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Wonderful Compactification of an Arrangement of Subvarieties

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1. Introduction

The purpose of this paper is to define the so-called wonderful compactification of an arrangement of subvarieties, to prove its expected properties, to give a construction by a sequence of blow-ups, and to discuss the order in which the blow-ups can be carried out.

Fix a nonsingular algebraic variety Y over an algebraically closed field (of arbitrary characteristic). An *arrangement* of subvarieties $\mathcal{S}$ is a finite collection of nonsingular subvarieties such that all nonempty scheme-theoretic intersections of subvarieties in $\mathcal{S}$ are again in $\mathcal{S}$ or, equivalently, such that any two subvarieties intersect cleanly and the intersection is either empty or a subvariety in this collection (see Definition 2.1).

Let $\mathcal{S}$ be an arrangement of subvarieties of Y. A subset $\mathcal{G} \subseteq \mathcal{S}$ is called a *building set of* $\mathcal{S}$ if, for all $S \in \mathcal{S} \setminus \mathcal{G}$, the minimal elements in $\{G \in \mathcal{G} : G \supseteq S\}$ intersect transversally and the intersection is S. A set of subvarieties $\mathcal{G}$ is called a building set if all the possible intersections of subvarieties in $\mathcal{G}$ form an arrangement $\mathcal{S}$ (called the *induced arrangement* of $\mathcal{G}$) and $\mathcal{G}$ is a building set of $\mathcal{S}$ (see Definition 2.2).

For any building set $\mathcal{G}$, the wonderful compactification of $\mathcal{G}$ is defined as follows.

Definition 1.1. Let $\mathcal{G}$ be a nonempty building set and $Y^\circ = Y \setminus \bigcup_{G \in \mathcal{G}} G$. The closure of the image of the natural locally closed embedding

$$Y^\circ \hookrightarrow \prod_{G \in \mathcal{G}} Bl_G Y$$

is called the *wonderful compactification* of the arrangement $\mathcal{G}$ and is denoted by $Y_{\mathcal{G}}$.

The following description of $Y_{\mathcal{G}}$ is the main theorem and is proved at the end of Section 2.3. A *$\mathcal{G}$-nest* is a subset of the building set $\mathcal{G}$ satisfying some inductive condition (see Definition 2.3).

Theorem 1.2. *Let Y be a nonsingular variety and let $\mathcal{G}$ be a nonempty building set of subvarieties of Y. Then the wonderful compactification $Y_{\mathcal{G}}$ is a nonsingular variety. Moreover, for each $G \in \mathcal{G}$ there is a nonsingular divisor $D_G \subset Y_{\mathcal{G}}$ such that*:

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(i) *the union of these divisors is* $Y_{\mathcal{G}} \setminus Y^\circ$;
(ii) *any set of these divisors meets transversally. An intersection of divisors* $D_{T_1} \cap \cdots \cap D_{T_r}$ *is nonempty exactly when* $\{T_1, \ldots, T_r\}$ *form a* $\mathcal{G}$*-nest.*

This theorem is proved by a construction of $Y_{\mathcal{G}}$ through an explicit sequence of blow-ups of Y along nonsingular centers (see Definition 2.12 and Theorem 2.13).

Here are some examples of wonderful compactifications of an arrangement (see Section 4 for details).

(1) De Concini–Procesi's wonderful model of subspace arrangements (Section 4.1). In this case, Y is a vector space, $\mathcal{S}$ is a finite set of proper subspaces of Y, and $\mathcal{G}$ is a building set with respect to $\mathcal{S}$.
(2) Suppose X is a nonsingular algebraic variety, n is a positive integer, and Y is the Cartesian product X^n. A diagonal of X^n is

$$\Delta_I = \{(p_1, \ldots, p_n) \in X^n \mid p_i = p_j \ \forall i, j \in I\}$$

for $I \subseteq [n]$, $|I| \geq 2$. A polydiagonal is an intersection of diagonals

$$\Delta_{I_1} \cap \cdots \cap \Delta_{I_k}$$

for $I_i \subseteq [n]$, $|I_i| \geq 2$ $(1 \leq i \leq k)$.
   (a) The Fulton–MacPherson configuration space $X[n]$ (Section 4.2). This is the wonderful compactification $Y_{\mathcal{G}}$ where $\mathcal{G}$ is the set of all diagonals in Y and the induced arrangement $\mathcal{S}$ is the set of all polydiagonals. It is a special example of Kuperberg–Thurston's compactification X^Γ when Γ is the complete graph with n vertices.
   (b) Ulyanov's polydiagonal compactification $X\langle n\rangle$ (Section 4.5). It is the wonderful compactification $Y_{\mathcal{G}}$ where $\mathcal{S} = \mathcal{G}$ are the set of all polydiagonals.
   (c) Kuperberg–Thurston's compactification X^Γ when Γ is a connected graph with n labeled vertices (Section 4.3). Here X^Γ is the wonderful compactification $Y_{\mathcal{G}}$, where $\mathcal{G}$ is the set of diagonals in Y corresponding to vertex-2-connected subgraphs of Γ and where $\mathcal{S}$ is the set of polydiagonals generated by intersections of diagonals in $\mathcal{G}$.
(3) The moduli space of rational curves with n marked points $\overline{M}_{0,n}$ (Section 4.4). It is the wonderful compactification $Y_{\mathcal{G}}$, where $Y = (\mathbb{P}^1)^{n-3}$ and $\mathcal{G}$ is set of all diagonals and augmented diagonals $\Delta_{I,a}$ defined as

$$\Delta_{I,a} := \{(p_4, \ldots, p_n) \in (\mathbb{P}^1)^{n-3} \mid p_i = a \ \forall i \in I\}$$

for $I \subseteq \{4, \ldots, n\}$, $|I| \geq 2$, and $a \in \{0, 1, \infty\}$.

The moduli space $\overline{M}_{0,n}$ is also the wonderful compactification $Y_{\mathcal{G}}$ where $Y = \mathbb{P}^{n-3}$ and $\mathcal{G}$ is the set of all projective subspaces of $\mathbb{P}^{n-3}$ spanned by any subset of fixed $n-1$ generic points [Ka].
(4) Hu's compactification of open varieties (Section 4.6). It is a wonderful compactification of $(Y, \mathcal{S}, \mathcal{G})$, where Y is a nonsingular algebraic variety and $\mathcal{S} = \mathcal{G}$ is an arrangement of subvarieties of Y [Hu].

During the study of the sequence of blow-ups, a natural question arises: In which order can we carry out the blow-ups to obtain the wonderful compactification? For example, neither the original construction of the Fulton–MacPherson configuration space $X[n]$ nor Keel's construction of $\overline{M}_{0,n}$ nor Kapranov's construction of $\overline{M}_{0,n}$ is obtained by blowing up along the centers with increasing dimensions. If we change the order of blow-ups, do we still get the same variety?

We answer this question with the following theorem, which is proved in Section 3. The notation $\widetilde{G}$ stands for the *dominant tranform* of G (see Definition 2.7), which is similar to but slightly different from the strict tranform: for a subvariety G contained in the center of a blow-up, the strict transform of G is empty but the dominant transform $\widetilde{G}$ is the preimage of G.

THEOREM 1.3. *Let Y be a nonsingular variety and let $\mathcal{G} = \{G_1, \dots, G_N\}$ be a nonempty building set of subvarieties of Y. Let $\mathcal{I}_i$ be the ideal sheaf of $G_i \in \mathcal{G}$.*

(i) *The wonderful compactification $Y_{\mathcal{G}}$ is isomorphic to the blow-up of Y along the ideal sheaf $\mathcal{I}_1\mathcal{I}_2 \cdots \mathcal{I}_N$.*

(ii) *If we arrange $\mathcal{G} = \{G_1, \dots, G_N\}$ in such an order that*

$$\text{the first } i \text{ terms } G_1, \dots, G_i \text{ form a building set for any } 1 \le i \le N, \qquad (*)$$

then

$$Y_{\mathcal{G}} = Bl_{\widetilde{G}_N} \cdots Bl_{\widetilde{G}_2} Bl_{G_1} Y,$$

where each blow-up is along a nonsingular subvariety.

EXAMPLE. By Keel's construction [Ke1] and the preceding theorem, $\overline{M}_{0,n}$ is isomorphic to the wonderful compactification $Y_{\mathcal{G}}$ when Y is $(\mathbb{P}^1)^{n-3}$ and $\mathcal{G}$ is set of all diagonals and augmented diagonals. In other words, we can blow up along the centers in any order satisfying $(*)$ (e.g., of increasing dimension). As a consequence, we have the following corollary.

COROLLARY 1.4. *Let $\psi: \mathbb{P}^1[n] \to (\mathbb{P}^1)^3$ be the composition of the natural morphism $\mathbb{P}^1[n] \to (\mathbb{P}^1)^n$ and let $\pi_{123}: (\mathbb{P}^1)^n \to (\mathbb{P}^1)^3$ be the projection to the first three components. Then $\overline{M}_{0,n}$ is isomorphic to the fiber of ψ over the point $(0, 1, \infty) \in (\mathbb{P}^1)^3$. Equivalently, $\overline{M}_{0,n}$ is isomorphic to the fiber over any point (p_1, p_2, p_3), where p_1, p_2, p_3 are three distinct points in $\mathbb{P}^1$.*

Similarly, Kapranov's construction does not delicately depend on the order of the blow-ups; for example, we can blow up along the centers in any order of increasing dimension.

This article is built on the following previous works: Fulton and MacPherson [FM], De Concini and Procesi [DP], MacPherson and Procesi [MP], Ulyanov [U], and Hu [Hu].

The inspiring paper by De Concini and Procesi [DP] gives a thorough discussion of an arrangement of linear subspaces of a vector space. Given a vector space Y and an arrangement of subspaces $\mathcal{S}$, De Concini and Procesi give a condition for a subset $\mathcal{G} \subseteq \mathcal{S}$ such that there exists a *wonderful model* $Y_{\mathcal{G}}$ of the arrangement in which the elements in $\mathcal{G}$ are replaced by simple normal crossing divisors. De Concini and Procesi call $\mathcal{G}$ a *building set*. Their paper also gives a criterion of whether the intersection of a collection of such divisors is nonempty by introducing the notion of a *nest*.

This idea was later generalized by MacPherson and Procesi to nonsingular varieties over $\mathbb{C}$ with conical stratifications. They consider conical stratifications in place of the subspace arrangments in [DP]. The notions of *building set* and *nest* are generalized in this setting. The idea of the construction of wonderful compactifications of arrangement of subvarieties in our paper is largely inspired by the beautiful paper [MP]. In our paper we give definitions of arrangements of subvarieties, building sets, and nests. The wonderful compactifications are shown to have properties analogous to those in [DP] and [MP].

The paper is organized as follows. In Section 2 we give the construction of the wonderful compactification $Y_{\mathcal{G}}$. We begin by defining arrangements, building sets, and nests and then describe how they vary under one blow-up; we finish by giving the actual construction of $Y_{\mathcal{G}}$. In Section 3 we discuss the order in which the blow-ups could be carried out to obtain $Y_{\mathcal{G}}$. Section 4 gives some examples of wonderful compactifications.

Section 5 comprises the Appendix. In Section 5.1 we discuss clean intersections and transversal intersections, and in Section 5.2 we give the proofs of previous statements. In Section 5.3 we discuss how different choices of blow-ups change the codimension of the centers. Finally, in Section 5.4 we give the statements for a general (nonsimple) arrangement (proofs omitted).

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2. Arrangements of Subvarieties and the Wonderful Compactifications

By a *variety* we shall mean a reduced and irreducible algebraic scheme defined over a fixed algebraically closed field (of arbitrary characteristic). A *subvariety* of a variety is a closed subscheme that is a variety. By a *point* of a variety we shall mean a closed point of that variety. By the *intersection* of subvarieties $Z_1, \dots, Z_k$ we shall mean the set-theoretic intersection (denoted by $Z_1 \cap \cdots \cap Z_k$). We denote the ideal sheaf of a subvariety V of a variety Y by $\mathcal{I}_V$.

In this section we discuss the arrangements, building sets, and nests upon which is based our definition of the wonderful compactifications of an arrangement. The idea is inspired by [DP] and [MP].

2.1. Arrangement, Building Set, Nest

The following definition of arrangement is adapted from [Hu]. For a brief review of the definitions of clean intersection and transversal intersection, see Section 5.1.

DEFINITION 2.1. A simple *arrangement* of subvarieties of a nonsingular variety Y is a finite set $\mathcal{S} = \{S_i\}$ of nonsingular closed subvarieties S_i properly contained in Y and satisfying the following conditions:

(i) S_i and S_j intersect cleanly (i.e., their intersection is nonsingular and the tangent bundles satisfy $T(S_i \cap S_j) = T(S_i)|_{(S_i \cap S_j)} \cap T(S_j)|_{(S_i \cap S_j)}$);
(ii) $S_i \cap S_j$ is either equal to some S_k or is empty.

This definition is equivalent to stating that $\mathcal{S}$ is an arrangement if and only if it is closed under scheme-theoretic intersections (cf. Lemma 5.1).

For the sake of clarity we discuss only the simple arrangement, but most statements still hold (with minor revision) for general arrangements. For example, instead of condition (ii) we may allow $S_i \cap S_j$ to be a disjoint union of some S_k (see Section 5.4).

For a simple arrangement, the condition of transversality can be checked at one point (instead of at every point) of the intersection (Lemma 5.2).

DEFINITION 2.2. Let $\mathcal{S}$ be an arrangement of subvarieties of Y. A subset $\mathcal{G} \subseteq \mathcal{S}$ is called a *building set* of $\mathcal{S}$ if, for all $S \in \mathcal{S}$, the minimal elements in $\{G \in \mathcal{G} : G \supseteq S\}$ intersect transversally and their intersection is S (by our definition of transversality in Section 5.1, the condition is satisfied if $S \in \mathcal{G}$). In this case, these minimal elements are called the *$\mathcal{G}$-factors* of S.

A finite set $\mathcal{G}$ of nonsingular subvarieties of Y is called a building set if the set of all possible intersections of collections of subvarieties from $\mathcal{G}$ forms an arrangement $\mathcal{S}$ and if $\mathcal{G}$ is a building set of $\mathcal{S}$. In this situation, $\mathcal{S}$ is called the *arrangement induced by $\mathcal{G}$*.

EXAMPLE. Let X be a nonsingular variety of positive dimension and let Y be the Cartesian product X^3.

(1) The set $\mathcal{G} = \{\Delta_{12}, \Delta_{13}, \Delta_{23}, \Delta_{123}\}$ is a building set whose induced arrangement is $\mathcal{G}$ itself.
(2) The set $\mathcal{G} = \{\Delta_{12}, \Delta_{13}\}$ is a building set whose induced arrangement is $\{\Delta_{12}, \Delta_{13}, \Delta_{123}\}$. On the other hand, $\mathcal{G}$ is not a building set of the arrangement $\{\Delta_{12}, \Delta_{13}, \Delta_{23}, \Delta_{123}\}$.
(3) The set $\mathcal{G} = \{\Delta_{12}, \Delta_{13}, \Delta_{23}\}$ is not a building set, because the set of all possible intersections from $\mathcal{G}$ is $\{\Delta_{12}, \Delta_{13}, \Delta_{23}, \Delta_{123}\}$ yet Δ_{123} is not a transversal intersection of Δ_{12}, Δ_{13}, and Δ_{23}.

REMARK. The building set $\mathcal{G}$ defined here is related to the one defined in [DP] as follows. For any point $y \in Y$, define $\mathcal{S}_y^* = \{T_{S,y}^{\perp}\}_{S \in \mathcal{S}}$ and $\mathcal{G}_y^* = \{T_{S,y}^{\perp}\}_{S \in \mathcal{G}}$. We claim that the set $\mathcal{G}$ is a building set if and only if $\mathcal{G}_y^*$ is a building set for all $y \in Y$ in the sense of De Concini and Procesi.

Indeed, $\mathcal{S}$ being an arrangement is equivalent to the condition that, for any $y \in Y$, $\mathcal{S}_y^*$ is a finite set of nonzero linear subspaces of T_y^* that is closed under sum and such that each element of $\mathcal{S}_y^*$ is equal to $T_{S,y}^{\perp}$ for a unique $S \in \mathcal{S}$. The subset $\mathcal{G} \subseteq \mathcal{S}$ being a building set is equivalent to the following condition: for all $S \in \mathcal{S}$ and for all $y \in S$, suppose $T_1^{\perp}, \dots, T_k^{\perp}$ are all the maximal elements of $\mathcal{G}_y^*$ contained in $T_{S,y}^{\perp}$; then they form a direct sum and

$$T_1^{\perp} \oplus T_2^{\perp} \oplus \cdots \oplus T_k^{\perp} = T_{S,y}^{\perp},$$

which is exactly the definition of building set in [DP, Sec. 2.3, Thm. (2)].

DEFINITION 2.3 (cf. [MP, Sec. 4]). A subset $\mathcal{T} \subseteq \mathcal{G}$ is called *$\mathcal{G}$-nested* (or a *$\mathcal{G}$-nest*) if it satisfies one of the following equivalent conditions.

(i) There is a flag of elements in $\mathcal{S}$: $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_\ell$ such that

$$\mathcal{T} = \bigcup_{i=1}^{\ell} \{A : A \text{ is a } \mathcal{G}\text{-factor of } S_i\}.$$

(We say $\mathcal{T}$ is *induced* by the flag $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_\ell$.)

(ii) Let $A_1, \dots, A_k$ be the minimal elements of $\mathcal{T}$; then they are all the $\mathcal{G}$-factors of a certain element in $\mathcal{S}$. For any $1 \le i \le k$, the set $\{A \in \mathcal{T} : A \supsetneq A_i\}$ is also $\mathcal{G}$-nested as defined by induction.

EXAMPLE. Let X be a nonsingular variety of positive dimension and let Y be the Cartesian product X^4. Take the building set $\mathcal{G}$ to be the set of all diagonals in X^4.

(1) The set $\mathcal{T} = \{\Delta_{12}, \Delta_{123}\}$ is a $\mathcal{G}$-nest, since it can be induced by the flag $\Delta_{123} \subseteq \Delta_{12}$.

(2) The set $\mathcal{T} = \{\Delta_{12}, \Delta_{34}, \Delta_{1234}\}$ is a $\mathcal{G}$-nest, since it can be induced by the flag $\Delta_{1234} \subseteq (\Delta_{12} \cap \Delta_{34})$.

(3) The set $\mathcal{T} = \{\Delta_{12}, \Delta_{13}\}$ is not a $\mathcal{G}$-nest. Indeed, the intersection of the minimal elements in $\mathcal{T}$ is Δ_{123}, which has only one $\mathcal{G}$-factor: Δ_{123} itself. By condition (ii) of the definition, $\mathcal{T}$ is not a $\mathcal{G}$-nest.

Note that the intersection of elements in a $\mathcal{G}$-nest $\mathcal{T}$ is nonempty by (ii). Now we explain why the two conditions (i) and (ii) are equivalent. Given a set $\mathcal{T}$ satisfying (ii), we can construct a flag as follows. Define $S_1 = A_1 \cap \cdots \cap A_k$, which is the intersection of all subvarieties in $\mathcal{T}$. Let S_2 be the intersection of the subvarieties in $\mathcal{T}$ that are not minimal elements in $\mathcal{T}$ containing S_1. Then inductively let S_{j+1} be the intersection of those that are not minimal elements in $\mathcal{T}$ containing S_j. It is easy to show that $\mathcal{T}$ is induced by the flag $S_1 \subseteq S_2 \subseteq \cdots$, so (ii) $\Rightarrow$ (i). Conversely, let $S_{1j} = A_j$ $(1 \le j \le k)$ be the $\mathcal{G}$-factors of S_1. Observe that, for any $1 \le i \le \ell$, a $\mathcal{G}$-factor of S_i must contain exactly one element of $A_1, \dots, A_k$; otherwise, the A_i would not intersect transversally. Let S_{ij} be the $\mathcal{G}$-factor of S_i that contains A_j. (Define $S_{ij} = Y$ if there is no such a $\mathcal{G}$-factor.) Then, for each $1 \le j \le k$, there is a flag of elements $S_{2j} \subseteq S_{3j} \subseteq \cdots \subseteq S_{\ell j}$, which induces the $\mathcal{G}$-nest $\{A \in \mathcal{T} : A \supsetneq A_i\}$. This shows that (i) $\Rightarrow$ (ii).

We now state some basic properties about arrangements and building sets.

LEMMA 2.4. *Let Y be a nonsingular variety and let $\mathcal{G}$ be a building set with the induced arrangement $\mathcal{S}$. Suppose $S \in \mathcal{S}$ and $G_1, \dots, G_k$ are all the $\mathcal{G}$-factors of S (Definition 2.2). Then the following statements hold.*

(i) *For any $1 \le m \le k$, the subvarieties $G_1, \dots, G_m$ are all the $\mathcal{G}$-factors of the subvariety $G_1 \cap \cdots \cap G_m$.*

(ii) *Suppose $F \in \mathcal{G}$ is minimal such that $F \cap S \neq \emptyset$, $F \subseteq G_1, \dots, G_m$, and $F \nsubseteq G_{m+1}, \dots, G_k$. Then $F, G_{m+1}, \dots, G_k$ are all the $\mathcal{G}$-factors of the subvariety $F \cap S$.*

Proof. See Section 5.2. $\square$

Here is an immediate consequence of Lemma 2.4.

LEMMA 2.5. *If $G_1, \dots, G_k \in \mathcal{G}$ are all minimal and their intersection S is nonempty, then $G_1, \dots, G_k$ are all the $\mathcal{G}$-factors of S.*

Next we introduce the notion of F-factorization, which turns out to be a convenient terminology for the proof of the construction of wonderful compactifications.

LEMMA 2.6. *Suppose $F \in \mathcal{G}$ is minimal.*

(i) *Any $G \in \mathcal{G}$ either contains F or intersects transversally with F.*

(ii) *Every $S \in \mathcal{S}$ satisfying $S \cap F \neq \emptyset$ can be uniquely expressed as $A \cap B$, where $A, B \in \mathcal{S} \cup \{Y\}$ satisfy $A \supseteq F$ and $B \pitchfork F$ (hence $A \pitchfork B$). We call this expression $S = A \cap B$ the F-*factorization *of S.*

(iii) *Suppose the $\mathcal{G}$-factors of S are $G_1, \dots, G_k$, where $G_1, \dots, G_m$ contain F ($0 \le m \le k$; the case $m = 0$ is understood to mean that no $\mathcal{G}$-factors of S contain F). Let the F-factorization of S be $A \cap B$. Then $G_1, \dots, G_m$ are all the $\mathcal{G}$-factors of A and $G_{m+1}, \dots, G_k$ are all the $\mathcal{G}$-factors of B, so $A = \bigcap_{i=1}^{m} G_i$ and $B = \bigcap_{i=m+1}^{k} G_i$. (Here we assume that $A = Y$ if $m = 0$ and that $B = Y$ if $m = k$.)*

(iv) *Suppose $S' \in \mathcal{S}$ such that $S' \cap S \cap F \neq \emptyset$. Let $S' = A' \cap B'$ be the F-factorization of S'. Then $F \pitchfork (B \cap B')$ and therefore the F-factorization of $S \cap S'$ is $(A \cap A') \cap (B \cap B')$.*

Proof. See Section 5.2. $\square$

2.2. *Change of an Arrangement after a Blow-up*

Before considering a sequence of blow-ups, we first consider a single blow-up. Let Y be a nonsingular variety and let $\mathcal{G}$ be a building set with the induced arrangement $\mathcal{S}$. In Proposition 2.8 we show that, if $F \in \mathcal{G}$ is minimal, then there exists a natural arrangement $\widetilde{\mathcal{S}}$ in $Bl_F Y$ induced from $\mathcal{S}$ as well as a natural building set $\widetilde{\mathcal{G}}$ induced from $\mathcal{G}$.

DEFINITION 2.7. Let Z be a nonsingular subvariety of a nonsingular variety Y and let $\pi : Bl_Z Y \to Y$ be the blow-up of Y along Z.

For any irreducible subvariety V of Y, we define the *dominant transform* of V, denoted by $\widetilde{V}$ or $V^{\sim}$, to be the strict transform of V if $V \nsubseteq G$ and to be the scheme-theoretic inverse $\pi^{-1}(V)$ if $V \subseteq G$.

For a sequence of blow-ups, we still denote the iterated dominant transform $(\cdots((V^{\sim})^{\sim})\cdots)^{\sim}$ by $\widetilde{V}$ or $V^{\sim}$.

REMARK. We introduce the notion of dominant transform because the strict transform does not behave as expected: the strict transform of a subvariety contained in the center of a blow-up is empty, which is not what we need.

PROPOSITION 2.8. *Let Y be a nonsingular variety and let $\mathcal{G}$ be a building set with the induced arrangement $\mathcal{S}$. Let F be a minimal element in $\mathcal{G}$ and let $\pi : Bl_F Y \to Y$ be the blow-up of Y along F. Denote the exceptional divisor by E.*

(i) *The collection $\widetilde{\mathcal{S}}$ of subvarieties in $Bl_G Y$ defined as*

$$\widetilde{\mathcal{S}} := \{\widetilde{S}\}_{S\in\mathcal{S}} \cup \{\widetilde{S} \cap E\}_{\emptyset \subsetneq S\cap F \subsetneq S}$$

is a (simple) arrangement of subvarieties in $Bl_G Y$.

(ii) *$\widetilde{\mathcal{G}} := \{\widetilde{G}\}_{G\in\mathcal{G}}$ is a building set of $\widetilde{\mathcal{S}}$.*

(iii) *Given a subset $\mathcal{T}$ of $\mathcal{G}$, we define $\widetilde{\mathcal{T}} := \{\widetilde{A}\}_{A\in\mathcal{T}}$. Then $\mathcal{T}$ is a $\mathcal{G}$-nest if and only if $\widetilde{\mathcal{T}}$ is a $\widetilde{\mathcal{G}}$-nest.*

The proof is in Section 5.2, and its main ingredient is the following lemma.

LEMMA 2.9. *Assume the same notation as in Proposition 2.8. Assume that A, A_1, A_2, B, B_1, B_2, and G are nonsingular subvarieties of Y.*

(i) *Suppose $A \supsetneq F$. Then $\widetilde{A} \cap E$ intersect transversally (hence cleanly).*

(ii) *Suppose that $A_1 \nsubseteq A_2$ and $A_2 \nsubseteq A_1$, and suppose that $A_1 \cap A_2 = F$ and the intersection is clean. Then $\widetilde{A}_1 \cap \widetilde{A}_2 = \emptyset$.*

(iii) *Suppose that A_1 and A_2 intersect cleanly and that $F \subsetneq A_1 \cap A_2$. Then $\widetilde{A}_1 \cap \widetilde{A}_2 = (A_1 \cap A_2)^{\sim}$. Moreover, $\widetilde{A}_1$ and $\widetilde{A}_2$ intersect cleanly.*

(iv) *Suppose that B_1 and B_2 intersect cleanly and that G is transversal to B_1, B_2, and $B_1 \cap B_2$. Then $\widetilde{B}_1 \cap \widetilde{B}_2 = (B_1 \cap B_2)^{\sim}$. Moreover, $\widetilde{B}_1$ and $\widetilde{B}_2$ intersect cleanly.*

(v) *Suppose $A \pitchfork B$, $F \subseteq A$, and $F \pitchfork B$. Then $\widetilde{A} \cap \widetilde{B} = (A \cap B)^{\sim}$. Moreover, $\widetilde{A} \pitchfork \widetilde{B}$ and $(E \cap \widetilde{A}) \pitchfork \widetilde{B}$.*

(vi) *Assume that $F \subseteq A$, $F \pitchfork B_1 \pitchfork B_2$, $G \subseteq F \cap B_1$, and $G \pitchfork B_2$. Then $\widetilde{G} \cap \widetilde{A} \cap (B_1 \cap B_2)^{\sim} = \widetilde{G} \cap \widetilde{A} \cap \widetilde{B}_2$, where the latter is a transversal intersection.*

Proof. See Section 5.2. □

2.3. A Sequence of Blow-ups and the Construction of Wonderful Compactifications

Now we study a sequence of blow-ups, give different descriptions of a wonderful compactification, and study the relations of the arrangements occurring in the sequence of blow-ups.

Given k morphisms between algebraic varieties with the same domain $f_i: X \to Y_i$, we adopt the notation $(f_1, f_2, \ldots, f_k): X \to Y_1 \times \cdots \times Y_k$ to signify the composition of the diagonal morphism $X \to X \times \cdots \times X$ with the morphism $f_1 \times \cdots \times f_k$.

LEMMA 2.10. *Let V and W be two nonsingular subvarieties of a nonsingular variety Y such that either V and W intersect transversally or one of V and W contains the other. Let $f: Y_1 \to Y$ (resp. $g: Y_2 \to Y$) be the blow-up of Y along W (resp. V). Let $g': Y_3 \to Y_1$ be the blow-up of Y_1 along the dominant transform $\widetilde{V}$. Then there exists a morphism $f': Y_3 \to Y_2$ such that the following diagram commutes.*

$$\begin{array}{ccc} Y_3 & \xrightarrow{f'} & Y_2 \\ \Big\downarrow{\scriptstyle g'} & & \Big\downarrow{\scriptstyle g} \\ Y_1 & \xrightarrow{f} & Y \end{array}$$

Moreover, $(g', f'): Y_3 \to Y_1 \times Y_2$ is a closed embedding.

Proof. Because of the universal property of blowing up [H, Prop. 7.14], in order to show the existence of f' we need only show that $(fg')^{-1}\mathcal{I}_V \cdot \mathcal{O}_{Y_3}$ is an invertible sheaf of ideals on Y_3. But this is true since, by our choice of V and W, the sheaf $f^{-1}\mathcal{I}_V \cdot \mathcal{O}_{Y_1}$ is either $\mathcal{I}_{\widetilde{V}}$ or $\mathcal{I}_{\widetilde{V}}\mathcal{I}_E$, where E is the exceptional divisor of the blow-up $f: Y_1 \to Y$. Hence the ideal sheaf

$$(fg')^{-1}\mathcal{I}_V \cdot \mathcal{O}_{Y_3} = g'^{-1}(f^{-1}\mathcal{I}_V \cdot \mathcal{O}_{Y_1}) \cdot \mathcal{O}_{Y_3}$$

is either $(g'^{-1}\mathcal{I}_{\widetilde{V}}) \cdot \mathcal{O}_{Y_3}$ or $g'^{-1}(\mathcal{I}_{\widetilde{V}} \cdot \mathcal{I}_E) \cdot \mathcal{O}_{Y_3}$, both of which are invertible by the construction of g'; therefore, the ideal sheaf $(fg')^{-1}\mathcal{I}_V \cdot \mathcal{O}_{Y_3}$ is invertible. That (g', f') is a closed embedding can be checked using local parameters. $\square$

LEMMA 2.11. *Suppose $X_1, X_2, X_3, Y_1, Y_2, Y_3$ are nonsingular varieties such that the following diagram commutes.*

$$\begin{array}{ccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 \\ \Big\downarrow{\scriptstyle g_1} & & \Big\downarrow{\scriptstyle g_2} & & \Big\downarrow{\scriptstyle g_3} \\ Y_1 & \xrightarrow{h_1} & Y_2 & \xrightarrow{h_2} & Y_3 \end{array}$$

If $(g_1, f_1): X_1 \to Y_1 \times X_2$ and $(g_2, f_2): X_2 \to Y_2 \times X_3$ are closed embeddings, then $(g_1, f_2 f_1): X_1 \to Y_1 \times X_3$ is also a closed embedding.

As a consequence, if we have the commutative diagram

$$\begin{array}{ccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{k-1}} & X_k \\ \Big\downarrow{\scriptstyle g_1} & & \Big\downarrow{\scriptstyle g_2} & & & & \Big\downarrow{\scriptstyle g_k} \\ Y_1 & \xrightarrow{h_1} & Y_2 & \xrightarrow{h_2} & \cdots & \xrightarrow{h_{k-1}} & Y_k \end{array}$$

and if $(g_i, f_i)\colon X_i \to Y_i \times X_{i+1}$ *are closed embeddings for all* $1 \le i \le k-1$, *then* $(g_1, f_{k-1}\cdots f_1)\colon X_1 \to Y_1 \times X_k$ *is also a closed embedding.*

Proof. The composition of two closed embeddings is still a closed embedding, so

$$\phi := (g_1, g_2 f_1, f_2 f_1)\colon X_1 \to Y_1 \times Y_2 \times X_3$$

is a closed embedding whose image $\phi(X_1)$ is a closed subvariety of $Y_1 \times Y_2 \times X_3$, which is isomorphic to X_1. Consider the projection $\pi_{13}\colon Y_1 \times Y_2 \times X_3 \to Y_1 \times X_3$ and the morphism $\Gamma_{h_1} \times 1_{X_3}\colon Y_1 \times X_3 \to Y_1 \times Y_2 \times X_3$.

$$\begin{tikzcd} X_1 \arrow[r, "\phi"] \arrow[dr, "(g_1, f_2 f_1)"'] & Y_1 \times Y_2 \times X_3 \arrow[d, shift right, "\pi_{13}"'] \\ & Y_1 \times X_3 \arrow[u, shift right, "\Gamma_{h_1} \times 1_{X_3}"'] \end{tikzcd}$$

Notice that $\pi_{13} \circ (\Gamma_{h_1} \times 1_{X_3})$ is the identity automorphism of $Y_1 \times X_3$ and that $(\Gamma_{h_1} \times 1_{X_3}) \circ \pi_{13}|_{\phi(X_1)}$ is the identity automorphism of $\phi(X_1)$. It follows that $(g_1, f_2 f_1)\colon X_1 \to Y_1 \times X_3$ is a closed embedding. $\square$

DEFINITION 2.12. (Inductive construction of $Y_{\mathcal{G}}$). Let Y be a nonsingular variety, $\mathcal{S}$ an arrangement of subvarieties, and $\mathcal{G}$ a building set of $\mathcal{S}$. Suppose $\mathcal{G} = \{G_1, \dots, G_N\}$ is indexed in an order that is compatible with inclusion relations (i.e., $i \le j$ if $G_i \subseteq G_j$). We define $(Y_k, \mathcal{S}^{(k)}, \mathcal{G}^{(k)})$ inductively with respect to k as follows.

(i) For $k = 0$, define $Y_0 = Y$, $\mathcal{S}^{(0)} = \mathcal{S}$, $\mathcal{G}^{(0)} = \mathcal{G} = \{G_1, \dots, G_N\}$, and $G_i^{(0)} = G_i$ for $1 \le i \le N$.

(ii) Assume that $(Y_{k-1}, \mathcal{S}^{(k-1)}, \mathcal{G}^{(k-1)})$ is constructed.
- Define Y_k to be the blow-up of Y_{k-1} along the nonsingular subvariety $G_k^{(k-1)}$.
- Define $G^{(k)} := (G^{(k-1)})^{\sim}$ for $G \in \mathcal{G}$ and define

$$\mathcal{G}^{(k)} := \{G^{(k)}\}_{G \in \mathcal{G}}.$$

- Define $\mathcal{S}^{(k)}$ to be the induced arrangement of $\mathcal{G}^{(k)}$.

(iii) Continue the inductive construction until $k = N$. We obtain

$$(Y_N, \mathcal{S}^{(N)}, \mathcal{G}^{(N)}),$$

where all the subvarieties in the building set $\mathcal{G}^{(N)}$ are divisors.

REMARK. In step (ii) we need Proposition 2.8. Indeed, since $G_i^{(k-1)}$ for $i < k$ are all divisors and hence are too large to be contained in $G_k^{(k-1)}$, it follows that $G_k^{(k-1)}$ is minimal in $\mathcal{G}^{(k-1)}$. Proposition 2.8 then asserts the existence of a naturally induced arrangement $\mathcal{S}^{(k)}$ and that $\mathcal{G}^{(k)} = \{G^{(k)}\}_{G \in \mathcal{G}}$ is a building set with respect to $\mathcal{S}^{(k)}$.

PROPOSITION 2.13. *The variety* Y_N *constructed in Definition 2.12 is isomorphic to the wonderful compactification* $Y_{\mathcal{G}}$ *defined in Definition 1.1.*

Proof. We prove by induction that Y_k is the closure of the inclusion

$$Y^\circ \hookrightarrow \prod_{i=1}^{k} Bl_{G_i} Y.$$

The proposition is then the special case $k = N$.

Let $0 \le i \le k-1$. Since $G_{i+1}^{(i)}$ is minimal in $\mathcal{G}^{(i)}$, Lemma 2.6(i) asserts that there are only two possible relations between the nonsingular subvarieties $G_k^{(i)}$ and $G_{i+1}^{(i)}$ of Y_i: either $G_k^{(i)} \supseteq G_{i+1}^{(i)}$ or $G_k^{(i)} \pitchfork G_{i+1}^{(i)}$. Therefore, Lemma 2.10 applies. Since $G_k^{(i+1)} = (G_k^{(i)})^{\sim}$, there exists a morphism f' such that following diagram commutes.

$$\begin{tikzcd}
Bl_{G_k^{(i+1)}} Y_{i+1} \arrow[r, "f'"] \arrow[d, "g'"] & Bl_{G_k^{(i)}} Y_i \arrow[d, "g"] \\
Y_{i+1} \arrow[r, "f"] & Y_i
\end{tikzcd}$$

The morphism $(g', f')\colon Bl_{G_k^{(i+1)}} Y_{i+1} \to Y_{i+1} \times Bl_{G_k^{(i)}} Y_i$ is a closed embedding. Using Lemma 2.11 on the diagram

$$\begin{tikzcd}
Bl_{G_k^{(k-1)}} Y_{k-1} \arrow[r] \arrow[d] & Bl_{G_k^{(k-2)}} Y_{k-2} \arrow[r] \arrow[d] & \cdots \arrow[r] & Bl_{G_k^{(0)}} Y_0 \arrow[d] \\
Y_{k-1} \arrow[r] & Y_{k-2} \arrow[r] & \cdots \arrow[r] & Y_0
\end{tikzcd}$$

and using that $Y_k = Bl_{G_k^{(k-1)}} Y_{k-1}$, $G_k^{(0)} = G_k$, and $Y_0 = Y$, we conclude that the morphism

$$Y_k \to Y_{k-1} \times Bl_{G_k} Y$$

is a closed embedding. Because the composition of closed embeddings is still a closed embedding, the morphism

$$Y_k \to \prod_{i=1}^{k} Bl_{G_i} Y$$

is a closed embedding. Then, since Y° is an open subset of Y_k and since Y_k is irreducible, from the composition

$$Y^\circ \hookrightarrow Y_k \hookrightarrow Y \times \prod_{i=1}^{k} Bl_{G_i} Y$$

we see that the closure of Y° in $Y \times \prod_{i=1}^{k} Bl_{G_i} Y$ is Y_k. $\square$

Proof of Theorem 1.2. Since $Y_{\mathcal{G}} \cong Y_N$, it follows that $Y_{\mathcal{G}}$ is nonsingular, $D_G := G^{(N)}$ are codimension-1 nonsingular subvarieties of $Y_{\mathcal{G}}$, and $Y_{\mathcal{G}} \setminus Y^\circ = \bigcup D_G$. Hence (i) is clear.

For any $T_1, \dots, T_r$ in $\mathcal{G}$ that form a $\mathcal{G}$-nest, $D_{T_1}, \dots, D_{T_r}$ form a $\mathcal{G}^{(N)}$-nest; as a result,

$$D_{T_1} \cap \cdots \cap D_{T_r} \neq \emptyset$$

by the definition of nest. Conversely, given $T_1, \dots, T_r$ in $\mathcal{G}$ such that the displayed intersection is nonempty, Lemma 2.5 implies that $D_{T_1}, \dots, D_{T_r}$ are all the $\mathcal{G}^{(N)}$-factors of the intersection and therefore intersect transversally. Moreover, by the definition of nest, $D_{T_1}, \dots, D_{T_r}$ form a $\mathcal{G}^{(N)}$-nest. Proposition 2.8 then implies that $T_1, \dots, T_r$ form a $\mathcal{G}$-nest. So (ii) is clear. □

3. Order of Blow-ups

In this section we shall prove Theorem 1.3, which is subsequently used in Sections 4.2–4.4. For the proof we need the following proposition, which is stronger than Proposition 2.8(ii) in the sense that a building set still induces a building set after a blow-up even when the center of the blow-up is not assumed to be minimal.

PROPOSITION 3.1. *Suppose that $\mathcal{G} = \{G_1, \dots, G_k\}$ is a building set of an arrangement $\mathcal{S}$ in Y and that $F \in \mathcal{G}$ is minimal. Let $\phi\colon Bl_F Y \to Y$ be the blow-up of Y along F, let $\widetilde{\mathcal{G}}$ be the induced building set, and let $\widetilde{\mathcal{S}}$ be the arrangement induced by $\widetilde{\mathcal{G}}$. Suppose $\mathcal{G}_+ = \{G_0, \dots, G_k\}$ is a building set and $\mathcal{S}_+$ is the arrangement induced by $\mathcal{G}_+$.*

Then $\widetilde{\mathcal{G}}_+ := \widetilde{\mathcal{G}} \cup \{\widetilde{G}_0\}$ is a building set of the induced arrangement

$$\widetilde{\mathcal{S}}_+ := \widetilde{\mathcal{S}} \cup \{\widetilde{S} \cap \widetilde{G}_0\}_{S \in \mathcal{S}}.$$

Proof. As the proof of Proposition 2.8, we need to discuss different types of intersections of subvarieties. See Section 5.2. □

LEMMA 3.2. *Let $\mathcal{I}_1, \mathcal{I}_2$ be two ideal sheaves on a variety Y. Define $Bl_{\mathcal{I}_2} Bl_{\mathcal{I}_1} Y$ to be the blow-up of $Y' = Bl_{\mathcal{I}_1} Y$ along the ideal sheaf $\phi^{-1}\mathcal{I}_2 \cdot \mathcal{O}_{Y'}$, where ϕ is the blow-up morphism $\phi\colon Y' \to Y$. Define $Bl_{\mathcal{I}_1} Bl_{\mathcal{I}_2} Y$ symmetrically. Then*

$$Bl_{\mathcal{I}_1\mathcal{I}_2} Y \cong Bl_{\mathcal{I}_2} Bl_{\mathcal{I}_1} Y \cong Bl_{\mathcal{I}_1} Bl_{\mathcal{I}_2} Y.$$

Proof. We show the existence of two natural morphisms

$$f\colon Bl_{\mathcal{I}_2} Bl_{\mathcal{I}_1} Y \to Bl_{\mathcal{I}_1\mathcal{I}_2} Y,$$
$$g\colon Bl_{\mathcal{I}_1\mathcal{I}_2} Y \to Bl_{\mathcal{I}_2} Bl_{\mathcal{I}_1} Y,$$

from which we obtain the isomorphism $Bl_{\mathcal{I}_1\mathcal{I}_2} Y \cong Bl_{\mathcal{I}_2} Bl_{\mathcal{I}_1} Y$. The other isomorphism $Bl_{\mathcal{I}_1\mathcal{I}_2} Y \cong Bl_{\mathcal{I}_1} Bl_{\mathcal{I}_2} Y$ follows symmetrically.

For simplicity of notation, denote $Y_1 = Bl_{\mathcal{I}_1} Y$, $Y_2 = Bl_{\mathcal{I}_2} Bl_{\mathcal{I}_1} Y$, and $Y_3 = Bl_{\mathcal{I}_1\mathcal{I}_2} Y$.

(i) We show the existence of f.

$$\begin{tikzcd}
 & & Bl_{\mathcal{I}_1\mathcal{I}_2} Y \arrow[d, "\phi"] \\
Bl_{\mathcal{I}_2} Bl_{\mathcal{I}_1} Y \arrow[r, "\phi_2"'] \arrow[urr, dotted, "f"] & Bl_{\mathcal{I}_1} Y \arrow[r, "\phi_1"'] & Y
\end{tikzcd}$$

By the universal property of blowing up, it suffices to show that

$$(\phi_1\phi_2)^{-1}(\mathcal{I}_1\mathcal{I}_2)\cdot\mathcal{O}_{Y_2}$$

is an invertible sheaf. Indeed,

$$\begin{aligned}(\phi_1\phi_2)^{-1}(\mathcal{I}_1\mathcal{I}_2)\cdot\mathcal{O}_{Y_2} &= \phi_2^{-1}((\phi_1^{-1}\mathcal{I}_1\cdot\mathcal{O}_{Y_1})\cdot(\phi_1^{-1}\mathcal{I}_2\cdot\mathcal{O}_{Y_1}))\cdot\mathcal{O}_{Y_2}\\ &= (\phi_2^{-1}(\phi_1^{-1}\mathcal{I}_1\cdot\mathcal{O}_{Y_1})\cdot\mathcal{O}_{Y_2})\cdot(\phi_2^{-1}(\phi_1^{-1}\mathcal{I}_2\cdot\mathcal{O}_{Y_1})\cdot\mathcal{O}_{Y_2}).\end{aligned}$$

In the last expression, both factors are invertible sheaves, so the product is also invertible.

(ii) We show the existence of g.

$$\begin{tikzcd} Bl_{\mathcal{I}_1\mathcal{I}_2}Y \arrow[r, dotted, "g"] \arrow[dr, dotted, "h"] \arrow[ddr, "\phi"'] & Bl_{\mathcal{I}_2}Bl_{\mathcal{I}_1}Y \arrow[d, "\phi_2"] \\ & Bl_{\mathcal{I}_1}Y \arrow[d, "\phi_1"] \\ & Y \end{tikzcd}$$

Because $(\phi^{-1}\mathcal{I}_1\cdot\mathcal{O}_{Y_3})\cdot(\phi^{-1}\mathcal{I}_2\cdot\mathcal{O}_{Y_3}) = \phi^{-1}(\mathcal{I}_1\mathcal{I}_2)\cdot\mathcal{O}_{Y_3}$ is invertible, both $(\phi^{-1}\mathcal{I}_1\cdot\mathcal{O}_{Y_3})$ and $(\phi^{-1}\mathcal{I}_2\cdot\mathcal{O}_{Y_3})$ are invertible. The invertibility of $(\phi^{-1}\mathcal{I}_1\cdot\mathcal{O}_{Y_3})$ implies the existence of h by the universal property of blowing up. Then, since ϕ_2 is the blow-up of the ideal sheaf $(\phi_1^{-1}\mathcal{I}_2\cdot\mathcal{O}_{Y_1})$ and since

$$h^{-1}(\phi_1^{-1}\mathcal{I}_2\cdot\mathcal{O}_{Y_1})\cdot\mathcal{O}_{Y_3} = \phi^{-1}\mathcal{I}_2\cdot\mathcal{O}_{Y_3}$$

is invertible, we can lift h to g by again applying the universal property of blowing up. This completes the proof. $\square$

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. (i) We fix the indices of $\{G_i\}$ in an order that is compatible with inclusion relations (i.e., $i < j$ if $G_i \subset G_j$). Consider the blow-up $\phi\colon \widetilde{Y} := Bl_{\mathcal{I}_1}Y \to Y$, where $\mathcal{I}_1$ is the ideal sheaf of G_1. Since G_i $(i > 1)$ either contains G_1 or is transversal to G_1 by Lemma 2.6, the ideal sheaf $\phi^{-1}\mathcal{I}_{G_i}\cdot\mathcal{O}_{\widetilde{Y}}$ is either $\mathcal{I}_{\widetilde{G}_i}\cdot\mathcal{I}_E$ or $\mathcal{I}_{\widetilde{G}_i}$. Because $\mathcal{I}_E$ is invertible, the blow-up of $\mathcal{I}_{\widetilde{G}_i}\cdot\mathcal{I}_E$ is isomorphic to the blow-up of $\mathcal{I}_{\widetilde{G}_i}$—that is, the blow-up along the nonsingular subvariety $\widetilde{G}_i$. By the same argument, each blow-up $Y_{k+1}\to Y_k$ is isomorphic to the blow-up of the ideal sheaf $\psi^{-1}\mathcal{I}_{k+1}\cdot\mathcal{O}_{Y_k}$, where $\psi: Y_k \to Y$ is the natural morphism. Therefore, by Lemma 3.2,

$$Y_{\mathcal{G}} \cong Bl_{\mathcal{I}_N}\cdots Bl_{\mathcal{I}_2}Bl_{\mathcal{I}_1}Y \cong Bl_{\mathcal{I}_1\cdots\mathcal{I}_N}Y.$$

(ii) Now assume that the order of $\{G_i\}$ is not necessarily compatible with inclusion relations but that it does satisfy $(*)$.

The proof is by induction with respect to N. The statement is obviously true for $N = 1$. Assume the statement (ii) is true for N, and consider $\mathcal{G}_+ = \mathcal{G}\cup\{G_{N+1}\}$.

We need to show that $Y_{\mathcal{G}_+}$ is isomorphic to the blow-up of $Y_{\mathcal{G}}$ along a nonsingular subvariety $\widetilde{G}_{N+1}$.

Suppose F is minimal in $\mathcal{G}$, $\widetilde{Y} = Bl_F Y$, and $\phi\colon \widetilde{Y} \to Y$ is the natural morphism. Proposition 3.1 implies that $\widetilde{\mathcal{G}}_+ := \widetilde{\mathcal{G}} \cup \{\widetilde{G}_{N+1}\}$ is a building set in $\widetilde{Y}$. There are two cases. If F is not minimal in $\mathcal{G}_+$, then G_{N+1} must be minimal and $G_{N+1} \subsetneq F$; in this case, $\phi^{-1}\mathcal{I}_{G_{N+1}} \cdot \mathcal{O}_{\widetilde{Y}} = \mathcal{I}_{\widetilde{G}_{N+1}}$. Next consider the case where F is minimal in $\mathcal{G}_+$. Now G_{N+1} either contains F or is transversal to F, so $\phi^{-1}\mathcal{I}_{G_{N+1}} \cdot \mathcal{O}_{\widetilde{Y}}$ is either $\mathcal{I}_{\widetilde{G}_{N+1}} \cdot \mathcal{I}_E$ or $\mathcal{I}_{\widetilde{G}_{N+1}}$. In each situation, $\phi^{-1}\mathcal{I}_{G_{N+1}} \cdot \mathcal{O}_{\widetilde{Y}}$ is isomorphic to $\mathcal{I}_{\widetilde{G}_{N+1}}$ up to an invertible sheaf. Continue this procedure until all elements in $\mathcal{G}$ have been blown up. Let $\psi\colon Y_{\mathcal{G}} \to Y$ be the natural morphism. Then $\psi^{-1}\mathcal{I}_{G_{N+1}} \cdot \mathcal{O}_{Y_{\mathcal{G}}}$ is isomorphic to the ideal sheaf of the nonsingular subvariety $\widetilde{G}_{N+1} \subset Y_{\mathcal{G}}$ up to an invertible sheaf, and hence the blow-up of $Y_{\mathcal{G}}$ along $\psi^{-1}\mathcal{I}_{G_{N+1}} \cdot \mathcal{O}_{Y_{\mathcal{G}}}$ is isomorphic to the blow-up of $Y_{\mathcal{G}}$ along $\widetilde{G}_{N+1}$. This completes the proof. $\square$

4. Examples of Wonderful Compactifications

4.1. Wonderful Model of Subspace Arrangements

If we let $Y = V$ be a finite-dimensional vector space, let $\mathcal{S}$ be any finite collection of subspaces of V, and construct the wonderful compactification of any building set of subspaces of V, then we recover the *wonderful model of subspace arrangements* by De Concini and Procesi.

It was discovered by De Concini and Procesi [DP] that, if a subset $\mathcal{G} \subseteq \mathcal{S}$ forms a so-called *building set,* then the closure of the natural locally closed embedding

$$i\colon V \setminus \bigcup_{W \in \mathcal{G}} W \hookrightarrow V \times \prod_{W \in \mathcal{G}} \mathbb{P}(V/W)$$

is a nonsingular variety birational to V. Moreover, the subspaces in $\mathcal{S}$ are replaced by a normal crossing divisor.

Remark. This idea motivated a generalized definition of the so-called wonderful conical compactifications for a complex manifold given by MacPherson and Procesi [MP]. Our definition of wonderful compactification is neither strictly general nor strictly less general than the wonderful compactification defined in [MP]. On the one hand, our compactification does not include the conic case: all the subvarieties involved in this paper are assumed to be nonsingular. On the other hand, even over the complex field $\mathbb{C}$, many arrangements of nonsingular varieties are not conical.

4.2. Fulton–MacPherson Configuration Spaces

Let X be a nonsingular variety, let $Y = X^n$, and let $\mathcal{G}$ be the set of diagonals of X^n. Our wonderful compactification gives the Fulton–MacPherson configuration space $X[n]$.

In [FM], Fulton and MacPherson constructed a compactification $X[n]$ of the configuration space $F(X, n)$ of n distinct labeled points in a nonsingular algebraic variety X. This compactification is related to several areas of mathematics. In [FM], Fulton and MacPherson used their compactification to construct a differential graded algebra that is a model for the configuration space $F(X,n)$ in the sense of Sullivan. Axelrod and Singer [AxS] used an analogous construction in the setting of real smooth manifolds in Chern–Simons perturbation theory. Now we give a brief review of Fulton and MacPherson's construction.

The configuration space $F(X,n)$ is an open subset of the Cartesian product X^n defined as the complement of all diagonals:

$$F(X,n) := X^n \setminus \bigcup_{|I|\geq 2} \Delta_I = \{(p_1,\dots,p_n) \in X^n \mid p_i \neq p_j \ \forall i \neq j\}.$$

The construction of $X[n]$ by Fulton and MacPherson is inductive. They define $X[1]$ to be X and $X[n+1]$ to be the variety that results from a sequence of blow-ups of $X[n] \times X$ along nonsingular subvarieties corresponding to all diagonals Δ_I, where $I \subseteq [n+1]$, $|I| \geq 2$, and I contains the number $n+1$.

For example, $X[2]$ is the blow-up of X^2 along the diagonal Δ_{12}. The variety $X[3]$ is obtained from a sequence of blow-ups of $X[2] \times X$ along nonsingular subvarieties corresponding to $\{\Delta_{123}; \Delta_{13}, \Delta_{23}\}$. More specifically, denoting by π the blow-up $X[2] \times X \to X^3$, we blow up first along $\pi^{-1}(\Delta_{123})$ and then along the strict transforms of Δ_{13} and Δ_{23} (the two strict transforms are disjoint, so they can be blown up in any order). In general, the order of blow-ups in the construction of $X[n]$ can be expressed as

$$\Delta_{12}, \Delta_{123}, \Delta_{13}, \Delta_{23}, \Delta_{1234}, \Delta_{124}, \Delta_{134}, \Delta_{234}, \Delta_{14}, \Delta_{24}, \Delta_{34}, \Delta_{12345}, \Delta_{1235}, \dots.$$

It is easy to verify that this sequence satisfies $(*)$ in Theorem 1.3, so the resulting variety $X[n]$ is indeed the wonderful compactification $Y_{\mathcal{G}}$. Theorem 1.3 also implies that $X[n]$ can be obtained from a more symmetric sequence of blow-ups in the order of ascending dimension:

$$\Delta_{12,\dots,n}, \Delta_{12,\dots,(n-1)}, \dots, \Delta_{23,\dots,n}, \dots, \Delta_{12}, \dots, \Delta_{(n-1),n}.$$

This more symmetric order of blow-ups is given by De Concini and Procesi [DP], MacPherson and Procesi [MP], and Thurston [T].

In fact, graphs can be used to clarify condition $(*)$ by using Kuperberg–Thurston's compactification (cf. the discussion after Proposition 4.2).

4.3. Kuperberg–Thurston's Compactification

In [KuT], Kuperberg and Thurston constructed an interesting compactification of the configuration space $F(X,n)$. Their construction is for real smooth manifolds and in this section we adapt their compactification to a nonsingular algebraic variety.

Let Γ be a (not necessarily connected) graph with n labeled vertices such that Γ has no self-loops or multiple edges. Denote by Δ_Γ the polydiagonal in X^n, where $x_i = x_j$ if i, j are connected in Γ. We call a graph Γ *vertex-2-connected* if the graph is connected and if it will remain connected after we remove any vertex. In particular, a single edge is vertex-2-connected.

In [KT] the authors state and sketch a proof that blowing up along $\Delta_{\Gamma'}$ for all vertex-2-connected subgraphs $\Gamma' \subseteq \Gamma$ gives a compactification X^{Γ}. If Γ is the complete graph with n vertices (i.e., any two vertices are joined with an edge), then the compactification X^{Γ} is exactly the Fulton–MacPherson compactication $X[n]$.

Kuperberg–Thurston's compactification X^{Γ} is a special case of the wonderful compactification of an arrangement of subvarieties given in this paper. Indeed, let $Y = X^n$, let

$$\mathcal{G} := \{\Delta_{\Gamma'} : \Gamma' \subseteq \Gamma \text{ is vertex-2-connected}\},$$

and let $\mathcal{S}$ be the set of polydiagonals of X^n obtained by intersecting only the diagonals in $\mathcal{G}$.

PROPOSITION 4.1. *In the notation just given, $\mathcal{G}$ is a building set with respect to $\mathcal{S}$. Therefore, Kuperberg–Thurston's compactification is the wonderful compactification $Y_{\mathcal{G}}$.*

Proof. The proof is in two steps.

(1) We call $\Gamma' \subseteq \Gamma$ a *full subgraph* if the following conditions are satisfied:

- Γ' contains all vertices in Γ;
- for any edge $e \in \Gamma$, if its endpoints p and q are in the same connected component of Γ' then $e \in \Gamma'$.

Then there is a one-to-one correspondence between the set of all full subgraphs of Γ and the set $\mathcal{S}$. The correspondence is given by mapping a full subgraph Γ' to $\Delta_{\Gamma'}$.

(2) Any full subgraph Γ' has a unique decomposition into vertex-2-connected subgraphs $\Gamma_1, \dots, \Gamma_k$. Observe that $\Delta_{\Gamma_1}, \dots, \Delta_{\Gamma_k}$ are the minimal elements in $\mathcal{G}$ containing $\Delta_{\Gamma'}$ and that they intersect transversally with the intersection $\Delta_{\Gamma'}$. Therefore, $\mathcal{G}$ is a building set by Definition 2.2. □

REMARK. It is also easy to describe a $\mathcal{G}$-nest. It corresponds to a set of vertex-2-connected subgraphs of Γ where, for any two subgraphs Γ_1 and Γ_2, one of the following statements holds:

(i) Γ_1 and Γ_2 are disjoint; or
(ii) Γ_1 and Γ_2 intersect at one vertex; or
(iii) $\Gamma_1 \subseteq \Gamma_2$ or $\Gamma_2 \subseteq \Gamma_1$.

The next proposition describes the relation between X^{Γ_1} and X^{Γ_2} for $\Gamma_1 \subsetneq \Gamma_2$, which will help us understand the construction of Fulton and MacPherson configuration spaces.

PROPOSITION 4.2. *Let $\Gamma_1 \subsetneq \Gamma_2$ be two (not necessarily connected) graphs with n labeled vertices without self-loops and multiple edges. Then X^{Γ_2} can be obtained by a sequence of blow-ups of X^{Γ_1} along nonsingular centers.*

One such order is given as follows. Let $\{\Gamma_j''\}_{j=1}^{t}$ be the set of all vertex-2-connected subgraphs of Γ_2 that are not contained in Γ_1. Arrange the index such that $i < j$ if the number of vertices of Γ_i'' is greater than the number of vertices *of* Γ_j''. *Then* X^{Γ_2} *can be obtained by blowing up along the nonsingular centers* $\widetilde{\Delta}_{\Gamma_1''}, \widetilde{\Delta}_{\Gamma_2''}, \ldots, \widetilde{\Delta}_{\Gamma_t''}$.

Proof. Let $\{\Gamma_i'\}_{i=1}^s$ be the set of all vertex-2-connected subgraphs of Γ_1. Arrange the indices such that $i < j$ if the number of vertices of Γ_i' is greater than the number of vertices of Γ_j'.

It is easy to verify that $\{\Delta_{\Gamma_1'}, \ldots, \Delta_{\Gamma_s'}, \Delta_{\Gamma_1''}, \ldots, \Delta_{\Gamma_t''}\}$ satisfies $(*)$ in Theorem 1.3. Apply Theorem 1.3, we know that X^{Γ_2} is the blow-up of X^n along the nonsingular centers

$$\Delta_{\Gamma_1'}, \widetilde{\Delta}_{\Gamma_2'}, \ldots, \widetilde{\Delta}_{\Gamma_s'}, \widetilde{\Delta}_{\Gamma_1''}, \ldots, \widetilde{\Delta}_{\Gamma_t''}.$$

On the other hand, after the first s blow-ups we get X^{Γ_1}. Therefore X^{Γ_2} can be obtained by a sequence of blow-ups along nonsingular centers $\widetilde{\Delta}_{\Gamma_1''}, \ldots, \widetilde{\Delta}_{\Gamma_t''}$. This completes the proof. □

In light of Proposition 4.2, Fulton and MacPherson's original construction of $X[n]$ can be understood as specifying a chain of graphs. Indeed, by the proposition, the first arrow corresponds to blowing up $X[4]$ along Δ_{12}, the second corresponds to blowing up along $\widetilde{\Delta}_{123}$, $\widetilde{\Delta}_{13}$, and $\widetilde{\Delta}_{23}$ (which correspond to all the vertex-2-connected subgraphs that are not in the previous graph), and the last arrow corresponds to blowing up along $\widetilde{\Delta}_{1234}$, $\widetilde{\Delta}_{124}$, $\widetilde{\Delta}_{134}$, $\widetilde{\Delta}_{234}$, $\widetilde{\Delta}_{14}$, $\widetilde{\Delta}_{24}$, and $\widetilde{\Delta}_{34}$. On the other hand, the symmetric construction of $X[4]$ corresponds to the chain containing only two graphs: the first graph and last graph in Figure 1.

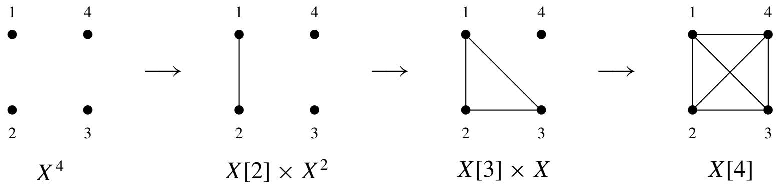

Figure 1 Fulton and MacPherson's construction of $X[4]$

To illustrate the idea a little more, in Figure 2 we construct $X[3]$ corresponding to the chain of graphs. The first step is to blow up along Δ_{12}, the second step is to blow up along $\widetilde{\Delta}_{23}$, and the final step is to blow up along $\widetilde{\Delta}_{123}$ and $\widetilde{\Delta}_{13}$. Each blow-up is along a nonsingular subvariety.

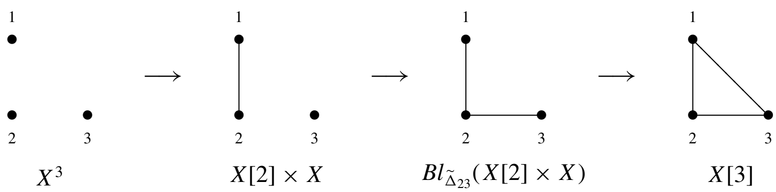

Figure 2 A new construction of $X[3]$

4.4. Moduli Space $\overline{M}_{0,n}$ of Rational Curves with n Marked Points

The moduli space $\overline{M}_{0,n}$ is the wonderful compactification of $((\mathbb{P}^1)^{n-3}, \mathcal{S}, \mathcal{G})$, where $\mathcal{G}$ is set of all diagonals and augmented diagonals

$$\Delta_{I,a} := \{(p_4, \dots, p_n) \in (\mathbb{P}^1)^{n-3} \mid p_i = a \ \forall i \in I\}$$

for $I \subseteq \{4, \dots, n\}$, $|I| \geq 2$, and $a \in \{0, 1, \infty\}$. Here $\mathcal{S}$ is the set of all intersections of elements in $\mathcal{G}$.

This result is an immediate consequence of Theorem 1.3 applied to Keel's construction [Ke1]. Indeed, Keel gives the construction of $\overline{M}_{0,n}$ by a sequence of blow-ups in the following order:

$$\Delta_{45,0}, \Delta_{45,1}, \Delta_{45,\infty}, \Delta_{456,0}, \Delta_{456,1}, \Delta_{456,\infty}, \dots, \Delta_{46,0}, \dots, \Delta_{456}, \dots.$$

To be more precise: for I such that $\max I = 5$, blow up along $\Delta_{I,a}$ for those I such that $|I| = 2$; for $\max I = 6$, blow up $\Delta_{I,a}$ for $|I| = 3$ and then Δ_I for $|I| = 3$. In general, for $\max I = k$, blow up $\Delta_{I,a}$ for $|I| = k - 3$, then $\Delta_{I,a}$ for $|I| = k - 4$ and Δ_I for $|I| = n - 3$, then $\Delta_{I,a}$ for $|I| = k - 5$ and Δ_I for $|I| = n - 4$, and so forth. It is easy to check that the order satisfies $(*)$ in Theorem 1.3. Therefore, $\overline{M}_{0,n}$ is a wonderful compactification.

Notice that the preceding diagonals and augmented diagonals in $\mathbb{P}^{n-3}$ are just the restrictions of diagonals in $(\mathbb{P}^1)^n$ to the codimension-3 subvariety

$$Y = \{(p_1, p_2, \dots, p_n) \in (\mathbb{P}^1)^n \mid p_1 = 0,\ p_2 = 1,\ p_3 = \infty\}.$$

Now, by blowing up all the diagonals of $(\mathbb{P}^1)^n$ in order of increasing dimension and then comparing with the construction of Fulton–MacPherson configuration space, we obtain a relation between $\overline{M}_{0,n}$ and the Fulton–MacPherson space $\mathbb{P}^1[n]$ in Corollary 1.4.

4.5. Ulyanov's Compactification

Closely related to Fulton and MacPherson's compactification is another compactification of the configuration space $F(X, n)$, which Ulyanov [U] discovered and denoted by $X\langle n\rangle$. The construction consists of blowing up more subvarieties in X^n than Fulton–MacPherson's construction does; in particular, Ulyanov blows up not only diagonals but also polydiagonals. The order of the blow-ups in [U] is the ascending order of the dimension. For example, $X\langle 4\rangle$ is the blow-up of X^4 along polydiagonals in the following order:

$$(1234), (123), (124), (134), (234), (12, 34), (13, 24), (14, 23), (12), \dots, (34).$$

The polydiagonal compactification $X\langle n\rangle$ shares many similar properties with Fulton–MacPherson's compactification. However, one difference is that, in the case of characteristic 0, the isotropy group of any point in $X\langle n\rangle$ is abelian under the symmetric group action, whereas the isotropy group of a point in $X[n]$ is not necessarily abelian (but is always solvable) under the symmetric group action.

4.6. Hu's Compactification versus Minimal Compactifications

We now consider the general situation where Y is nonsingular with an arrangement of subvarieties $\mathcal{S}$. By blowing up all $S \in \mathcal{S}$ in order of ascending dimension, we obtain a nonsingular variety $Bl_{\mathcal{S}}Y$ [Hu]. Define $Y^\circ := Y \setminus \bigcup_{S\in\mathcal{S}} S$, the open stratum of Y. It is isomorphic to an open subset of $Bl_{\mathcal{S}}Y$. Hu showed that:

(1) the boundary $Bl_{\mathcal{S}}Y \setminus Y^\circ = \bigcup_{S\in\mathcal{S}} D_S$ is a simple normal crossing divisor; and
(2) for any $S_1, \dots, S_n \in \mathcal{S}$, the intersection of $D_{S_1}, \dots, D_{S_k}$ is nonempty if and only if $\{S_i\}$ forms a chain—that is, $S_1 \subseteq \cdots \subseteq S_k$, with a rearrangement of indices if necessary.

Hu's compactification generalized Ulyanov's polydiagonal compactification and is a special case of the wonderful compactification of arrangement of subvarieties given in this paper where the building set $\mathcal{G} = \mathcal{S}$. (In this special case, a $\mathcal{G}$-nest is simply a chain of subvarieties.)

Fixing an arrangement $\mathcal{S}$, Hu's compactification $Y_{\mathcal{S}}$ is the maximal wonderful compactification. Indeed, it is not hard to show that, for any building set $\mathcal{G}$ of $\mathcal{S}$, the natural birational map $Y_{\mathcal{S}} \to Y_{\mathcal{G}}$ is a morphism. At the other extreme, there exists a minimal wonderful compactification for $\mathcal{S}$ that can be defined by the set of so-called irreducible elements in $\mathcal{S}$.

Definition 4.3. An element G in $\mathcal{S}$ is called *reducible* if there are $G_1, \dots, G_k \in \mathcal{S}$ $(k \geq 2)$ with $G = G_1 \pitchfork \cdots \pitchfork G_k$ and if, for every $G' \supseteq G$ in $\mathcal{S}$, there exist $G_i' \in \mathcal{S}$ with $G_i' \supseteq G_i$ for $1 \leq i \leq k$ such that $G' = G_1' \pitchfork \cdots \pitchfork G_k'$. An element $G \in \mathcal{S}$ is called *irreducible* if it is not reducible.

By the same method as in [DP] we can show that the irreducible elements in $\mathcal{S}$ form a building set, denoted by $\mathcal{G}_{\min}$, and that every building set $\mathcal{G}$ of the arrangement $\mathcal{S}$ contains $\mathcal{G}_{\min}$. It is not hard to show that the natural birational map $Y_{\mathcal{G}} \to Y_{\mathcal{G}_{\min}}$ is a morphism.

Of the previous examples, the Fulton–MacPherson configuration spaces, Kuperberg–Thurston's compactification, and the moduli space $\overline{M}_{0,n}$ are minimal wonderful compactifications. Ulyanov's polydiagonal compactification is maximal.

5. Appendix

5.1. Clean Intersection versus Transversal Intersection

Let Y be a nonsingular variety. For a nonsingular subvariety A (more generally, a subscheme whose connected components are nonsingular subvarieties) of Y, denote by T_A the total space of the tangent bundle of A and by $T_{A,y}$ the tangent space of A at the point $y \in A$. For a point $y \notin A$, define $T_{A,y}$ to be T_y, the tangent space of Y at y. (This stipulation will simplify the definition of transversal intersection.) In this paper, $T_{A,y}$ is viewed as a subspace of T_y and T_A is viewed as a subvariety of T_Y.

5.1.1. *Clean Intersection*
The notion of cleanness can be traced back to Bott [B] in the setting of differential geometry.

We say that the intersection of two nonsingular subvarieties A and B is *clean* if the set-theoretic intersection $A \cap B$ is a nonsingular subvariety (or, more generally, is a scheme whose connected components are nonsingular subvarieties) and also satisfies the condition

$$T_{A\cap B,y} = T_{A,y} \cap T_{B,y} \quad \forall y \in A \cap B.$$

The following lemma gives a useful criterion for the cleanness of intersections.

LEMMA 5.1. *Suppose that A and B are nonsingular closed subvarieties of Y and that the intersection $C = A \cap B$ is a disjoint union of nonsingular subvarieties. Let $\mathcal{I}_A$ (resp. $\mathcal{I}_B, \mathcal{I}_C$) denote the ideal sheaf of A (resp. B, C). Then the following statements are equivalent*:

(i) *the subvarieties A and B intersect cleanly*;
(ii) $\mathcal{I}_A + \mathcal{I}_B = \mathcal{I}_C$.

In other words, two subvarieties intersect cleanly if and only if their scheme-theoretic intersection is nonsingular.

Proof. Condition (i) is equivalent to

$$T_{A,y} \cap T_{B,y} = T_{C,y} \quad \forall y \in A \cap B. \tag{5.1}$$

By definition of tangent space,

$$T_{A,y} = \{v \in T_y \mid df(v) = 0 \ \forall f \in (\mathcal{I}_A)_y\}.$$

Define $\phi\colon \mathfrak{m}_y \to \mathfrak{m}_y/\mathfrak{m}_y^2$ to be the natural quotient. Then $T_{A,y} = \phi((\mathcal{I}_A)_y)^\perp$, the annihilator of $\phi((\mathcal{I}_A)_y)$ in the dual space $(\mathfrak{m}_y/\mathfrak{m}_y^2)^* \cong T_y$. Therefore, (5.1) is equivalent to

$$\phi((\mathcal{I}_A)_y)^\perp \cap \phi((\mathcal{I}_B)_y)^\perp = \phi((\mathcal{I}_C)_y)^\perp \quad \forall y \in A \cap B,$$

which is equivalent to

$$\phi((\mathcal{I}_A)_y) + \phi((\mathcal{I}_B)_y) = \phi((\mathcal{I}_C)_y) \quad \forall y \in A \cap B.$$

Since $\phi((\mathcal{I}_A)_y) = ((\mathcal{I}_A)_y + \mathfrak{m}_y^2)/\mathfrak{m}_y^2$ (and similarly for B and C), the preceding condition is equivalent to

$$(\mathcal{I}_A)_y + (\mathcal{I}_B)_y + \mathfrak{m}_y^2 = (\mathcal{I}_C)_y + \mathfrak{m}_y^2 \quad \forall y \in A \cap B. \tag{5.2}$$

On the other hand, two ideal sheaves on Y are the same if and only if their germs coincide at every closed point $y \in Y$. So condition (ii) is equivalent to

$$(\mathcal{I}_A)_y + (\mathcal{I}_B)_y = (\mathcal{I}_C)_y \quad \forall y \in Y, \tag{5.3}$$

where $\mathcal{I}_y$ denotes the germ of a sheaf $\mathcal{I}$ at point y. Therefore it suffices to show that (5.2) $\Leftrightarrow$ (5.3).

Obviously (5.3) $\Rightarrow$ (5.2). To see the implication (5.2) $\Rightarrow$ (5.3), observe first that condition (5.3) holds for $y \notin A \cap B$ and that the inclusion $(\mathcal{I}_A)_y + (\mathcal{I}_B)_y \subseteq (\mathcal{I}_C)_y$ holds for $y \in A \cap B$. Thus it remains to show that $(\mathcal{I}_A)_y + (\mathcal{I}_B)_y \supseteq (\mathcal{I}_C)_y$ holds for $y \in A \cap B$. Using local parameters allows us to check that $(\mathcal{I}_C)_y \cap \mathfrak{m}_y^2 = (\mathcal{I}_C)_y \mathfrak{m}_y$. Condition (5.2) then implies the surjection

$$(\mathcal{I}_A)_y + (\mathcal{I}_B)_y \twoheadrightarrow ((\mathcal{I}_C)_y + \mathfrak{m}_y^2)/\mathfrak{m}_y^2 \xrightarrow{\cong} (\mathcal{I}_C)_y/((\mathcal{I}_C)_y \cap \mathfrak{m}_y^2)$$
$$\xrightarrow{\cong} (\mathcal{I}_C)_y/(\mathcal{I}_C)_y\mathfrak{m}_y.$$

Hence $(\mathcal{I}_A)_y + (\mathcal{I}_B)_y + (\mathcal{I}_C)_y\mathfrak{m}_y = (\mathcal{I}_C)_y$. Applying Nakayama's lemma then yields $(\mathcal{I}_A)_y + (\mathcal{I}_B)_y = (\mathcal{I}_C)_y$, which completes the proof. □

5.1.2. *Transversal Intersection*

By definition, A and B intersect transversally (denoted by $A \pitchfork B$) if $T_{A,y}^{\perp} + T_{B,y}^{\perp}$ form a direct sum in the dual space $T_y^* \cong \mathfrak{m}_y/\mathfrak{m}_y^2$ of T_y for any point $y \in Y$ or, equivalently, if

$$T_y = T_{A,y} + T_{B,y} \quad \forall y \in Y.$$

More generally, we state that a finite collection of k nonsingular subvarieties $A_1, \ldots, A_k$ intersect transversally (denoted by $A_1 \pitchfork A_2 \pitchfork \cdots \pitchfork A_k$) if $k = 1$ or if, for any $y \in Y$,

$$T_{A_1,y}^{\perp} + T_{A_2,y}^{\perp} + \cdots + T_{A_k,y}^{\perp}$$

form a direct sum in T_y^*; or, equivalently, if

$$\operatorname{codim}\left(\bigcap_{i=1}^{k} T_{A_i,y}, T_y\right) = \sum_{i=1}^{k} \operatorname{codim}(A_i, Y);$$

or, equivalently, if for any $y \in Y$ there exist (a) a system of local parameters $x_1, \ldots, x_n$ on Y at y that are regular on an affine neighborhood U of y such that y is defined by the maximal ideal $(x_1, \ldots, x_n)$ as well as (b) integers $0 = r_0 \le r_1 \le \cdots \le r_k \le n$ such that the subvariety A_i is defined by the ideal

$$(x_{r_{i-1}+1}, x_{r_{i-1}+2}, \ldots, x_{r_i}) \quad \text{for all } 1 \le i \le k.$$

(If $r_{i-1} = r_i$ then the ideal is assumed to be the ideal containing units, which means geometrically that the restriction of A_i to U is empty.)

5.1.3. *Transversal Intersection $\Rightarrow$ Clean Intersection*

If A and B intersect transversally, then we can choose local parameters around any point $y \in A \cap B$ such that (a) y is the origin and (b) the restrictions of A and B are defined by local parameters. Then it is obvious that $T_{A\cap B,y} = T_{A,y} \cap T_{B,y}$ for all $y \in A \cap B$.

5.1.4. *Transversal Intersection at One Point + Clean Intersection $\Rightarrow$ Transversal Intersection*

Lemma 5.2. *Let A_1 and A_2 be two nonsingular closed subvarieties of Y that intersect cleanly along a closed nonsingular subvariety A. If A_1 and A_2 intersect* *transversally at a point $y_0 \in A$, then they intersect transversally (at every point $y \in A$).*

In general, let $A_1, \ldots, A_k$ be subvarieties in a simple arrangement $\mathcal{S}$ (cf. Definition 2.1) and let $A = \bigcap_{i=1}^{k} A_i$. If $A_1, \ldots, A_k$ intersect transversally at a point $y_0 \in A$, then they intersect transversally (at every point).

Proof. We prove the general case. Without loss of generality, we need only prove the transversality for points in A. The irreducibility of A_i and A implies that $\dim T^{\perp}_{A_i,y} = \dim T^{\perp}_{A_i,y_0}$ and $\dim T^{\perp}_{A,y} = \dim T^{\perp}_{A,y_0}$. By the definition of clean intersection, we have

$$T^{\perp}_{A_1,y} + \cdots + T^{\perp}_{A_k,y} = T^{\perp}_{A,y}.$$

On the other hand, by the transversality condition at point y_0 it follows that

$$T^{\perp}_{A_1,y_0} \oplus \cdots \oplus T^{\perp}_{A_k,y_0} = T^{\perp}_{A,y_0}.$$

Comparing the dimensions of the two equalities just displayed, we see that the left-hand side of the first equality must form a direct sum; therefore, $A_1, \ldots, A_k$ intersect transversally at y. □

5.1.5. *Examples and Nonexamples of Clean and Transversal Intersections*

- k ($\leq n$) hyperplanes H_i in $\mathbb{A}^n$ defined by $x_i = 0$ intersect transversally; therefore, any two of them intersect cleanly.
- Two (not necessarily distinct) lines in $\mathbb{A}^3$ passing through the origin intersect cleanly but not transversally.
- In $\mathbb{A}^2$, the intersection of the parabola $y = x^2$ and the line $y = 0$ is not clean and thus not transversal.

5.2. *Proofs of Statements in Previous Sections*

Proof of Lemma 2.4. It is convenient to carry out the proof using the cotangent space T^*_y. We use the same notation $\mathcal{G}^*_y$, S, $T^{\perp}_{S,y}$, $T^{\perp}_i$ as in the remark after Definition 2.2. By [DP, Sec. 2.3, Thm. (2)], the definition of building set implies the following: if $S' \in \mathcal{S}$ is such that $S' \supseteq S$, then

$$T^{\perp}_{S',y} = \bigoplus_{i=1}^{k} (T^{\perp}_{S',y} \cap T^{\perp}_i);$$

moreover, if $T^{\perp}_{S',y} = T'^{\perp}_1 \oplus \cdots \oplus T'^{\perp}_s$, where $T'^{\perp}_1, \ldots, T'^{\perp}_s$ are the maximal elements in $\mathcal{G}^*_y$ contained in $T^{\perp}_{S',y}$, then each term $(T^{\perp}_{S',y} \cap T^{\perp}_i)$ is a direct sum of some $T'^{\perp}_j$.

Fix a point $y \in S$. To show (i), it is enough to show the following. Suppose $T^{\perp}_1, \ldots, T^{\perp}_k$ are all the maximal elements in $\mathcal{G}^*_y$ that are contained in $T^{\perp}_{S,y}$. Suppose m is an integer such that $1 \leq m \leq k$, and define $T^{\perp} := T^{\perp}_1 \oplus \cdots \oplus T^{\perp}_m$. Then $T^{\perp}_1, \ldots, T^{\perp}_m$ are all the maximal elements in $\mathcal{G}^*_y$ that are contained in $T^{\perp}$.

To show (ii), it is equivalent to show the following. Suppose $T_1^\perp, \dots, T_k^\perp$ are all the maximal elements in $\mathcal{G}_y^*$ that are contained in $T_{S,y}^\perp$. Suppose $T^\perp \in \mathcal{G}_y^*$ is maximal, $T^\perp \supseteq T_1^\perp, \dots, T_m^\perp$, and $T^\perp \not\supseteq T_{m+1}^\perp, \dots, T_k^\perp$. Then $T^\perp, T_{m+1}^\perp, \dots, T_k^\perp$ are all the maximal elements in $\mathcal{G}_y^*$ that are contained in $T^\perp + T_{S,y}^\perp$.

Both claims can be shown by routine linear algebra. $\square$

Proof of Lemma 2.6. Part (i) follows directly from the definition of building set: if F is disjoint from G then of course $F \pitchfork G$; otherwise, G contains some $\mathcal{G}$-factor of $F \cap G$. But a $\mathcal{G}$-factor of $F \cap G$ is either F or is transversal to F (which implies that $G \pitchfork F$).

(iii) Define $A = \bigcap_{i=1}^m G_i$ and $B = \bigcap_{i=m+1}^k G_i$. We claim that $A \supseteq F$ and $B \pitchfork F$. That $A = \bigcap_{i=1}^m G_i \supseteq F$ follows from the definition of m. Lemma 2.4(ii) asserts that $F \pitchfork G_{m+1} \pitchfork \cdots \pitchfork G_k$, so F is transversal to B. Then (iii) follows from Lemma 2.4(i).

(ii) The proof of (iii) demonstrates the existence of an F-factorization. Now we show that such an factorization is unique. Suppose we have another factorization $S = A' \cap B'$ such that $A' \supseteq F$ and $B' \pitchfork F$. Since $B' \supseteq F \cap B' = F \cap S$ and since, by Lemma 2.4(ii), the $\mathcal{G}$-factors of $F \cap S$ are $F, G_{m+1}, \dots, G_k$, it follows that each $\mathcal{G}$-factor G' of B' contains F or G_i for some $m+1 \le i \le k$. But $B' \pitchfork F$ implies $G' \pitchfork F$, so $G' \not\supseteq F$. Hence $G' \supseteq G_i$ for some $m+1 \le i \le k$. Intersecting all the $\mathcal{G}$-factors G' of B' yields $B' = \bigcap G' \supseteq \bigcap_{i=m+1}^k G_i = B$. Fixing a point $y \in F \cap S$, we have

$$T_{F,y}^\perp \oplus T_{B,y}^\perp = T_{F,y}^\perp \oplus T_{B',y}^\perp$$

and $T_{B,y}^\perp \supseteq T_{B',y}^\perp$; therefore, $T_{B,y}^\perp = T_{B',y}^\perp$ and so $B = B'$. Similarly $A = A'$.

(iv) Suppose the F-factorization of $S \cap S'$ is $A'' \cap B''$. Then $F \cap B''$ is the F-factorization of the intersection. Since $B \supseteq (F \cap S) = (F \cap B'')$ but $B \pitchfork F$, we have $B \supseteq B''$. Similarly, $B' \supseteq B''$ and so $B \cap B' \supseteq B''$. By an analogous argument using the dual of the tangent space as in the proof of (ii), we can show that $B \cap B' = B''$; hence $F \pitchfork (B \cap B')$. Then it is easy to see that $A'' = A \cap A'$ and that the F-factorization of $S \cap S'$ is indeed $(A \cap A') \cap (B \cap B')$. $\square$

Proof of Lemma 2.9. We give only the proof of part (iii), because (ii) and (iv) can be proved similarly while (i), (v), and (vi) can be easily checked using a system of local parameters.

In the complement of the exceptional divisor E, we have

$$(\widetilde{A}_1 \cap \widetilde{A}_2) \setminus E \cong (A_1 \setminus F) \cap (A_2 \setminus F) = (A_1 \cap A_2) \setminus G = (A_1 \cap A_2)^\sim \setminus E.$$

Inside E, we have

$$\begin{aligned}(\widetilde{A}_1 \cap \widetilde{A}_2) \cap E &= \mathbb{P}(N_F A_1) \cap \mathbb{P}(N_F A_2) = \mathbb{P}(T_{A_1}/T_F) \cap \mathbb{P}(T_{A_2}/T_F) \\ &= \mathbb{P}((T_{A_1} \cap T_{A_2})/T_F) = \mathbb{P}(T_{A_1 \cap A_2}/T_F) = \mathbb{P}(N_F(A_1 \cap A_2)) \\ &= (A_1 \cap A_2)^\sim \cap E,\end{aligned}$$

where $N_F(A_1 \cap A_2)$ denotes the normal bundle of F in $A_1 \cap A_2$. (Note that in the fourth equality we have used the condition that A_1 and A_2 intersect cleanly.) Hence $\widetilde{A}_1 \cap \widetilde{A}_2 = (A_1 \cap A_2)^\sim$.

According to Lemma 5.1, $\widetilde{A}_1$ and $\widetilde{A}_2$ intersect cleanly if and only if

$$\mathcal{I}_{\widetilde{A}_1} + \mathcal{I}_{\widetilde{A}_2} = \mathcal{I}_{(A_1 \cap A_2)^\sim}. \tag{5.4}$$

But $\widetilde{A}_1 = \mathcal{R}(E, \pi^{-1}(A_1))$, where $\mathcal{R}(E, \pi^{-1}(A_1))$ is the residue scheme to E in $\pi^{-1}(A_1)$ (see [K2, Thm. 1] or [F, Sec.9.2]). By a property of residue schemes, we have

$$\mathcal{I}_{\mathcal{R}(E,\pi^{-1}(A_1))} \cdot \mathcal{I}_E = \mathcal{I}_{\pi^{-1}(A_1)},$$

which is the same as

$$\mathcal{I}_{\widetilde{A}_1} \cdot \mathcal{I}_E = \mathcal{I}_{\pi^{-1}(A_1)}.$$

Similarly, we have

$$\mathcal{I}_{\widetilde{A}_2} \cdot \mathcal{I}_E = \mathcal{I}_{\pi^{-1}(A_2)},$$

$$\mathcal{I}_{(A_1 \cap A_2)^\sim} \cdot \mathcal{I}_E = \mathcal{I}_{\pi^{-1}(A_1 \cap A_2)}.$$

Because A_1 and A_2 intersect cleanly, $\mathcal{I}_{A_1} + \mathcal{I}_{A_2} = \mathcal{I}_{A_1 \cap A_2}$, and this implies

$$\mathcal{I}_{\pi^{-1}(A_1)} + \mathcal{I}_{\pi^{-1}(A_2)} = \mathcal{I}_{\pi^{-1}(A_1 \cap A_2)}.$$

Thus we derive that equality

$$\mathcal{I}_{\widetilde{A}_1} \cdot \mathcal{I}_E + \mathcal{I}_{\widetilde{A}_2} \cdot \mathcal{I}_E = \mathcal{I}_{(A_1 \cap A_2)^\sim} \cdot \mathcal{I}_E.$$

Since $\mathcal{I}_E$ is an invertible sheaf, this equality implies (5.4) and hence (iii) is proved. $\square$

Proof of Proposition 2.8. (i) We need to check that any two elements in $\widetilde{\mathcal{S}}$ intersect cleanly and that the intersection is still in $\widetilde{\mathcal{S}}$. For this, we need to check three possibilities: $\widetilde{S} \cap \widetilde{S}'$, $\widetilde{S} \cap (\widetilde{S}' \cap E)$, and $(\widetilde{S} \cap E) \cap (\widetilde{S}' \cap E)$. We prove only the first because proofs for the other two possibilities are similar.

Suppose $S, S' \in \mathcal{S}$. We can assume that $S \cap S' \neq \emptyset$, for otherwise $\widetilde{S} \cap \widetilde{S}'$ is obviously empty. Suppose the F-factorizations of S and S' are $S = A \cap B$ and $S' = A' \cap B'$, respectively. By Lemma 2.6(iv), the F-factorization of $S \cap S'$ is $(A \cap A') \cap (B \cap B')$. Lemma 2.9(v) asserts that $\widetilde{S} = \widetilde{A} \cap \widetilde{B}$ and $\widetilde{S}' = \widetilde{A}' \cap \widetilde{B}'$. To show that $\widetilde{S}$ and $\widetilde{S}'$ intersect cleanly along a subvariety in $\widetilde{\mathcal{S}}$, we consider three cases as follows.

Case 1: $F \subsetneq A \cap A'$. In this case, $(S \cap S')^\sim = (A \cap A')^\sim \cap (B \cap B')^\sim$ and

$$\widetilde{S} \cap \widetilde{S}' = (\widetilde{A} \cap \widetilde{A}') \cap (\widetilde{B} \cap \widetilde{B}') = (A \cap A')^\sim \cap (B \cap B')^\sim = (S \cap S')^\sim.$$

Moreover,

$$T_{\widetilde{S}} \cap T_{\widetilde{S}'} = T_{\widetilde{A}} \cap T_{\widetilde{B}} \cap T_{\widetilde{A}'} \cap T_{\widetilde{B}'} = T_{(A \cap A')^\sim} \cap T_{(B \cap B')^\sim} = T_{(S \cap S')^\sim},$$

where the first and third equalities hold by Lemma 2.9(v) and the second equality holds by parts (iii) and (iv) of Lemma 2.9. Thus $\widetilde{S}$ intersects $\widetilde{S}'$ cleanly along $(S \cap S')^\sim \in \widetilde{\mathcal{S}}$.

Case 2: $F = A \cap A'$ but $F \neq A$ and $F \neq A'$. By Lemma 2.9(ii) we have $\widetilde{A} \cap \widetilde{A}' = \emptyset$; hence

$$\widetilde{S} \cap \widetilde{S} = (\widetilde{A} \cap \widetilde{A}') \cap (\widetilde{B} \cap \widetilde{B}') = \emptyset.$$

Case 3: $F = A$ or A'. Without loss of generality, we assume $F = A$. Then

$$\widetilde{S} \cap \widetilde{S}' = (\widetilde{A} \cap \widetilde{A}') \cap (\widetilde{B} \cap \widetilde{B}') = E \cap (\widetilde{A}' \cap \widetilde{B} \cap \widetilde{B}') = E \cap (A' \cap B \cap B')^{\sim}.$$

By parts (i) and (v) of Lemma 2.9,

$$T_{\widetilde{S}} \cap T_{\widetilde{S}'} = (T_E \cap T_{\widetilde{B}}) \cap (T_{\widetilde{A}'} \cap T_{\widetilde{B}'}) = (T_E \cap T_{\widetilde{A}'}) \cap T_{(B \cap B')^{\sim}} = T_{E \cap (A' \cap B \cap B')^{\sim}},$$

so again $\widetilde{S}$ intersects $\widetilde{S}'$ cleanly along $E \cap (A \cap B \cap B')^{\sim} \in \widetilde{\mathcal{S}}$. We have thus shown that $\widetilde{S}$ and $\widetilde{S}'$ intersect cleanly along a subvariety in $\widetilde{\mathcal{S}}$ in all possible cases.

(ii) To show that $\widetilde{\mathcal{G}} := \{\widetilde{G}\}_{G \in \mathcal{G}}$ forms a building set we first need to show that, for all $\widetilde{S}$ (resp. $(\widetilde{S} \cap E)) \in \widetilde{\mathcal{S}}$, the $\widetilde{\mathcal{G}}$-factors of $\widetilde{S}$ (resp. of $(\widetilde{S} \cap E)$) intersect transversally along $\widetilde{S}$ (resp. along $(\widetilde{S} \cap E)$).

By Lemma 2.6, we can assume $S = (G_1 \pitchfork \cdots \pitchfork G_m) \pitchfork (G_{m+1} \pitchfork \cdots \pitchfork G_k)$, $F \subseteq G_1, \ldots, G_m$, and $F \pitchfork G_{m+1}, \ldots, G_k$. Define $A = G_1 \pitchfork \cdots \pitchfork G_m$ and $B = G_{m+1} \pitchfork \cdots \pitchfork G_k$. Then $\widetilde{S} = \widetilde{A} \cap \widetilde{B}$ by Lemma 2.9(v).

Two cases need to be considered: $F \subsetneq A$ and $F = A$. We prove only the first case because the second case can be proved analogously. So assume that $F \subsetneq A$.

First we show that, for all $\widetilde{S} \in \widetilde{\mathcal{S}}$, the $\widetilde{\mathcal{G}}$-factors of $\widetilde{S}$ (resp. of $(\widetilde{S} \cap E)$) intersect transversally along $\widetilde{S}$. Lemma 2.9 implies that

$$\widetilde{S} = \widetilde{G}_1 \pitchfork \cdots \pitchfork \widetilde{G}_k$$

and that $\widetilde{G}_1, \ldots, \widetilde{G}_k$ are all the $\widetilde{\mathcal{G}}$-factors of $\widetilde{S}$. (Indeed, if some $\widetilde{G} \in \widetilde{\mathcal{G}}$ contains $\widetilde{S}$, then $G = \pi(\widetilde{G})$ contains $S = \pi(\widetilde{S})$. Since $G_1, \ldots, G_k$ are all the minimal elements in $\mathcal{G}$ that contain S, it follows that G contains G_r for some $1 \le r \le k$. The inclusion of their dominant transforms still holds: $\widetilde{G} \supseteq \widetilde{G}_r$.) Therefore, the $\widetilde{\mathcal{G}}$-factors of $\widetilde{S}$ intersect transversally.

Next we show that, for all $(\widetilde{S} \cap E) \in \widetilde{\mathcal{S}}$, the $\widetilde{\mathcal{G}}$-factors of $(\widetilde{S} \cap E)$ intersect transversally along $(\widetilde{S} \cap E)$. Observing that

$$\widetilde{S} \cap E = E \pitchfork \widetilde{A} \pitchfork \widetilde{B} = E \pitchfork \widetilde{G}_1 \pitchfork \cdots \pitchfork \widetilde{G}_k,$$

we assert that $E, \widetilde{G}_1, \ldots, \widetilde{G}_k$ are all the $\widetilde{\mathcal{G}}$-factors of $(\widetilde{S} \cap E)$ and so the conclusion follows. Indeed it is enough to show that, given any $\widetilde{G} \in \widetilde{\mathcal{G}}$ containing $(\widetilde{S} \cap E)$, either $\widetilde{G} = E$ or $\widetilde{G} \supseteq \widetilde{G}_r$ for some $1 \le r \le k$. The inclusion $\widetilde{G} \supseteq (\widetilde{S} \cap E)$ implies $G \supseteq (S \cap F)$ if we take the image of π. By Lemma 2.4(ii), we know that $F, G_{m+1}, \ldots, G_k$ are all the $\mathcal{G}$-factors of $(S \cap F)$. Hence G contains either F or one of G_r for $m + 1 \le r \le k$. In the latter case we immediately get the conclusion, so we assume that G contains F. If $G = F$ then $\widetilde{G} = E$, from which the conclusion follows.

Now we assume that $G \supsetneq F$. Since

$$\widetilde{G} \cap E \cong \mathbb{P}(N_F G), \qquad \widetilde{S} \cap E \cong \mathbb{P}(N_F A|_{F \cap B})$$

and since $\widetilde{G} \cap E$ contains $\widetilde{S} \cap E$, we have $(N_F G)_y \supseteq (N_F A)_y$ for any $y \in F \cap B$. But $(N_F G)_y \cong T_{G,y}/T_{F,y}$ and $(N_F A)_y \cong T_{A,y}/T_{F,y}$, so $T_{G,y} \supseteq T_{A,y}$ and G contains A. Since $G_1, \ldots, G_m$ are the $\mathcal{G}$-factors of A (by Lemma 2.4(i)), we have that G contains G_r for some $1 \le r \le m$.

(iii) "$\mathcal{T}$ is a nest $\Rightarrow \widetilde{\mathcal{T}}$ is a nest". Suppose $\mathcal{T}$ is induced by the flag $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k$. If $S_1 \nsubseteq F$ or $S_k \subseteq F$, then $\widetilde{\mathcal{T}}$ is induced by the flag $\widetilde{S}_1 \subseteq \widetilde{S}_2 \subseteq \cdots \subseteq \widetilde{S}_k$; otherwise, there exists an integer m $(1 \le m \le k-1)$ where $S_m \subseteq F$ but $S_{m+1} \nsubseteq F$. In this case it can be easily checked that $\widetilde{\mathcal{T}}$ is generated by the flag

$$(\widetilde{S}_1 \cap \widetilde{S}_{m+1}) \subseteq \cdots \subseteq (\widetilde{S}_m \cap \widetilde{S}_{m+1}) \subseteq (\widetilde{S}_{m+1} \cap E) \subseteq \cdots \subseteq (\widetilde{S}_k \cap E).$$

"$\mathcal{T}$ is a nest $\Leftarrow \widetilde{\mathcal{T}}$ is a nest". Suppose $\widetilde{\mathcal{T}}$ is induced by the flag $S'_1 \subseteq S'_2 \subseteq \cdots \subseteq S'_k$. If $S'_1 \nsubseteq E$, then $\mathcal{T}$ is induced by the flag $\pi(S'_1) \subseteq \pi(S'_2) \subseteq \cdots \subseteq \pi(S'_k)$ and we are done. Now assume that $S'_1 \subseteq E$ and denote by m the maximal integer satisfying $S'_m \subseteq E$. Since E is both minimal and maximal in $\widetilde{\mathcal{G}}$, the E-factorization of S'_i must be of the form $E \cap B'_i$ for $1 \le i \le m$. Then it can be checked that $\mathcal{T}$ is induced by the following flag:

$$(G \cap \pi(B'_1)) \subseteq \pi(B'_1) \subseteq \cdots \subseteq \pi(B'_m) \subseteq \pi(S'_{m+1}) \subseteq \cdots \subseteq \pi(S'_k). \qquad \square$$

Proof of Proposition 3.1. The proof is similar to the proof of Proposition 2.8. The only new case is when F is not minimal in $\mathcal{G}_+$. So, throughout the proof we assume that G_0 is minimal and that $G_0 \subsetneq F$.

We begin by showing that $\widetilde{\mathcal{S}}_+$ is an arrangement. First we prove that the intersection $(\widetilde{G}_0 \cap \widetilde{S}) \cap \widetilde{S}'$ is clean for $S, S' \in \mathcal{S}$. Take the F-factorizations $S = A \cap B$ and $S' = A' \cap B'$ in the arrangement $\mathcal{S}$, and take the G_0-factorization $B = B_1 \cap B_2$ in the arrangement $\mathcal{S}_+$. Similarly to proving Proposition 2.8(i), we must consider three cases.

Case 1: $F \subsetneq A \cap A'$. Then

$$\begin{aligned}(\widetilde{G}_0 \cap \widetilde{S}) \cap \widetilde{S}' &= \widetilde{G}_0 \cap \widetilde{A} \cap \widetilde{B}_1 \cap \widetilde{B}_2 \cap \widetilde{A}' \cap \widetilde{B}' \\ &= \widetilde{G}_0 \cap \widetilde{A} \cap \widetilde{A}' \cap \widetilde{B}_2 \cap \widetilde{B}' \\ &= \widetilde{G}_0 \cap (A \cap A')^{\sim} \cap (B_2 \cap B')^{\sim} \\ &= \widetilde{G}_0 \cap (A \cap A' \cap B_2 \cap B')^{\sim}.\end{aligned}$$

The second equality holds because $\mathcal{I}_{\widetilde{B}_1} = \phi^{-1}\mathcal{I}_{B_1} \subseteq \phi^{-1}\mathcal{I}_{G_0} = \mathcal{I}_{\widetilde{G}_0}$, so $\widetilde{B}_1 \supseteq \widetilde{G}_0$. The third and fourth equalities follow from Lemma 2.9.

Moreover, we have

$$\begin{aligned}T_{\widetilde{G}_0 \cap \widetilde{S} \cap \widetilde{S}'} &= T_{\widetilde{G}_0} \cap T_{\widetilde{A} \cap \widetilde{A}'} \cap T_{\widetilde{B}_2 \cap \widetilde{B}'} \\ &= T_{\widetilde{G}_0} \cap T_{\widetilde{A}} \cap T_{\widetilde{A}'} \cap T_{\widetilde{B}_2} \cap T_{\widetilde{B}'} \\ &= (T_{\widetilde{G}_0} \cap T_{\widetilde{A}} \cap T_{\widetilde{B}_1} \cap T_{\widetilde{B}_2}) \cap (T_{\widetilde{A}'} \cap T_{\widetilde{B}'}) \\ &= T_{\widetilde{G}_0 \cap \widetilde{S}} \cap T_{\widetilde{S}'}.\end{aligned}$$

Case 2: $F = A \cap A'$ but $F \neq A$ and $F \neq A'$. In this case it is easy to verify that $(\widetilde{G}_0 \cap \widetilde{S}) \cap \widetilde{S}' = \emptyset$.

Case 3: $F = A'$. The proof is similar to that for Case 1 and so we omit it.

Similarly, we can check that $(\widetilde{G}_0 \cap \widetilde{S}) \cap (\widetilde{G}_0 \cap \widetilde{S}')$ and $(\widetilde{G}_0 \cap \widetilde{S}) \cap (E \cap \widetilde{S}')$ are clean intersections along some elements in $\widetilde{\mathcal{S}}_+$.

For our second step, we show that $\widetilde{\mathcal{G}}_+$ is a building set. It is enough to demonstrate that the minimal elements in $\widetilde{\mathcal{G}}_+$ that contain $\widetilde{G}_0 \cap \widetilde{S}$ intersect transversally along $\widetilde{G}_0 \cap \widetilde{S}$. Assume the F-factorization of S is $A \cap B$, where $G \subsetneq A$. (If $G = A$, then $\widetilde{G}_0 \subseteq E$ and so we can replace S by B and keep $\widetilde{G}_0 \cap \widetilde{S}$ unchanged.) Assume the G_0-factorization of B is $B_1 \cap B_2$.

We claim that the set of $\widetilde{\mathcal{G}}_+$-factors of $\widetilde{G}_0 \cap \widetilde{S}$ is

$$\mathcal{P} := \{\widetilde{G}_0\} \cup \{\widetilde{\mathcal{G}}\text{-factors of } \widetilde{A}\} \cup \{\widetilde{\mathcal{G}}\text{-factors of } \widetilde{B}_2\}.$$

It is easy to check that the subvarieties in $\mathcal{P}$ intersect transversally. To show that the subvarieties in $\mathcal{P}$ are all the $\widetilde{\mathcal{G}}_+$-factors, it suffices to show that any minimal element $\widetilde{G} \in \widetilde{\mathcal{G}}_+$ that contains $\widetilde{G}_0 \cap \widetilde{S}$ $(= \widetilde{G}_0 \cap \widetilde{A} \cap \widetilde{B}_2)$ belongs to $\mathcal{P}$.

Since $G = \phi(\widetilde{G}) \supseteq \phi(\widetilde{G}_0 \cap \widetilde{A} \cap \widetilde{B}_2) = G_0 \cap B_2$ and since the set of $\mathcal{G}_+$-factors of $G_0 \cap B_2$ consists of G_0 and all the $\mathcal{G}$-factors of B_2, we have either $G \supseteq G_0$ or $G \supseteq B_2$. If $G \supseteq B_2$ then the conclusion follows, so we can assume that $G \supseteq G_0$. Then either $G \pitchfork F$ or $G \supseteq F$.

If $G \pitchfork F$, then $G \supseteq G_0$ implies $\widetilde{G} \supseteq \widetilde{G}_0$. Since $\widetilde{G}$ is chosen to be minimal, it follows that $\widetilde{G} = \widetilde{G}_0$ belongs to $\mathcal{P}$. If $G \supseteq F$, then $\widetilde{G} = \mathbb{P}(N_F G)$ and $\widetilde{G}_0 \cap \widetilde{S} = \mathbb{P}(N_F A|_{G_0 \cap B})$. Hence $G \supseteq A$, which implies that G is a $\mathcal{G}$-factor of A and that $\widetilde{G}$ is a $\widetilde{\mathcal{G}}$-factor of $\widetilde{A}$. Therefore, $\widetilde{G} \in \mathcal{P}$. □

5.3. Codimensions of the Centers

(Thanks to the referee for suggesting this question.) In the original construction of the Fulton–MacPherson configuration space $X[n]$, each blow-up is along a nonsingular center of codimension m or $m+1$, where m is the dimension of X. But if we construct $X[n]$ by blowing up centers in ascending dimension, then the codimensions of the centers are much larger: the first blow-up is along the smallest diagonal $\Delta_{[n]}$, which is of codimension $m(n-1)$.

In general, given a specified order of blow-ups, we can easily find the dimension (and hence the codimension) of the centers.

PROPOSITION 5.3. *Let Y be a nonsingular variety and let $\mathcal{G} = \{G_1, \dots, G_N\}$ be a nonempty building set of subvarieties of Y satisfying the condition* $(*)$ *in Theorem 1.3. Let j be an integer between* 1 *and N. Define $\mathcal{G}' := \{G_1, \dots, G_{j-1}\}$, and define $\mathcal{F} = \{G_{i_1}, G_{i_2}, \dots, G_{i_\ell}\}$ to be the minimal elements of $\{G \in \mathcal{G}' : G \supseteq G_j\}$.*

Then, in the construction of $Y_{\mathcal{G}}$ by blowing up along $G_1, \dots, G_N$ in order, the center of the jth *blow-up is of dimension*

$$\dim G_j + \sum_{k=1}^{\ell} (d - 1 - \dim G_{i_k})$$

if $\mathcal{F} \neq \emptyset$ and is of dimension $\dim G_j$ *if $\mathcal{F} = \emptyset$.*

Proof. The set $\mathcal{G}' := \{G_1, \dots, G_{j-1}\}$ is a building set by condition $(*)$. By Theorem 1.3, we can assume that $G_1, \dots, G_{j-1}$ is in order of ascending dimension. Denote by Y_i the variety obtained after the ith blow-up. We want to find the dimension of $\widetilde{G}_j$ in Y_{j-1}.

Since blowing up a center that does not contain $\widetilde{G}_j$ will not change the dimension of $\widetilde{G}_j$ and since $\widetilde{G}$ does not contain $\widetilde{G}_j$ if G does not contain G_j, we can focus on the subset $\mathcal{G}'' \subseteq \mathcal{G}'$ of subvarieties that contain G_j. Let $\mathcal{F} := \{G_{i_1}, G_{i_2}, \dots, G_{i_\ell}\}$ be the set of minimal elements in $\mathcal{G}''$. Define $S := G_{i_1} \cap G_{i_2} \cap \dots \cap G_{i_\ell}$. Then $\mathcal{F}$ is also the set of minimal elements in $\mathcal{G}'$ that contain S. By the definition of building set, $G_{i_1} \dots G_{i_\ell}$ intersect transversally. A subvariety $G \in \mathcal{G}'' \setminus \mathcal{F}$ must contain a subvariety (say, G_i) in $\mathcal{F}$. Then $G \supsetneq G_i \supsetneq G_j$, and in the variety Y_{i-1} we have $\widetilde{G} \supsetneq \widetilde{G}_i \supsetneq \widetilde{G}_j$. It can easily be checked that, in the variety Y_j (obtained by blowing up along $\widetilde{G}_i$), $\widetilde{G}$ will no longer contain $\widetilde{G}_j$; hence the blow-up along $\widetilde{G}$ will not change the dimension of $\widetilde{G}_j$. In other words, we only need to find the change of $\dim \widetilde{G}_j$ for the blow-ups along the transversal subvarieties in $\mathcal{F}$, which is simply

$$\sum_{k=1}^{\ell} (d - 1 - \dim G_{i_k}). \qquad \square$$

EXAMPLE. Let $m = \dim X$. In the original construction of the Fulton–MacPherson configuration space $X[4]$ (cf. Section 4.2), the dimension of the first center is $\dim \Delta_{12} = 3m$ and the dimension of the second center, $\widetilde{\Delta}_{123}$, is

$$\dim \Delta_{123} + (4m - 1 - \dim \Delta_{12}) = 2m + (m - 1) = 3m - 1.$$

It can be easily checked that $\dim \widetilde{\Delta}_I$ is $3m$ if $|I| = 2$ and is $3m - 1$ otherwise, so the codimension is either m or $m + 1$. In general, when using the order of blow-ups in the orginal construction of $X[n]$ for $n \geq 2$, the codimension of each center $\widetilde{\Delta}_I$ is m if $|I| = 2$ and is $m + 1$ otherwise.

EXAMPLE. The codimension is 2 for each blow-up center in Keel's construction of $\overline{M}_{0,n}$. Since the blow-ups in Keel's construction can be obtained by restricting the blow-ups in the original Fulton–MacPherson construction of $X[n]$ to a fiber of π_{123} (defined in Corollary 1.4), it follows that each blow-up center is of codimension $m = 1$ or $m + 1 = 2$. But blowing up along a center of codimension 1 does nothing, so we need only carry out blow-ups along codimension-2 centers.

5.4. The Statements for a General Arrangement

DEFINITION 5.4. An *arrangement* of subvarieties of a nonsingular variety Y is a finite set $\mathcal{S} = \{S_i\}$ of nonsingular closed subvarieties S_i, properly contained in Y, that satisfy the following conditions:

(i) S_i and S_j intersect cleanly;
(ii) $S_i \cap S_j$ either is equal to a disjoint union of some S_k or is empty.

DEFINITION 5.5. Let $\mathcal{S}$ be an arrangement of subvarieties of Y. A subset $\mathcal{G} \subseteq \mathcal{S}$ is called a *building set* of $\mathcal{S}$ if there is an open cover $\{U_i\}$ of Y such that (a) the restriction of the arrangement $\mathcal{S}|_{U_i}$ is simple for each i and (b) $\mathcal{G}|_{U_i}$ is a building set of $\mathcal{S}|_{U_i}$.

A finite set $\mathcal{G}$ of nonsingular subvarieties of Y is called a building set if the set of all possible intersections of collections of subvarieties from $\mathcal{G}$ forms an arrangement $\mathcal{S}$ and if $\mathcal{G}$ is a building set of $\mathcal{S}$ as defined previously. In this situation, $\mathcal{S}$ is called the *arrangement induced by* $\mathcal{G}$.

Definition 5.6. A subset $\mathcal{T} \subseteq \mathcal{G}$ is called *$\mathcal{G}$-nested* (or a *$\mathcal{G}$-nest*) if there is an open cover $\{U_i\}$ of Y such that (a) the restriction of the arrangement induced by $\mathcal{G}$ to each U_i is simple and (b) $\mathcal{T}|_{U_i}$ is a $\mathcal{G}|_{U_i}$-nest.

We define the wonderful compactification as in Definition 1.1. Then Theorem 1.2 and Theorem 1.3 still hold.

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