x^i(X, Z_p) = 0 \quad \text{for } i \neq n_p, \qquad H_x^{n_p}(X, Z_p) \approx Z_p.$$

Consider the exact sequence

$$\cdots \to H^{i-1}(X, Z_p) \to H^i(X, Z) \xrightarrow{p} H^i(X, Z) \to H^i(X, Z_p) \to \cdots$$

of Theorem 7.2. Each $H_x^i(X, Z)$ is finitely generated. Suppose some $H_x^i(X, Z)$ contains an element of finite order. Consideration of the exact sequence then shows that $H_x^{i-1}(X, Z_p) \neq 0$ and $H_x^i(X, Z_p) \neq 0$ for some prime p. This being impossible, each $H_x^i(X, Z)$ is a finitely generated free abelian group. Hence p: $H_x^i(X, Z) \to H_x^i(X, Z)$ is an isomorphism into. For each i,

$$H_x^i(X, Z)/pH_x^i(X, Z) \approx H_x^i(X, Z_p).$$

It follows that $H_x^i(X, Z) = 0$ for $i \neq n_p$, and $H_x^{n_p}(X, Z) \approx Z$. The assertion follows.

COROLLARY 7.4. *If* X *is a locally compact Hausdorff space which is cohomology locally connected over* Z, *has locally constant local cohomology groups over* Z (*or equivalently, has property* Q *over* Z), *and if* dim X *is finite, then the conclusion of Theorem* 7.3 *holds.*

Since X is cohomology locally connected, for all open U, V with $\overline{V} \subset U$ and $\overline{V}$ compact, j_{uv}^*: $H^n(V, Z) \to H^n(U, Z)$ has a finitely generated image [2, 6.3]. Hence $H_x^*(X, Z)$ is finitely generated, and we may apply Theorem 7.3.

COROLLARY 7.5. *Suppose* X *is a connected, locally compact Hausdorff space with* dim X $< \infty$, *and suppose* X *is cohomology locally connected over* Z *and has locally constant local cohomology groups over* Z. *Then* X *is a locally orientable generalized* n-*manifold over* Z, *for some* n.

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University of Michigan
and
University of Virginia