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Positivity of Cotangent Bundles

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1. Introduction

Let X be a projective scheme over an algebraically closed field. Given a vector bundle $\mathscr{E}$ on X, we can consider various notions of positivity for $\mathscr{E}$, such as ample, nef, and big. As a particular example, consider a smooth projective variety X and its cotangent bundle Ω_X. When Ω_X is ample, X has some very nice properties. For example, all subvarieties of X are of general type and X is algebraically hyperbolic; so, in particular, X does not contain rational or elliptic curves and there do not exist nonconstant maps $f\colon A \to X$ where A is an abelian variety and X is Kobayashi hyperbolic [3]. Requiring that the cotangent bundle be ample is certainly a very strong property, and for a long time there were few examples of such varieties even though they were expected to be reasonably abundant. One such example was constructed by Michael Schneider.

Theorem 1.1 [17]. *Let $f\colon X \to Y$ be a smooth projective nonisotrivial morphism, where X and Y are smooth projective varieties over $\mathbb{C}$ of dimensions 2 and 1, respectively. Suppose that, for all $y \in Y$, the Kodaira–Spencer map $\rho_{f,y}\colon T_{Y,y} \to H^1(X_y, T_{X_y})$ is nonzero. Then Ω_X is ample.*

Note that certain Kodaira surfaces satisfy the stipulated conditions. In this paper, we generalize Theorem 1.1 to varieties of higher dimensions. To do so, we will introduce a slightly weaker notion of ampleness, which we call "quasi-ample" and "quasi-ample with respect to an open subset U" (see Definitions 1.9 and 1.13). Using this notion, we extend Schneider's result to varieties of higher dimension.

Theorem 1.2. *Let*

$$X^n \xrightarrow{f_n} X^{n-1} \xrightarrow{f_{n-1}} X^{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_3} X^2 \xrightarrow{f_2} X^1,$$

where each X^i is a smooth projective variety over $\mathbb{C}$ of dimension i and each $f_i\colon X^i \to X^{i-1}$ is a smooth projective morphism with $\operatorname{Var}(f_i) = i-1$. *Then Ω_{X^n} is nef and quasi-ample with respect to an open U_n (as defined precisely in Theorem 2.11), and $\mathscr{O}_{\mathbb{P}(\Omega_{X^n})}(1)$ is a big line bundle on $\mathbb{P}(\Omega_{X^n})$.*

We also extend this result to towers of varieties

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$$X^n \xrightarrow{f_n} X^{n-1} \xrightarrow{f_{n-1}} X^{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_3} X^2 \xrightarrow{f_2} X^1,$$

where the f_i are not necessarily smooth (Theorem 2.16), and show that $\Omega^1_{X^n}(\log D)$ is quasi-ample with respect to an open set, where D is a suitable divisor taking into account the singularities of the given morphisms.

In Section 3 we construct a tower of varieties satisfying the conditions of Theorem 2.12 by using a construction due to Kodaira that, for any n, produces a g for which $\mathscr{M}_g$ contains a complete n-dimensional subvariety.

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Definitions and Examples

Recall that a vector bundle $\mathscr{E}$ on a proper scheme X is *ample* if, for every coherent sheaf $\mathscr{F}$, there is an integer $m_0 > 0$ such that, for every $m \geq m_0$, the sheaf $\mathscr{F} \otimes \mathrm{Sym}^m \mathscr{E}$ is generated as an $\mathscr{O}_X$-module by its global sections. Equivalently, $\mathscr{E}$ is ample if the tautological line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ on $\mathbb{P}(\mathscr{E})$ is ample.

A vector bundle $\mathscr{E}$ on a proper scheme X is *nef* (or *semipositive*) if, for every complete nonsingular curve C and map $\gamma : C \to X$, every quotient bundle $\mathscr{Q}$ of $\gamma^*\mathscr{E}$ has degree at least 0. Equivalently, $\mathscr{E}$ is nef if the tautological line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ on $\mathbb{P}(\mathscr{E})$ is nef.

Vector bundles that are ample or nef have many nice properties. For example, quotients of ample (resp. nef) vector bundles are ample (resp. nef), and extensions of ample (resp. nef) vector bundles are ample (resp. nef). For more properties, see [8] or [15, Sec. 6.1A and 6.1B].

Let $\mathscr{E}$ be a vector bundle of rank r on an irreducible projective variety X of dimension n. Following the work of Fulton and Lazarsfeld [5], we can introduce a type of numerical positivity. More precisely, starting with a weighted homogeneous polynomial $P \in \mathbb{Q}[c_1, \ldots, c_r]$, we get a Chern number

$$\int_X P(c(\mathscr{E})) := \int_X P(c_1(\mathscr{E}), \ldots, c_r(\mathscr{E})).$$

DEFINITION 1.3. Let $\Lambda(n, r)$ be the set of all partitions of n by nonnegative integers less than or equal to r. Then for every $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(n, r)$ we can form the *Schur polynomial,* $s_\lambda \in \mathbb{Q}[c_1, \ldots, c_r]$, of weighted degree n, which is the determinant of the $n \times n$ matrix

$$\begin{vmatrix} c_{\lambda_1} & c_{\lambda_1+1} & \cdots & c_{\lambda_1+n-1} \\ c_{\lambda_2-1} & c_{\lambda_2} & \cdots & c_{\lambda_2+n-2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{\lambda_n-n} & c_{\lambda_n-n+1} & \cdots & c_{\lambda_n} \end{vmatrix},$$

where $c_0 = 1$ and $c_i = 0$ if $i \notin [0, r]$.

In particular, if P is a positive linear combination of Schur polynomials and $\mathscr{E}$ is ample (resp. nef), then $\int_X P(c(\mathscr{E})) > 0$ (resp. ≥ 0).

We next generalize the notion of "big" for vector bundles. Recall that a divisor D is *big* if there exists a $c > 0$ such that $h^0(X, \mathscr{O}_X(mD)) > cm^n$ for $m \gg 1$. Generalizing this notion to vector bundles is not consistent in the literature. As a first definition, we generalize the notion as we did with ample and nef. This is the definition given, for example, in [15] and [2].

DEFINITION 1.4 [15, 6.1.23]. Let $\mathscr{E}$ be a vector bundle on X. Then $\mathscr{E}$ is *L-big* if $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ is a big line bundle on $\mathbb{P}(\mathscr{E})$. (Here, *L-big* is used instead of *big* to avoid confusion with Definition 1.6.)

EXAMPLE 1.5. As an example, consider the rank-2 vector bundle

$$\mathscr{E} = \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1)$$

on $\mathbb{P}^1$. Since a direct sum of line bundles is L-big if and only if some $\mathbb{N}$-linear combination of the direct summands is a big line bundle [15, 2.3.2(iv)], we see that $\mathscr{E}$ is L-big. Note that in this example $\mathscr{E}$ is L-big but has a quotient $\mathscr{O}_{\mathbb{P}^1}$, which is not big.

It is also useful to see that $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ is big in a slightly different way (which we will refer to in Example 1.7). First note that $\mathbb{P}(\mathscr{E}) \simeq \mathbb{P}(\mathscr{E}')$, where $\mathscr{E}' := \mathscr{E} \otimes \mathscr{O}_{\mathbb{P}^1}(-1) \simeq \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$. Fix a section C_0 of $X = \mathbb{P}(\mathscr{E}')$ with $\mathscr{O}_X(C_0) \simeq \mathscr{O}_{\mathbb{P}(\mathscr{E}')}(1)$, and let f be a fiber. Then $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ corresponds to a section C_1, which is linearly equivalent to $C_0 + f$. Given $A = aC_0 + bf$ ample, it follows that $a > 0$ and $b > a$ [10, V.2.18]; hence $bC_1 - A = (b-a)C_0$ is effective and so $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ is big.

We next turn to a different generalization of big to vector bundles that was introduced by Viehweg. To do so, we first need a generalization of nef.

DEFINITION 1.6 [19]. Let $\mathscr{E}$ be vector bundle on a projective variety X, and let $\mathscr{H}$ be an ample line bundle.

(i) $\mathscr{E}$ is *weakly positive over an open U* if, for every $a > 0$, there exists a $b > 0$ such that $\mathrm{Sym}^{ab}(\mathscr{E}) \otimes \mathscr{H}^b$ is globally generated over U; that is, $H^0(X, \mathrm{Sym}^{ab}(\mathscr{E}) \otimes \mathscr{H}^b) \otimes \mathscr{O}_X \to \mathrm{Sym}^{ab}(\mathscr{E}) \otimes \mathscr{H}^b$ is surjective over U.

(ii) $\mathscr{E}$ is *V-big* (or *ample with respect to U*) if there exist an open dense U in X and a $c > 0$ such that $\mathrm{Sym}^c \mathscr{E} \otimes \mathscr{H}^{-1}$ is weakly positive over U.

Equivalently, a vector bundle $\mathscr{E}$ is ample with respect to U if and only if the tautological bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ is ample with respect to $\pi^{-1}(U)$ (where $\pi : \mathbb{P}(\mathscr{E}) \to X$); see [21, 3.4]. At first glance it may seem that these two generalizations of big to vector bundles may be equivalent. However, V-big is strictly stronger than L-big.

EXAMPLE 1.7. Consider Example 1.5, where we view $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ as $C_1 = C_0 + f$. Then C_1 is ample with respect to $V = X - C_0$. But $\pi(V) = \mathbb{P}^1$, so there is no open $U \subset \mathbb{P}^1$ with $\pi^{-1}(U) = V$. Thus $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1)$ is L-big but not V-big.

In general, we have that if $\mathscr{E} \to \mathscr{Q}$ is surjective over an open set U and if $\mathscr{E}$ is ample with respect to U, then $\mathscr{Q}$ is also ample with respect to U [21, 3.30]. As noted in Example 1.5, L-big does not have this property, so this again shows that the notion of V-big is stronger than that of L-big.

EXAMPLE 1.8. As a second example, let C be a curve of genus > 1 and let $X = C \times C$, with projections p_1 and p_2. In this case the cotangent bundle Ω_X is L-big (i.e., $\mathcal{O}_{\mathbb{P}(\Omega_X)}(1)$ is a big line bundle on $\mathbb{P}(\Omega_X)$), but since Ω_X surjects onto $p_1^*\omega_C$ we see that Ω_X is not V-big. Similarly, we can see that the two definitions of big are not equivalent for the tangent bundle. Consider the smooth quadric surface Q; here T_Q is L-big, but T_Q surjects onto $p_1^*\omega_{\mathbb{P}^1}^{-1}$ and so is not V-big.

In view of these different definitions, we will avoid saying that a vector bundle $\mathscr{E}$ on X is "big" and instead say either that $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ is a big line bundle or that $\mathscr{E}$ is ample with respect to some open set U.

We next define a new notion of positivity that is slightly weaker than ample but stronger than nef.

DEFINITION 1.9. A vector bundle $\mathscr{E}$ on X is *quasi-ample* if, for every nonconstant morphism $\gamma\colon C \to X$ from a complete nonsingular curve C, $\gamma^*\mathscr{E}$ is ample on C.

In the case where $\mathscr{E}$ is a line bundle, the terminology *strictly nef* has been used [18]. Many properties of ample vector bundles carry over to quasi-ample vector bundles, as the following theorem shows.

THEOREM 1.10 [11, 4.3]. *Let $\mathscr{E}$ and $\mathscr{E}'$ be vector bundles on X.*

(i) *If $\mathscr{E}$ is quasi-ample, then any quotient of $\mathscr{E}$ is quasi-ample.*
(ii) *If $\mathrm{Sym}^m \mathscr{E}$ is quasi-ample for some m, then $\mathscr{E}$ is quasi-ample.*
(iii) *If $\mathscr{E}$ is quasi-ample, then $\mathrm{Sym}^m \mathscr{E}$ and $\mathscr{E}^m$ are quasi-ample for every $m > 0$ and $\Lambda^m\mathscr{E}$ is quasi-ample for $m = 1, 2, \ldots, r$, where r is the rank of $\mathscr{E}$.*
(iv) *If $\mathscr{E}$ and $\mathscr{E}'$ are quasi-ample, then $\mathscr{E} \otimes \mathscr{E}'$ is quasi-ample.*
(v) *Let*
$$0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$$
be an exact sequence of vector bundles on X. If $\mathscr{E}'$ and $\mathscr{E}''$ are quasi-ample, then $\mathscr{E}$ is quasi-ample.
(vi) *Let $\mathscr{E}$ be quasi-ample, and let Y be a subscheme of X. Then $\mathscr{E}|_Y$ is quasi-ample on Y.*

Proof. We give the proof of (i); the others follow similarly. Let $\mathscr{E}$ be a quasi-ample vector bundle on X and let $\mathscr{Q}$ be a quotient of $\mathscr{E}$. If $\gamma\colon C \to X$ is any nonconstant morphism from a complete nonsingular curve, then $\gamma^*\mathscr{E}$ is an ample vector bundle with quotient $\gamma^*\mathscr{Q}$ and so $\gamma^*\mathscr{Q}$ is also ample. Hence $\mathscr{Q}$ is quasi-ample. $\square$

We also have the following criteria for when a quasi-ample bundle is ample, originally due to Gieseker; see [15, 6.1.7] for Gieseker's original statement and proof (or [11, 4.7]).

THEOREM 1.11. *Let $\mathscr{E}$ be a vector bundle on X, where X is proper over a field k. Then $\mathscr{E}$ is ample if and only if the two following conditions are satisfied*:

(i) *there exists an $m_0 > 0$ such that* $\mathrm{Sym}^m \mathscr{E}$ *is generated by global sections for all $m \geq m_0$; and*
(ii) *$\mathscr{E}$ is quasi-ample.*

COROLLARY 1.12. *Let X be a complex projective variety with at most canonical singularities. If ω_X is quasi-ample and big, then ω_X is ample.*

Proof. Let K_X be a canonical divisor corresponding to ω_X. Since K_X is quasi-ample, it is nef. Thus $2K_X - K_X = K_X$ is nef and big, so by the base point free theorem [13, 3.3] it follows that $|bK_X|$ has no base points for $b \gg 0$. Thus ω_X^b is generated by global sections and so, by (1.11), ω_X is ample. □

In general, quasi-ample does not imply ample. Mumford constructs an example of a quasi-ample vector bundle that is not ample; see [9, Ex. 10.6]. Namely, he starts with a curve of genus > 2 and a rank-2 vector bundle $\mathscr{E}$ of degree 0 such that $\mathrm{Sym}^m(\mathscr{E})$ is stable for all $m \geq 0$. Then $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ is a quasi-ample line bundle that is not big and hence cannot be ample. Ramanujam extends this example to produce a quasi-ample and big line bundle that is not ample; see [9, Ex. 10.8].

However, in certain cases quasi-ample implies ample. For example, it is not difficult to see that, for the tangent bundle of a projective variety, T_X being quasi-ample implies that T_X is ample and hence $X \simeq \mathbb{P}^n$. In the case of the cotangent bundle, it is unknown whether Ω_X being quasi-ample implies that Ω_X is ample. It also is unknown whether Ω_X being quasi-ample with $\mathscr{O}_{\mathbb{P}(\Omega_X)}(1)$ big implies that Ω_X is ample with respect to an open set. (*Note:* In Example 1.8, where $X = C \times C$, Ω_X is not quasi-ample.)

We can weaken the condition of quasi-ample as follows.

DEFINITION 1.13. If $U \subseteq X$ is an open set, then $\mathscr{E}$ is *quasi-ample with respect to U* if, for every nonconstant morphism $\gamma : C \to X$ from a complete nonsingular curve C with $\gamma(C) \cap U \neq \emptyset$, $\gamma^*\mathscr{E}$ is ample on C.

We end this section by recalling the notions of isotriviality and maximum variation.

DEFINITION 1.14. A morphism $X \to S$, where S is a complete nonsingular curve, is *isotrivial* if $X_s \simeq X_t$ for general $s, t \in S$.

Note that if $X \to S$ is a smooth projective isotrivial morphism then there exists an étale cover $S' \to S$ such that $X \times_S S' \to S'$ is trivial. If $f : X \to S$ is nonisotrivial then, for general $t \in S$, the Kodaira–Spencer map at t, $\rho_{f,t} : T_{S,t} \to H^1(X_t, T_{X_t})$, is nonzero.

More generally, let $f : X \to Y$ be a surjective morphism between smooth projective varieties; then $\mathrm{Var}(f)$ denotes the number of effective parameters of the birational equivalence classes of the fibers. For the rigorous definition of $\mathrm{Var}(f)$, see [12, 2.8] or [19, p. 329]. If $\mathrm{Var}(f) = 0$, then $X_y \simeq X_t$ for general $y, t \in Y$. If $\mathrm{Var}(f) = \dim Y$, we say that f has *maximum variation*. If f is smooth and if for

all $y \in Y$ the set $\{p \in Y \mid X_p \simeq X_y\}$ is finite, then $\mathrm{Var}(f) = \dim Y$. Conversely, if f is smooth and if $\mathrm{Var}(f) = \dim Y$, then there exists a moduli space for the fibers of f and hence there exists an open set $U \subseteq Y$ such that, for all $y \in U$, the set $\{p \in U \mid X_p \simeq X_y\}$ is finite. In the case where a moduli space exists for the fibers of f, the variation of f is equal to the rank of the Kodaira–Spencer map at a general point of Y; so, in particular, if $\mathrm{Var}(f) = \dim Y$ then the Kodaira–Spencer map at a general point of Y is injective.

2. Positivity of Cotangent Bundles

If X is a complex smooth projective variety, we can consider the case where the cotangent bundle Ω_X is ample. In this case, X is both algebraically and Kobayashi hyperbolic [3] and so (a) X contains no rational or elliptic curves and (b) any map from an abelian variety to X is constant. If we consider the weaker case of Ω_X being only quasi-ample, then X still has some nice properties.

LEMMA 2.1. *Let X be a smooth projective variety with quasi-ample cotangent bundle Ω_X. Then the following statements hold.*

(i) *If $Y \subset X$ is a nonsingular subvariety, then Y has a quasi-ample cotangent bundle Ω_Y.*
(ii) *If $f : Y \to X$ is any morphism where Y is an abelian variety or $\mathbb{P}^1$, then f is constant.*

Proof. Let $i : Y \hookrightarrow X$, where X has a quasi-ample cotangent bundle. Using $\mathscr{I}$ to denote the ideal sheaf of Y, we have the short exact sequence

$$0 \to \mathscr{I}/\mathscr{I}^2 \to \Omega_X|_Y \to \Omega_Y \to 0.$$

Because Ω_X is quasi-ample, $\Omega_X|_Y$ is quasi-ample and hence so is Ω_Y.

Suppose $f : Y \to X$ is a nonconstant morphism from an abelian variety Y to a smooth projective variety X, with Ω_X quasi-ample. Let C be a complete nonsingular curve, and let $\gamma : C \to Y$ be a nonconstant morphism such that $f\gamma : C \to X$ is also nonconstant. Then we have the commutative diagram

$$\begin{tikzcd} \gamma^* f^* \Omega_X \arrow[r, "\beta"] \arrow[d, "\alpha"'] & \Omega_C \arrow[d, equal] \\ \gamma^* \Omega_Y \arrow[r, "\beta'"] & \Omega_C \end{tikzcd}$$

with β and β' nonzero. Hence $\alpha : \gamma^* f^* \Omega_X \to \gamma^* \Omega_Y$ must also be nonzero. Since Ω_X is quasi-ample, $\mathrm{Hom}(\gamma^* f^* \Omega_X, \mathscr{O}_C) = 0$, and since Y is an abelian variety, $\gamma^* \Omega_Y \simeq \mathscr{O}_C^{\oplus d}$, where d is the dimension of Y. Thus $\mathrm{Hom}(\gamma^* f^* \Omega_X, \gamma^* \Omega_Y) = 0$, which forces α to be zero; this is a contradiction, so $f : Y \to X$ must be constant. If $f : \mathbb{P}^1 \to X$ is nonconstant, we obtain a nonconstant map $\sigma : f^* \Omega_X \to \Omega_{\mathbb{P}^1}$. If $\mathscr{E}$ is the image of σ, then $\mathscr{E}$ is ample because it is a quotient of $f^* \Omega_X$. But then,

since $T_{\mathbb{P}^1}$ surjects onto $\mathscr{E}^\vee$, it follows that $\mathscr{E}^\vee$ is also ample, which leads to a contradiction. Therefore, any $f: \mathbb{P}^1 \to X$ must be constant. □

We now begin our generalization of Theorem 1.1.

2.1. Smooth Towers of Smooth Projective Varieties

THEOREM 2.2. *Let $f: X \to Y$ be a smooth projective morphism, where* $\dim X = \dim Y + 1$ *and* $\mathrm{Var}(f) = \dim Y$. *Then, for all $y \in Y$, X_y is a curve of genus at least 2, and if* $\dim Y = 1$ *then the genus of Y is also at least 2. In particular, ω_{X_y} is an ample line bundle on X_y for all $y \in Y$.*

Proof. Suppose first that $\dim Y = 1$, so $\dim X = 2$. Then, by [1, III.15.4], $g(Y) \geq 2$. Since f is flat, the genus of the fibers is constant. If the genus is 0 then all the fibers are isomorphic to $\mathbb{P}^1$; hence f is isotrivial. By the existence of the J-fibration, the genus of the fibers cannot be 1 [1, Chap. V, Secs. 9 and 14]. Thus the genus of the fibers is at least 2. If $\dim Y = n$ for $n > 1$, let S be a general curve in Y. Restricting f to $f^{-1}(S)$ puts us in the previous case, so X_y must have genus at least 2. □

Given a surjective map of smooth projective varieties, $f: X \to Y$, of relative dimension k, we will use the positivity of $f_*\omega_{X/Y}^m$ and, more generally, of $f_*\Omega_{X/Y}^k(\log \Delta)^m$, where Δ is a normal crossing divisor on X. These deep results are found in the work of Viehweg [20] and Kollár [12]. In particular, we will use the following formulation.

THEOREM 2.3. *Let $f: X \to Y$ be a surjective map of smooth projective varieties of relative dimension k, with* $\mathrm{Var}(f) = \dim Y$.

(i) [24, 3.4] *If $f: X \to Y$ is smooth and if $\omega_{X/Y}$ is f-ample, then $f_*\omega_{X/Y}^m$ is ample with respect to an open dense $V \subset Y$ for all $m > 1$, where $f_*\omega_{X/Y}^m \neq 0$. Furthermore, we can take V to be the open set where the moduli map $\eta: V \to M_h$ is quasi-finite over its image.*

(ii) [24, 3.6] *Let $S \subset Y$ be a reduced normal crossing divisor containing the discriminant locus, and let $\Delta := f^*S$ be a normal crossing divisor. Let $V := Y - S$ and $U := X - \Delta$, so we have a smooth family $U \to V$. Suppose that $\omega_{U/V}$ is f-semi-ample, that the smooth fibers of f are canonically polarized, and that the moduli map $\eta: V \to M_h$ is quasi-finite over its image. Then, for m sufficiently large and divisible, $f_*\Omega_{X/Y}^k(\log \Delta)^m$ is ample with respect to V.*

THEOREM 2.4. *Let $f: X \to Y$ be a surjective nonisotrivial morphism of smooth projective varieties with* $\dim Y = 1$, *and let ω_{X_y} be an ample line bundle for all $y \in Y$. Then $\omega_{X/Y}$ is an ample line bundle.*

Proof. The proof follows exactly as in [14, 2.5] once we know that $f_*\omega_{X/Y}^m$ is ample for some $m > 0$ (cf. [12; 20]). □

The following lemma is due to Gieseker.

LEMMA 2.5 [6, Prop. 2.2]. *Let C be a nonsingular curve. Suppose that $\mathscr{F}$ is ample on C and that we have a nontrivial extension*

$$0 \to \mathscr{O}_C \to \mathscr{E} \to \mathscr{F} \to 0.$$

Then $\mathscr{E}$ is ample.

We first consider Schneider's original setup—that is, a nonisotrivial smooth projective morphism from a surface to a curve.

LEMMA 2.6. *Let X and Y be smooth projective varieties with* $\dim Y = 1$, *and let $f\colon X \to Y$ be a smooth projective nonisotrivial morphism. Let $B := \{p \in Y \mid \alpha\colon H^0(X_p, f^*T_Y|_{X_p}) \to H^1(X_p, T_{X_p})$ is not injective$\}$. Then, for any $y \in Y - B$, the short exact sequence*

$$0 \to f^*\Omega_Y|_{X_y} \to \Omega_X|_{X_y} \to \Omega_{X/Y}|_{X_y} \to 0 \tag{1}$$

does not split.

Proof. Let $y \in Y - B$; then $\alpha\colon H^0(X_p, f^*T_Y|_{X_p}) \to H^1(X_p, T_{X_p})$ is injective. To show that (1) does not split, it suffices to show that

$$0 \to T_{X_y} \to T_X|_{X_y} \to f^*T_Y|_{X_y} \to 0 \tag{2}$$

does not split. Taking cohomology, we obtain

$$\begin{aligned} \cdots \to H^0(X_y, T_X|_{X_y}) &\to H^0(X_y, f^*T_Y|_{X_y}) \\ &\to H^1(X_y, T_{X_y}) \to H^1(X_y, T_X|_{X_y}) \to \cdots . \end{aligned}$$

If (2) splits, then $H^0(X_y, T_X|_{X_y}) \to H^0(X_y, f^*T_Y|_{X_y})$ is surjective and so the image of $\alpha\colon H^0(X_y, f^*T_Y|_{X_y}) \to H^1(X_y, T_{X_y})$ is zero. Thus $\operatorname{im}(\alpha) = \ker(\alpha) = 0$, so $H^0(X_y, f^*T_Y|_{X_y}) = 0$—a contradiction. Therefore, (1) must not split. □

COROLLARY 2.7. *Let X and Y be smooth projective varieties over $\mathbb{C}$ of dimensions 2 and 1, respectively, and let $f\colon X \to Y$ be a smooth projective nonisotrivial morphism. Suppose $\gamma\colon C \to X$ is a nonconstant morphism from a complete nonsingular curve with $\gamma(C)$ contained in a fiber of f, say X_y. Moreover, suppose that $\alpha\colon H^0(X_y, f^*T_Y|_{X_y}) \to H^1(X_y, T_{X_y})$ is injective. Then $\gamma^*\Omega_X$ is an ample vector bundle on C.*

Proof. Suppose $\gamma(C) \subseteq X_y$ for some $y \in Y$. Since $\gamma\colon C \to X_y$ is finite and since the pull-back of an ample line bundle by a finite map is ample, it suffices to show that $\Omega_X|_{X_y}$ is ample. By Lemma 2.6,

$$0 \to f^*\Omega_Y|_{X_y} \to \Omega_X|_{X_y} \to \Omega_{X/Y}|_{X_y} \to 0$$

does not split. Since $f^*\Omega_Y|_{X_y} \simeq \mathscr{O}_{X_y}$ and since $\Omega_{X/Y}|_{X_y} \simeq \omega_{X_y}$ is ample by Theorem 2.2, it follows from Lemma 2.5 that $\Omega_X|_{X_y}$ is ample. □

THEOREM 2.8. *Let $f\colon X \to Y$ be a smooth, projective nonisotrivial morphism, with X and Y projective varieties over $\mathbb{C}$ of dimensions 2 and 1, respectively. Let*

$B := \{p \in Y \mid \alpha\colon H^0(X_p, f^*T_Y|_{X_p}) \to H^1(X_p, T_{X_p})$ is not injective$\}$. Then Ω_X is quasi-ample with respect to $U := f^{-1}(Y - B)$ and, for all $\lambda \in \Lambda(2,2)$, the Schur polynomial is positive; that is, $\int_X s_\lambda(\Omega_X) > 0$. In particular, $\mathscr{O}_{\mathbb{P}(\Omega_X)}(1)$ is big.

Proof. We first show that Ω_X is quasi-ample with respect to U. Let $\gamma\colon C \to X$ be a nonconstant morphism from a complete nonsingular curve C such that $\gamma(C) \cap U \neq \emptyset$. If $\gamma(C)$ is in a fiber of f then, by (2.7), $\gamma^*\Omega_X$ is ample.

Now suppose $\gamma(C)$ is not in a fiber of f, so that $f \circ \gamma\colon C \to Y$ is nonconstant. We have the following short exact sequence:

$$0 \to \gamma^* f^* \omega_Y \to \gamma^* \Omega_X \to \gamma^* \omega_{X/Y} \to 0.$$

Because ω_Y is ample, $\gamma^* f^* \omega_Y$ is ample on C. By Theorem 2.4, $\omega_{X/Y}$ is ample on X; hence $\gamma^*\omega_{X/Y}$ is ample on C. Therefore, $\gamma^*\Omega_X$ is ample and so Ω_X is quasi-ample with respect to U. Note also that Ω_X is an extension of two nef line bundles (viz., $f^*\omega_Y$ and $\omega_{X/Y}$), so Ω_X is nef.

We now will show that the Schur polynomials are positive. Let $\lambda \in \Lambda(2,2)$; then $\lambda = (1,1)$ or $\lambda = (2,0)$, and by definition we have

$$s_{(1,1)}(\Omega_X) = c_1(\Omega_X)^2 - c_2(\Omega_X) \quad \text{and} \quad s_{(2,0)}(\Omega_X) = c_2(\Omega_X).$$

Using the short exact sequence

$$0 \to f^*\omega_Y \to \Omega_X \to \omega_{X/Y} \to 0,$$

we see that

$$c_1(\Omega_X) = c_1(f^*\omega_Y) + c_1(\omega_{X/Y}) \quad \text{and} \quad c_2(\Omega_X) = c_1(f^*\omega_Y) \cdot c_1(\omega_{X/Y}).$$

Thus,

$$s_{(1,1)}(\Omega_X) = c_1(\Omega_X)^2 - c_2(\Omega_X) = c_1(f^*\omega_Y) \cdot c_1(\omega_{X/Y}) + c_1(\omega_{X/Y})^2$$

and

$$s_{(2,0)}(\Omega_X) = c_1(f^*\omega_Y) \cdot c_1(\omega_{X/Y}).$$

Since ω_Y is ample, $c_1(\omega_Y) = \sum m_i[y_i]$, where $y_i \in X$ are points and $\sum m_i > 0$. Since f is flat, $c_1(f^*\omega_Y) = f^*c_1(\omega_Y) = \sum n_i[X_{y_i}]$. Now, $\omega_{X/Y}|_{X_y} = \omega_{X_y}$ is ample for all $y \in Y$ and $\deg(\omega_{X_y}) = 2g(X_y) - 2 \geq 2$ is constant for all $y \in Y$. Thus,

$$\int_X c_1(f^*\omega_Y) \cdot c_1(\omega_{X/Y}) = \sum n_i(2g-2) > 0.$$

Furthermore, $\omega_{X/Y}$ is an ample line bundle on X by Theorem 2.4, so $\int_X c_1(\omega_{X/Y})^2 > 0$. Thus,

$$\int_X s_{(1,1)}(\Omega_X) = \int_X c_1(f^*\omega_Y) \cdot c_1(\omega_{X/Y}) + c_1(\omega_{X/Y})^2 > 0,$$
$$\int_X s_{(2,0)}(\Omega_X) = \int_X c_1(f^*\omega_Y) \cdot c_1(\omega_{X/Y}) > 0.$$

Since Ω_X is nef, to show that $\mathscr{O}_{\mathbb{P}(\Omega_X)}(1)$ is big it suffices to show that $\mathscr{O}_{\mathbb{P}(\Omega_X)}(1)$ has positive top intersection. But by [6, Lemma 1.8], this is equivalent to showing that $\int_X s_{(1,1)}(\Omega_X) > 0$; hence $\mathscr{O}_{\mathbb{P}(\Omega_X)}(1)$ is big. $\square$

We will now consider the tower

$$X^n \xrightarrow{f_n} X^{n-1} \xrightarrow{f_{n-1}} X^{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_3} X^2 \xrightarrow{f_2} X^1,$$

where each X^i is a smooth projective variety over $\mathbb{C}$ of dimension i and where, for $2 \le i \le n$, $f_i\colon X^i \to X^{i-1}$ is a smooth projective morphism with $\mathrm{Var}(f_i) = \dim X^{i-1}$.

We first prove the following generalization.

THEOREM 2.9. *Let*

$$X^n \xrightarrow{f_n} X^{n-1} \xrightarrow{f_{n-1}} X^{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_3} X^2 \xrightarrow{f_2} X^1,$$

where each X^i is a smooth projective variety over $\mathbb{C}$ of dimension i and where each $f_i\colon X^i \to X^{i-1}$ is a smooth projective morphism with $\mathrm{Var}(f_i) = \dim X^{i-1}$. *Then Ω_{X^n} is nef and, for all $\lambda \in \Lambda(n,n)$, the corresponding Schur polynomial is positive*; *that is, $\int_X s_\lambda(\Omega_{X^n}) > 0$. In particular, $\mathscr{O}_{\mathbb{P}(\Omega_{X^n})}(1)$ is big.*

Proof. We will prove this by induction on n. By Theorem 2.8, the statement is true for $n = 2$, and we assume it holds for $n-1$. Let $X := X^n$; then we have the short exact sequence

$$0 \to f_n^*\Omega_{X^{n-1}} \to \Omega_X \to \omega_{X/X^{n-1}} \to 0.$$

Now $\Omega_{X^{n-1}}$ is nef by induction, so $f_n^*\Omega_{X^{n-1}}$ is nef. Since f_n is smooth and since $\omega_{X/X^{n-1}}$ is f_n-ample, it follows from [22, 6.22] that $(f_n)_*\omega^m_{X/X^{n-1}}$ is nef for all $m > 0$. Additionally, since $\omega_{X/X^{n-1}}$ is f_n-ample, for $m \gg 0$ the natural map

$$f_n^*(f_n)_*\omega^m_{X/X^{n-1}} \to \omega^m_{X/X^{n-1}}$$

is surjective. Thus $\omega_{X/X^{n-1}}$ is nef and hence Ω_X is nef.

Let $\lambda \in \Lambda(n,n)$. Define $d_i := c_i(\Omega_X)$, $\alpha_i := c_i(f_n^*\Omega_{X^{n-1}})$, and $\beta := c_1(\omega_{X/X^{n-1}})$. Then, from the short exact sequence

$$0 \to f_n^*\Omega_{X^{n-1}} \to \Omega_X \to \omega_{X/X^{n-1}} \to 0,$$

we have

$$d_i = \alpha_i + \alpha_{i-1}\beta, \quad \text{where } \alpha_0 = 1 \text{ and } \alpha_i = 0 \text{ for } i \notin [0, n-1].$$

Thus, by [4, 5.2, Ex. 4],

$$\begin{aligned} s_\lambda(\Omega_X) &= s_\lambda(d_1, \ldots, d_n) \\ &= s_\lambda(\alpha_1, \ldots, \alpha_{n-1}, \beta) \\ &= \sum_{\mu \subset \lambda} s_{\lambda/\mu}(\alpha_1, \ldots, \alpha_{n-1}) s_\mu(\beta). \end{aligned}$$

Now $\beta = c_1(\omega_{X/X^{n-1}})$, so if $\mu = (\mu_1, \ldots, \mu_n)$ and $\mu_1 \neq 1$ then $s_\mu(\beta) = 0$. Also, note that $s_{(1^k)}(\omega_{X/X^{n-1}}) = c_1(\omega_{X/X^{n-1}})^k$. Therefore,

$$s_\lambda(\Omega_X) = \sum_{k=1}^{n} s_{\lambda/(1^k)}(f_n^*\Omega_{X^{n-1}}) \cdot c_1(\omega_{X/X^{n-1}})^k.$$

Let $1 \le k \le n-1$ and $\mu \in \Lambda(n-k, n-k)$. Then

$$s_\mu \cdot s_{(1^k)} = \sum_\nu c^\nu_{\mu/(1^k)} s_\nu \quad \text{and} \quad s_{\nu/(1^k)} = \sum_\mu c^\nu_{\mu/(1^k)} s_\mu.$$

By [4, 1.1, Prop.], $c^\nu_{\mu/(1^k)} = 1$ if ν can be obtained from μ by adding k boxes, no two of which are in the same row, and equals zero otherwise. Thus, setting $\nu = \lambda$ yields

$$s_{\lambda/(1^k)} = \sum_\mu s_\mu,$$

where the sum is taken over $\mu \in \Lambda(n-k, n-k)$ such that μ can be obtained from λ by subtracting k boxes, no two of which are in the same row. Therefore,

$$s_\lambda(\Omega_X) = \sum_{k=1}^{n} \left(c_1(\omega_{X/X^{n-1}})^k \sum_{\mu \in \Lambda(n-k,n-k)} s_\mu(f_n^*\Omega_{X^{n-1}}) \right),$$

where the second sum is taken over $\mu \in \Lambda(n-k, n-k)$ such that μ can be obtained from λ by subtracting k boxes, no two of which are in the same row.

Consider the first term, $c_1(\omega_{X/X^{n-1}}) \sum_\mu s_\mu(f_n^*\Omega_{X^{n-1}})$. By induction, $s_\mu(\Omega_{X^{n-1}})$ is a positive polynomial for $\mu \in \Lambda(n-1, n-1)$; that is, $s_\mu(\Omega_{X^{n-1}}) = \sum m_i y_i$, where $y_i \in X^{n-1}$ are points and $\sum m_i > 0$. Now $\omega_{X/X^{n-1}}|_{X_y} \simeq \omega_{X_y}$ is ample on X_y for all $y \in X^{n-1}$, so

$$\int_X c_1(\omega_{X/X^{n-1}}) \cdot s_\mu(f_n^*\Omega_{X^{n-1}}) > 0.$$

Since $f_n^*\Omega_{X^{n-1}}$ and $\omega_{X/X^{n-1}}$ are nef, for $2 \le k \le n-1$ and $\mu \in \Lambda(n-k, n-k)$ as before we have

$$\int_X c_1(\omega_{X/X^{n-1}})^k \cdot s_\mu(f_n^*\Omega_{X^{n-1}}) \ge 0$$

and

$$\int_X c_1(\omega_{X/X^{n-1}})^n \ge 0.$$

Therefore, $\int_X s_\lambda(\Omega_X) > 0$. In particular, this holds for $\lambda = (1^n)$ and so, by [6, Lemma 1.8], $\mathcal{O}_{\mathbb{P}(\Omega_{X^n})}(1)$ is big. □

We continue to assume that we have a tower of varieties

$$X^n \xrightarrow{f_n} X^{n-1} \xrightarrow{f_{n-1}} X^{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_3} X^2 \xrightarrow{f_2} X^1,$$

where each X^i is a smooth projective variety over $\mathbb{C}$ of dimension i and where each $f_i \colon X^i \to X^{i-1}$ is a smooth projective morphism with the property that $\operatorname{Var}(f_i) = \dim X^{i-1}$. For $1 \le i \le n-1$, define

$$B_i := \{p \in X^i \mid \alpha_i \colon H^0(X_p^{i+1}, f_{i+1}^* T_{X^i}|_{X_p^{i+1}}) \to H^1(X_p^{i+1}, T_{X_p^{i+1}}) \text{ is not injective}\} \subset X^i.$$

Note that, since each f_{i+1} is of maximum variation, $X^i - B_i$ is an open dense set. Set $U_1 := X^1$, and for $2 \le i \le n$ define open sets $U_i \subset X_i$ as

$$U_i := f_i^{-1}(U_{i-1} - B_{i-1}).$$

LEMMA 2.10. *In the setting just described, the moduli map*

$$\mu_{i-1}\colon V_{i-1} := U_{i-1} - B_{i-1} \to M_g$$

induced by the family $f_i\colon X^i \to X^{i-1}$ *is quasi-finite onto its image.*

Proof. It suffices to show that $F_{i-1} := \{y \in X^{i-1} \mid X^i_y \simeq X^i_p$ for infinitely many $p \in X^{i-1}\} \subseteq B_{i-1}$. Let $y \in F_{i-1}$; then there exists a connected closed subscheme $Z \subset X^{i-1}$ with $y \in Z$ such that $f_Z\colon X_Z := X^i \times_{X^{i-1}} Z \to Z$ has variation 0. In particular, $H^0((X_Z)_y, f_Z^* T_Z|_{(X_Z)_y}) \to H^1((X_Z)_y, T_{(X_Z)_y})$ is not injective. Then, from the commutative diagram

$$\begin{tikzcd}
H^0((X_Z)_y, f_Z^* T_Z|_{(X_Z)_y}) \arrow[r] \arrow[d, hook] & H^1((X_Z)_y, T_{(X_Z)_y}) \arrow[d, equal] \\
H^0(X_y, f_i^* T_Y|_{X_y}) \arrow[r] & H^1(X_y, T_{X_y})
\end{tikzcd}$$

we see that $H^0(X_y, f_i^* T_Y|_{X_y}) \to H^1(X_y, T_{X_y})$ cannot be injective; hence $y \in B_{i-1}$. □

THEOREM 2.11. *In the setting of Lemma 2.10,* Ω_{X^n} *is quasi-ample with respect to* U_n.

Proof. We will prove this by induction on n. By Theorem 2.8, the statement is true for $n = 2$, and we assume it holds for $n - 1$. Let $X := X^n$; then we have the short exact sequence

$$0 \to f_n^* \Omega_{X^{n-1}} \to \Omega_X \to \omega_{X/X^{n-1}} \to 0.$$

Let $\gamma\colon C \to X$ be a nonconstant morphism from a complete nonsingular curve C such that $\gamma(C) \cap U_n \neq \emptyset$. Suppose first that $\gamma(C)$ is contained in a fiber of f_n, say $\gamma(C) \subseteq X_y$ for some $y \in X^{n-1}$. Since $\gamma\colon C \to X_y$ is finite, it suffices to show that $\Omega_X|_{X_y}$ is ample. Note also that since $\gamma(C) \cap U_n \neq \emptyset$, it follows that $y \in U_{n-1} - B_{n-1}$. Let $f_2 f_3 \cdots f_{n-1}(y) = s \in X^1$ and let $h = f_2 f_3 \cdots f_{n-1} f_n : X = X^n \to X^1$. Then we have the short exact sequence

$$0 \to T_{X_s}|_{X_y} \to T_X|_{X_y} \to h^* T_{X^1}|_{X_y} \to 0. \tag{3}$$

I claim that (3) doesn't split. Indeed, suppose it splits; then, given the long exact sequence

$$\begin{aligned} \cdots \to H^0(X_y, T_X|_{X_y}) &\to H^0(X_y, h^* T_{X^1}|_{X_y}) \\ &\to H^1(X_y, T_{X_s}|_{X_y}) \to H^1(X_y, T_X|_{X_y}) \to \cdots, \end{aligned}$$

we have a surjection $\beta\colon H^0(X_y, T_X|_{X_y}) \twoheadrightarrow H^0(X_y, h^* T_{X^1}|_{X_y})$. Consider the following commutative diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & f_n^* T_{X^{n-1}/X^1}|_{X_y} \arrow[d] & \\
0 \arrow[r] & T_{X/X^{n-1}}|_{X_y} \arrow[r] \arrow[d] & T_X|_{X_y} \arrow[r] \arrow[d, equal] & f_n^* T_{X^{n-1}}|_{X_y} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_{X/X^1}|_{X_y} \arrow[r] \arrow[d] & T_X|_{X_y} \arrow[r] & h^* T_{X^1}|_{X_y} \arrow[r] \arrow[d] & 0 \\
 & f_n^* T_{X^{n-1}/X^1}|_{X_y} \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}$$

Taking cohomology then gives

$$\begin{tikzcd}
\cdots \arrow[r] & H^0(X_y, T_X|_{X_y}) \arrow[r] \arrow[d, equal] & H^0(X_y, f_n^* T_{X^{n-1}}|_{X_y}) \arrow[r, "\alpha_{n-1}"] \arrow[d] & H^1(X_y, T_{X/X^{n-1}}|_{X_y}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^0(X_y, T_X|_{X_y}) \arrow[r, "\beta"] & H^0(X_y, h^* T_{X^1}|_{X_y}) \arrow[r] & H^1(X_y, T_{X/X^1}|_{X_y}) \arrow[r] & \cdots
\end{tikzcd}$$

Since $y \notin B_{n-1}$, α_{n-1} is injective and hence $\operatorname{im}\beta = 0$. But β is surjective, so $H^0(X_y, h^* T_{X^1}|_{X_y}) = 0$—a contradiction. Thus (3) does not split and so neither does

$$0 \to \mathscr{O}_{X_y} \to \Omega_X|_{X_y} \to \Omega_{X_s}|_{X_y} \to 0.$$

By Lemma 2.5, to show that $\Omega_X|_{X_y}$ is ample, it suffices to show that $\Omega_{X_s}|_{X_y}$ is ample.

We have

$$X_s = X_s^n \xrightarrow{(f_n)_s} X_s^{n-1} \xrightarrow{(f_{n-1})_s} \cdots \longrightarrow X_s^3 \xrightarrow{(f_3)_s} X_s^2,$$

where each X_s^i is a smooth projective variety over $\mathbb{C}$ of dimension $i-1$ and where each $(f_i)_s \colon X_s^i \to X_s^{i-1}$ is a smooth projective morphism with the property that $\operatorname{Var}(f_i)_s = \dim(X^{i-1})_s$. For $2 \le i \le n-1$, define

$$\begin{aligned} B_{i,s} := \{ p \in X_s^i \mid \alpha_{i,s} \colon H^0(X_p^{i+1}, (f_{i+1})_s^* T_{X_s^i}|_{X_p^{i+1}}) \\ \to H^1((X_s^{i+1})_p, T_{(X_s^{i+1})_p}) \text{ is not injective} \}. \end{aligned}$$

Define open $U_{i,s} \subseteq (X^i)_s$ as follows:

$$U_{2,s} := X_s^2,$$
$$U_{i,s} := (f_i)_s^{-1}(U_{i-1,s} - B_{i-1,s}) \quad \text{for } 3 \le i \le n.$$

Then, by induction, Ω_{X_s} is quasi-ample with respect to

$$U_{n,s} := (f_n)_s^{-1}(U_{n-1,s} - B_{n-1,s}).$$

I first claim that $B_{i,s} \subseteq B_i \cap X_s^i$. Indeed, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & T_{X_s^{i+1}/X_s^i}|_{X_p^{i+1}} \arrow[r] \arrow[d, equal] & T_{X_s^{i+1}}|_{X_p^{i+1}} \arrow[r] \arrow[d, hook] & (f_{i+1})_s^* T_{X_s^i}|_{X_p^{i+1}} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & T_{X^{i+1}/X^i}|_{X_p^{i+1}} \arrow[r] & T_{X^{i+1}}|_{X_p^{i+1}} \arrow[r] & f_{i+1}^* T_{X^i}|_{X_p^{i+1}} \arrow[r] & 0
\end{tikzcd}$$

Hence, taking cohomology gives

$$\begin{tikzcd}
H^0(X_p^{i+1}, (f_{i+1})_s^* T_{X_s^i}|_{X_p^{i+1}}) \arrow[r, "\alpha_{i,s}"] \arrow[d, hook] & H^1(X_p^{i+1}, T_{X_p^{i+1}}) \arrow[d, equal] \\
H^0(X_p^{i+1}, f_{i+1}^* T_{X^i}|_{X_p^{i+1}}) \arrow[r, "\alpha_i"] & H^1(X_p^{i+1}, T_{X_p^{i+1}})
\end{tikzcd}$$

Thus, if α_i is injective then $\alpha_{i,s}$ is injective, so $B_{i,s} \subseteq B_i \cap X_s^i$.

I next claim that $(X^i)_s \cap U_i \subseteq U_{i,s}$ for $i \ge 2$. Indeed, if $i = 2$ then this follows from the definition of U_2. Suppose it is true for $i - 1$; then

$$\begin{aligned} U_i \cap X_s^i &= (f_i)_s^{-1}((U_{i-1} \cap X_s^{i-1}) - (B_{i-1} \cap X_s^{i-1})) \\ &\subseteq (f_i)_s^{-1}(U_{i-1,s} - B_{i-1,s}) = U_{i,s}. \end{aligned}$$

Thus, since $y \in X_s \cap (U_{n-1} - B_{n-1}) \subseteq U_{n-1,s} - B_{n-1,s}$, we conclude that $\Omega_{X_s}|_{X_y}$ is ample.

Now suppose $\gamma(C)$ is not in the fiber of f_n, so that $f_n \circ \gamma : C \to X^{n-1}$ is non-constant. Since $\gamma(C) \cap f_n^{-1}(U_{n-1} - B_{n-1}) \neq \emptyset$, we have $f_n\gamma(C) \cap U_{n-1} \neq \emptyset$. Thus, by induction, $\gamma^*(f_n^*\Omega_{X^{n-1}})$ is ample. We have the short exact sequence

$$0 \to \gamma^*(f_n^*\Omega_{X^{n-1}}) \to \gamma^*\Omega_X \to \gamma^*\omega_{X/X^{n-1}} \to 0,$$

so to show that $\gamma^*\Omega_X$ is ample it suffices to show that $\gamma^*\omega_{X/X^{n-1}}$ is ample. By Theorem 2.3, $f_*\omega^m_{X/X^{n-1}}$ is ample with respect to $X_{n-1} - B_{n-1}$ for $m > 1$ where $f_*\omega^m_{X/X^{n-1}} \neq 0$. Thus, since $\gamma(C) \cap f_n^{-1}(U_{n-1} - B_{n-1}) \neq \emptyset$, we have that $\gamma^*f^*f_*\omega^m_{X/X^{n-1}}$ is ample with respect to $\gamma^{-1}f^{-1}(X_{n-1} - B_{n-1})$, an open dense subset of the curve C. Hence, $\gamma^*f^*f_*\omega^m_{X/X^{n-1}}$ is ample on C. Furthermore, since $\omega_{X/X^{n-1}}$ is f_n-ample, it follows that

$$\gamma^*f_n^*(f_n)_*\omega^m_{X/X^{n-1}} \to \gamma^*\omega^m_{X/X^{n-1}}$$

is surjective for sufficiently large m and so $\gamma^*\omega_{X/X^{n-1}}$ is ample. $\square$

COROLLARY 2.12. *Let*

$$X^n \xrightarrow{f_n} X^{n-1} \xrightarrow{f_{n-1}} X^{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_3} X^2 \xrightarrow{f_2} X^1,$$

where each X^i is a smooth projective variety over $\mathbb{C}$ of dimension i and where each $f_i\colon X^i \to X^{i-1}$ is a smooth projective morphism with the property that, for all $y \in X^{i-1}$, $\alpha_i\colon H^0(X^{i+1}_y, f^*_{i+1}T_{X^i}|_{X^{i+1}_y}) \to H^1(X^{i+1}_y, T_{X^{i+1}_y})$ is injective. Then, Ω_{X^n} is quasi-ample and, for all $\lambda \in \Lambda(n,n)$, the Schur polynomial is positive—that is, $\int_X s_\lambda(\Omega_X) > 0$. In particular, Ω_{X^n} is quasi-ample and $\mathscr{O}_{\mathbb{P}(\Omega_{X^n})}(1)$ is big.

Proof. By the assumption on each $f_i\colon X^i \to X^{i-1}$, the sets $B_i \subset X^i$ are empty. Hence each $U_i = X^i$ and so, by Theorem 2.11, Ω_{X^n} is quasi-ample. The second statement follows from Theorem 2.9. □

Let us also remark that the condition on the Kodaira–Spencer maps is necessary for the cotangent bundle to be quasi-ample on *all* of X^n. As an example, consider a nonisotrivial smooth projective morphism $f\colon X \to Y$ from a smooth projective surface to a smooth projective curve. Suppose there exists a $y \in Y$ such that the Kodaira–Spencer map $\rho_{f,y}\colon T_{Y,y} \to H^1(X_y, T_{X_y})$ is zero. Then $\Omega_X|_{X_y} \simeq \mathscr{O}_{X_y} \oplus \Omega_{X_y}$, so $\Omega_X|_{X_y}$ is not ample.

2.2. Towers of Varieties Where the Morphisms Are Not Smooth

We next weaken the hypothesis on the f_i. Let X be a smooth variety of dimension n and let $D \subset X$ be a reduced normal crossing divisor. Recall that $\Omega^1_X(\log D)$ is the *sheaf of 1-forms on X with logarithmic poles along D* and is defined as follows: If $z_1, \ldots, z_n$ are local analytic coordinates on X with $D = (z_1 \cdots z_l = 0)$, then $\Omega^1_X(\log D)$ is locally generated by $\frac{dz_1}{z_1}, \ldots, \frac{dz_l}{z_l}, dz_{l+1}, \ldots, dz_n$. If D has normal crossings but is not reduced, we abuse notation and write $\Omega^1_X(\log D)$ for $\Omega^1_X(\log D_{\mathrm{red}})$.

LEMMA 2.13. *Let $D = D_1 + D_2$ be a normal crossing divisor on a smooth variety X, and suppose that $\Omega^1_X(\log D_2)$ is quasi-ample with respect to $X - D_2$. Then $\Omega^1_X(\log D)$ is quasi-ample with respect to $X - D$.*

Proof. Let $\gamma\colon C \to X$ be a nonconstant morphism from a complete nonsingular curve such that $\gamma(C) \cap (X - D) \neq \emptyset$. Without loss of generality, we may assume that D_1 and D_2 do not contain any common components. If $\gamma(C) \cap D_1 = \emptyset$ then $\gamma^*\Omega^1_X(\log D_2) \simeq \gamma^*\Omega^1_X \log(D)$ is ample. If $\gamma(C) \cap D_1 \neq \emptyset$ then, since $\gamma(C) \not\subseteq D_1$, it follows that $\gamma(C) \cap D_1$ must consist of a finite number of points. Hence we have the short exact sequence

$$0 \to \gamma^*\Omega^1_X(\log D_2) \to \gamma^*\Omega^1_X(\log D) \to \gamma^*\mathscr{O}_{D_1}(D_2|_{D_1}) \to 0$$

with $\gamma^*\Omega^1_X(\log D_2)$ and $\gamma^*\mathscr{O}_{D_1}(D_2|_{D_1})$ ample. Thus $\gamma^*\Omega^1_X(\log D)$ is also ample. □

We consider the following:

$$X^n \xrightarrow{f_n} X^{n-1} \xrightarrow{f_{n-1}} X^{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_3} X^2 \xrightarrow{f_2} X^1,$$

where each X^i is a smooth projective variety over $\mathbb{C}$ of dimension i. Suppose that, for $2 \le i \le n$:

(i) $f_i : X^i \to X^{i-1}$ is a flat projective morphism with $\mathrm{Var}(f_i) = \dim X^{i-1}$; and
(ii) $\omega_{X^i_y}$ is ample for all $y \notin \Delta_{i-1,i}$, where

$$\Delta_{i-1,i} := \{y \in X^{i-1} \mid f_i^{-1}(y) \text{ is singular}\}.$$

Put $\Delta_{2,1} := f_2^*\Delta_{1,2}$, and define recursively, for all $i < n$,

$$\Delta_{i,i-1} := f_i^*((\Delta_{i-1,i-2} + \Delta_{i-1,i})_{\mathrm{red}}) \subset X^i.$$

We will also write $\Delta_{i-1} := (\Delta_{i-1,i-2} + \Delta_{i-1,i})_{\mathrm{red}} \subset X^{i-1}$, so that $\Delta_{i,i-1} = f_i^*(\Delta_{i-1})$. Let $\Delta_{1,i}$ be the discriminant locus of $f_2 \circ \cdots \circ f_i : X^i \to X^1$; that is, $\Delta_{1,i} := \{s \in X^1 \mid f_i^{-1} \cdots f_2^{-1}(s) \text{ is singular}\}$ and $\Delta_{i,1} := f_i^* \cdots f_2^*\Delta_{1,i}$.

Define

$$B_i' := \{p \in X^i - \Delta_{i,i+1} \mid \alpha_i : H^0(X_p^{i+1}, f_{i+1}^* T_{X^i}|_{X_p^{i+1}}) \to H^1(X_p^{i+1}, T_{X_p^{i+1}}) \text{ is not injective}\}.$$

Set $B_1 := B_1'$ and, for $2 \le i \le n$, set

$$B_i := B_i' + f_i^{-1}(B_{i-1}).$$

In this setting we will show that, if Δ_i and $\Delta_{i+1,i}$ are normal crossing divisors for $1 \le i \le n-1$, then $\Omega^1_{X^n}(\log \Delta_{n,n-1})$ is quasi-ample with respect to $X^n - \Delta_{n,n-1} - f_n^{-1}(B_{n-1})$. We follow the same ideas as before and will prove this by induction. To prove the case where $n = 2$, we need the following lemma.

LEMMA 2.14 [23, 1.4]. *Let $f : X \to Y$ be a nonisotrivial morphism between smooth projective varieties of dimensions 2 and 1, respectively. Let $\Delta_{1,2} \subset Y$ be the discriminant divisor and let $\Delta_{2,1} := f^*\Delta_{1,2}$. Then $2g(Y) - 2 + \deg \Delta_{1,2} \ge 1$.*

THEOREM 2.15. *Let $f : X \to Y$ be a flat nonisotrivial morphism between smooth projective varieties over $\mathbb{C}$ of dimensions 2 and 1, respectively. Let $\Delta_{1,2} \subset Y$ be the discriminant divisor and $\Delta_{2,1} := f^*\Delta_{1,2}$. Let $B := B_1$ be as before. Suppose that ω_{X_y} is ample for all $y \notin \Delta_{1,2}$ and that both $\Delta_{1,2}$ and $\Delta_{2,1}$ are normal crossing divisors. Then $\Omega^1_X(\log \Delta_{2,1})$ is quasi-ample with respect to $X - \Delta_{2,1} - f^{-1}(B)$.*

Proof. We have the short exact sequence

$$0 \to f^*\Omega^1_Y(\log \Delta_{1,2}) \to \Omega^1_X(\log \Delta_{2,1}) \to \Omega^1_{X/Y}(\log \Delta_{2,1}) \to 0.$$

Since $\Omega^1_{X/Y}(\log \Delta_{2,1})$ is locally free, taking determinants gives $\Omega^1_{X/Y}(\log \Delta_{2,1}) \simeq \omega_{X/Y} \otimes \mathcal{O}_Y((\Delta_{2,1})_{\mathrm{red}} - \Delta_{2,1}) \subseteq \omega_{X/Y}$. By Lemma 2.14, $\Omega^1_Y(\log \Delta_{1,2}) \simeq \omega_Y \otimes \mathcal{O}_Y(\Delta_{1,2})$ is an ample line bundle on Y.

Let $\gamma : C \to X$ be a nonconstant morphism from a complete nonsingular curve C such that $\gamma(C) \cap (X - \Delta_{2,1} - f^{-1}(B)) \neq \emptyset$. Suppose first that $\gamma(C)$ is contained in a fiber of f, say $\gamma(C) \subseteq X_y$. Since $\gamma : C \to X_y$ is finite, it suffices to show that $\Omega_X|_{X_y}$ is ample. Note that $y \notin \Delta_{1,2}$ and hence $X_y \cap \Delta_{2,1} = \emptyset$. Since

$y \notin B$, the same argument as in the proof of Corollary 2.7 shows that the short exact sequence

$$0 \to T_{X/Y}|_{X_y} \to T_X|_{X_y} \to f^*T_Y|_{X_y} \to 0$$

does not split. Hence,

$$0 \to f^*\Omega^1_Y(\log \Delta_{1,2})|_{X_y} \to \Omega^1_X(\log \Delta_{2,1})|_{X_y} \to \Omega^1_{X/Y}(\log \Delta_{2,1})|_{X_y} \to 0$$

does not split. Now $f^*\Omega^1_Y(\log \Delta_{1,2})|_{X_y} \simeq \mathscr{O}_{X_y}$, and $\Omega^1_{X/Y}(\log \Delta_{2,1})|_{X_y} \simeq \omega_{X_y}$ is ample since $y \notin \Delta_{1,2}$. Thus, by Lemma 2.5, $\Omega_X|_{X_y}$ is an ample vector bundle on X_y.

Next suppose $\gamma(C)$ is not contained in a fiber of f, so that $f\gamma : C \to Y$ is non-constant. Since $\omega_Y \otimes \mathscr{O}_Y(\Delta_{1,2})$ is ample, $\gamma^*f^*(\omega_Y \otimes \mathscr{O}_Y(\Delta_{1,2}))$ is also ample. To show that $\gamma^*\Omega^1_X(\log \Delta_{2,1})$ is ample, it suffices to show that $\gamma^*\Omega^1_{X/Y}(\log \Delta_{2,1})$ is ample. We have a smooth family $X - \Delta_{2,1} - f^{-1}B \to Y - \Delta_{1,2} - B$ that satisfies the conditions of Theorem 2.3(ii); thus, for m sufficiently large and divisible, $f_*\Omega^1_{X/Y}(\log \Delta_{2,1} + f^{-1}B)^m$ is ample with respect to $Y - \Delta_{1,2} - B$. Furthermore, for m sufficiently large,

$$f^*f_*\Omega^1_{X/Y}(\log \Delta_{2,1} + f^{-1}B)^m \to \Omega^1_{X/Y}(\log \Delta_{2,1} + f^{-1}B)^m$$

is surjective over $X - \Delta_{2,1} - f^{-1}B$. Since $\gamma(C) \cap (X - \Delta_{2,1} - f^{-1}(B)) \neq \emptyset$, we find that $\gamma^*\Omega^1_{X/Y}(\log \Delta_{2,1} + f^{-1}B)^m$ is ample with respect to a dense open set of the curve C and hence that $\gamma^*\Omega^1_{X/Y}(\log \Delta_{2,1} + f^{-1}B)$ is an ample line bundle on C. Since $f^{-1}B$ is reduced, $\gamma^*\Omega^1_{X/Y}(\log \Delta_{2,1} + f^{-1}B) \simeq \gamma^*\Omega^1_{X/Y}(\log \Delta_{2,1})$ is also ample.

Thus, we have the short exact sequence

$$0 \to \gamma^*f^*\Omega^1_Y(\log \Delta_{1,2}) \to \gamma^*\Omega^1_X(\log \Delta_{2,1}) \to \gamma^*\Omega^1_{X/Y}(\log \Delta_{2,1}) \to 0,$$

with the outer terms ample, so $\gamma^*\Omega^1_X(\log \Delta_{2,1})$ is ample. Therefore, $\Omega^1_X(\log \Delta_{2,1})$ is quasi-ample with respect to $X - \Delta_{2,1} - f^{-1}(B)$. □

We now prove the general case.

THEOREM 2.16. *Let*

$$X^n \xrightarrow{f_n} X^{n-1} \xrightarrow{f_{n-1}} X^{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_3} X^2 \xrightarrow{f_2} X^1,$$

where each X^i is a smooth projective variety over $\mathbb{C}$ of dimension i. Suppose that, for $2 \le i \le n$:

(i) *$f_i : X^i \to X^{i-1}$ is a flat projective morphism with* $\mathrm{Var}(f_i) = \dim X^{i-1}$; *and*
(ii) *$\omega_{X^i_y}$ is ample for all $y \notin \Delta_{i-1,i} := \{y \in X^{i-1} \mid f_i^{-1}(y)$ is singular$\}$.*

Then, if Δ_i and $\Delta_{i+1,i}$ (as defined previously) are normal crossing divisors, the sheaf $\Omega^1_{X^n}(\log \Delta_{n,n-1})$ is quasi-ample with respect to $X^n - \Delta_{n,n-1} - f_n^{-1}(B_{n-1})$.

Proof. We will prove this by induction on n. By Theorem 2.15, the theorem is true for $n = 2$. Suppose it is true for $n - 1$. Let $X := X^n$; then we have the short exact sequence

$$0 \to f_n^*\Omega^1_{X^{n-1}}(\log \Delta_{n-1}) \to \Omega^1_X(\log \Delta_{n,n-1}) \to \Omega^1_{X/X^{n-1}}(\log \Delta_{n,n-1}) \to 0.$$

Let $\gamma\colon C \to X$ be a nonconstant morphism from a complete nonsingular curve C such that $\gamma(C) \cap (X - \Delta_{n,n-1} - f_n^{-1}(B_{n-1})) \neq \emptyset$. Suppose first that $\gamma(C)$ is contained in a fiber of f_n, say $\gamma(C) \subseteq X_y$ for some $y \in X^{n-1}$. We must show that $\Omega_X(\log \Delta_{n,n-1})|_{X_y}$ is ample. Note that $y \notin B_{n-1} \cup \Delta_{n-1}$ and, in particular, that $X_y \cap \Delta_{n,n-1} = \emptyset$.

Observe that $\Delta_{i,1} \subseteq \Delta_{i,i-1}$ for all i, since if $t \in \Delta_{1,i}$, then $f_i^{-1} \cdots f_2^{-1}(t)$ is singular and hence there exists a $y \in X_t^j$ for some $1 \le j \le i-1$ such that X_y^{j+1} is singular (i.e., $y \in \Delta_{j,j+1}$). Thus, $\Delta_{i,1} \subseteq f_i^*\Delta_{i-1} = \Delta_{i,i-1}$. So if we define $s := f_2 f_3 \cdots f_{n-1}(y) \in X^1$, then $s \notin \Delta_{i,1}$ for all $i \le n$ and so each X_s^i is a smooth projective variety over $\mathbb{C}$ of dimension $i-1$. Let $h = f_2 f_3 \cdots f_{n-1} f_n\colon X = X^n \to X^1$. Then we have the short exact sequence

$$0 \to T_{X/X^1}(-\log \Delta_{n,n-1}) \to T_X(-\log \Delta_{n,n-1}) \to h^*T_{X^1}(-\log \Delta_{1,n}) \to 0.$$

Restricting to X_s gives

$$0 \to T_{X_s}(-\log(\Delta_{n,n-1}|_{X_s})) \to T_X(-\log \Delta_{n,n-1})|_{X_s} \to h^*T_{X^1}|_{X_s} \to 0,$$

and restricting further to X_y gives

$$0 \to T_{X_s}|_{X_y} \to T_X|_{X_y} \to h^*T_{X^1}|_{X_y} \to 0.$$

As in the smooth case of Theorem 2.11, we find that

$$0 \to h^*\Omega_{X^1}|_{X_y} \to \Omega_X|_{X_y} \to \Omega_{X_s}|_{X_y} \to 0$$

does not split. Thus,

$$0 \to \mathscr{O}_{X_y} \to \Omega^1_X(\log \Delta_{n,n-1})|_{X_y} \to \Omega^1_{X_s}(\log(\Delta_{n,n-1}|_{X_s}))|_{X_y} \to 0$$

does not split. To show that $\Omega^1_X(\log \Delta_{n,n-1})|_{X_y}$ is ample, it suffices to show that $\Omega^1_{X_s}(\log(\Delta_{n,n-1}|_{X_s})|_{X_y}$ is ample.

We have

$$X_s = X_s^n \xrightarrow{(f_n)_s} X_s^{n-1} \xrightarrow{(f_{n-1})_s} \cdots \longrightarrow X_s^3 \xrightarrow{(f_3)_s} X_s^2,$$

where each X_s^i is a smooth projective variety over $\mathbb{C}$ of dimension $i-1$ and where each $(f_i)_s\colon X_s^i \to X_s^{i-1}$ is a flat projective morphism with the property that $\mathrm{Var}(f_i)_s = \dim(X^{i-1})_s$. Furthermore, $\omega_{(X_s^i)_y}$ is ample for all $y \notin \Delta_{i-1,i}|_{X_s^{i-1}}$.

For $2 \le i \le n-1$, define

$$B'_{i,s} := \{p \in X_s^i - \Delta_{i,i+1}|_{X_s^i} \mid \alpha_{i,s}\colon H^0(X_p^{i+1}, (f_{i+1})_s^* T_{X_s^i}|_{X_p^{i+1}}) \to H^1((X_s^{i+1})_p, T_{(X_s^{i+1})_p}) \text{ is not injective}\}.$$

As seen in Theorem 2.11, $B'_{i,s} \subseteq B'_i \cap X_s^i$. Set $B_{2,s} := B'_{2,s}$, and for $3 \le i \le n$ set

$$B_{i,s} := B'_{i,s} + (f_i)_s^{-1}(B_{i-1,s}).$$

Then, by induction, $\Omega^1_{X_s}(\log(\Delta_{n,n-1}|_{X_s}))$ is quasi-ample with respect to

$$X_s - (f_n)_s^{-1}(B_{n-1,s}) - \Delta_{n,n-1}|_{X_s}.$$

I claim that $(f_i)_s^{-1}(B_{i-1,s}) \subseteq (f_i)_s^{-1}(B_{i-1} \cap X_s^{i-1})$; indeed for $i = 2$ this is true because $B_{2,s} = B'_{2,s}$, so

$$\begin{aligned}(f_i)_s^{-1}(B_{i-1,s}) &= (f_i)_s^{-1}(B'_{i-1,s} + (f_{i-1})_s^{-1}(B_{i-2,s}))\\ &\subseteq (f_i)_s^{-1}(B'_{i-1} \cap X_s^{i-1} + f_{i-1}^{-1}(B_{i-2} \cap X_s^{i-2}))\\ &= (f_i)_s^{-1}(B_{i-1} \cap X_s^{i-1}).\end{aligned}$$

Thus, since

$$\begin{aligned}\emptyset \neq X_y &\cap (X - \Delta_{n,n-1} - f_n^{-1}(B_{n-1}))\\ &= X_y \cap (X_s - \Delta_{n,n-1}|_{X_s} - f_n^{-1}(B_{n-1} \cap X_s^{i-1}))\\ &\subseteq X_y \cap (X_s - (f_n)_s^{-1}(B_{n-1,s}) - \Delta_{n,n-1}|_{X_s}),\end{aligned}$$

it follows that $\Omega^1_{X_s}(\log(\Delta_{n,n-1}|_{X_s})|_{X_y}$ is ample.

Now suppose $\gamma(C)$ is not in a fiber of f_n so that $f_n\gamma\colon C \to X^{n-1}$ is nonconstant. By induction, we have that $\Omega^1_{X^{n-1}}(\log \Delta_{n-1,n-2})$ is quasi-ample with respect to $X^{n-1} - \Delta_{n-1,n-2} - f_{n-1}^{-1}(B_{n-2})$. Then, since $\Delta_{n-1} := (\Delta_{n-1,n-2} + \Delta_{n-1,n})_{\mathrm{red}}$, it follows from Lemma 2.13 that $\Omega^1_{X^{n-1}}(\log \Delta_{n-1})$ is quasi-ample with respect to $X^{n-1} - \Delta_{n-1} - f_{n-1}^{-1}(B_{n-2})$.

Therefore, we have the short exact sequence

$$\begin{aligned}0 \to \gamma^* f_n^* \Omega^1_{X^{n-1}}(\log \Delta_{n-1}) &\to \gamma^* \Omega^1_X(\log \Delta_{n,n-1})\\ &\to \gamma^* \Omega^1_{X/X^{n-1}}(\log \Delta_{n,n-1}) \to 0\end{aligned}$$

with $\gamma^* f_n^* \Omega^1_{X^{n-1}}(\log \Delta_{n-1})$ ample, so it suffices to show $\gamma^* \Omega^1_{X/X^{n-1}}(\log \Delta_{n,n-1})$ is ample.

As in Theorem 2.15, we have a smooth family $X - \Delta_{n,n-1} - f_n^{-1}B_{n-1} \to Y - \Delta_{n-1} - B_{n-1}$ that satisfies the conditions of Theorem 2.3(ii); thus, for m sufficiently large and divisible, $f_* \Omega^1_{X/Y}(\log \Delta_{n,n-1} + f^{-1}B)^m$ is ample with respect to $Y - \Delta_{n-1} - B_{n-1}$. Furthermore, for m sufficiently large,

$$f^* f_* \Omega^1_{X/X^{n-1}}(\log \Delta_{n,n-1} + f^{-1}B_{n-1})^m \to \Omega^1_{X/X^{n-1}}(\log \Delta_{n-1} + f^{-1}B_{n-1})^m$$

is surjective over $X - \Delta_{n-1,n} - f^{-1}B_{n-1}$. Since $\gamma(C) \cap (X - \Delta_{n-1,n} - f^{-1}B_{n-1}) \neq \emptyset$, we find that $\gamma^* \Omega^1_{X/X^{n-1}}(\log \Delta_{n-1} + f^{-1}B_{n-1})$ is an ample line bundle on C. Because $f^{-1}B_{n-1}$ is reduced, it follows that $\gamma^* \Omega^1_{X/X^{n-1}}(\log \Delta_{n-1} + f^{-1}B_{n-1}) \simeq \gamma^* \Omega^1_{X/X^{n-1}}(\log \Delta_{n-1})$. Therefore, $\Omega^1_{X^n}(\log \Delta_{n,n-1})$ is quasi-ample with respect to $X^n - \Delta_{n,n-1} - f_n^{-1}(B_{n-1})$. □

3. Constructing Towers of Smooth Projective Varieties

In this section we construct a tower of smooth projective varieties over the complex numbers as well as smooth morphisms between them of maximal variation. We first recall a construction of Kodaira that, for any n, produces a g for which $\mathscr{M}_g$ contains a complete n-dimensional subvariety (see [7; 16]).

LEMMA 3.1 [16]. *Let K be a field, D a curve over K of genus $g(D) \geq 2$, and $Q \in D(K)$. Suppose* $\mathrm{char}(K) \neq 2, 3$. *Then there exists a finite extension $K \subset L$ and a covering $C \to D$ of degree 3, defined over L, that is totally ramified in $C \ni P \mapsto Q \in D$ and unramified elsewhere.*

By keeping D fixed and letting the point Q vary, we construct a family of nonsingular projective curves parameterized by a covering D' of D whose image is a complete curve in $\mathscr{M}_{g(C)}$, where $g(C) = 3g(D) - 1$. Iterating this construction—that is, considering all covers of degree 3 of curves in the family $\{C_\lambda\}$ ramified at one point—we obtain a complete 2-dimensional family of curves of genus $9g(D) - 4$. In general, we obtain a complete n-dimensional family of curves of genus $3^n g(D) - (3^n - 1)/2$.

We now work over $\mathbb{C}$ and fix a curve C_0 of genus 2. Consider $\{C_\lambda\}$ the set of degree-3 covers of C_0 ramified in one point. These covers are parameterized by some curve B_0, a cover of C_0. Thus we get a map $g' \colon \{C_\lambda\} \to B_0$ with fibers C_λ for $\lambda \in B_0$. Now $\{[C_\lambda]\}$, the image of $\{C_\lambda\}$ in $\mathscr{M}_5$, is a complete curve; hence g' must be nonisotrivial.

For each $\lambda \in B_0$, we can iterate this construction. For a fixed $\lambda_0 \in B_0$, consider $\{C_{\lambda_0,\mu}\}$, the set of degree-3 covers of C_{λ_0} ramified in one point and parameterized by some curve B_{λ_0}. For this fixed λ_0, $\{[C_{\lambda_0,\mu}]\}$ is a complete curve in $\mathscr{M}_{14}$, so $f_{\lambda_0} \colon \{C_{\lambda_0,\mu}\} \to B_{\lambda_0}$ is nonisotrivial. Letting λ vary, we obtain a smooth projective morphism $f \colon X \to Y$, where $X = \{C_{\lambda,\mu}\}$ and $Y = \{B_\lambda\}$ are projective varieties of dimension 3 and 2, respectively. I claim that for any $p \in Y$ the set $\{q \in Y \mid X_p \simeq X_q\}$ is finite, so in particular $\mathrm{Var}\, f = 2$. Indeed, if $p, q \in B_{\lambda_0}$ for some λ_0 then, since f_{λ_0} is nonisotrivial, X_p is not isomorphic to X_q. Next suppose that F_1 is any fiber of f with $F_1 \in C_{\lambda_1,\mu}$; then, in particular, F_1 is a covering of C_{λ_1}. But F_1 can cover only finitely many curves, so F_1 can be isomorphic to at most finitely many other fibers $F_i \in C_{\lambda_i,\mu}$. Thus any fiber of f is isomorphic to finitely many other fibers and hence, for any $p \in Y$, the set $\{q \in Y \mid X_p \simeq X_q\}$ is finite.

Let $g \colon Y \to B_0$ be the composition $Y = \{B_\lambda\} \to \{C_\lambda\} \to B_0$. If $B_\lambda \simeq B_{\lambda'}$ for general $\lambda, \lambda' \in B_0$, then either $C_\lambda \simeq C_{\lambda'}$ for general $\lambda, \lambda' \in B_0$ or B_λ covers infinitely many nonisomorphic curves, which leads to a contradiction in either case. Hence $g \colon Y \to B_0$ is nonisotrivial. Thus we have the tower

$$X \xrightarrow{f} Y \xrightarrow{g} B_0$$

of smooth projective varieties with $\dim X = 3$, $\dim Y = 2$, $\dim B_0 = 1$, and smooth morphisms of maximal variation. One can iterate this construction to get a tower

$$X^n \xrightarrow{f_n} X^{n-1} \xrightarrow{f_{n-1}} X^{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_3} X^2 \xrightarrow{f_2} X^1,$$

where each X^i is a smooth projective variety over $\mathbb{C}$ of dimension i and where, for $2 \leq i \leq n$, $f_i \colon X^i \to X^{i-1}$ is a smooth projective morphism satisfying the property that, for all $y \in X^{i-1}$, the set $\{p \in Y \mid X^i_p \simeq X^i_y\}$ is finite and so, in particular, $\mathrm{Var}(f_i) = \dim X^{i-1}$.

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