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Dirac Operators with Periodic δ-Interactions: Spectral Gaps and Inhomogeneous Diophantine Approximation

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1. Introduction and Summary

Let $\kappa \in (0, 2\pi)$, $\Gamma = \{0, \kappa\} + 2\pi\mathbf{Z}$, $m \geq 0$, and $\beta \in \mathbf{R} \setminus \{0\}$. Let σ_1 and σ_3 stand for the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We are concerned with the spectrum of the Dirac operator H in $(L^2(\mathbf{R}))^2$, which is defined as

$$(Hf)(x) = -i\sigma_1 \frac{d}{dx} f(x) + m\sigma_3 f(x), \quad x \in \mathbf{R} \setminus \Gamma;$$

$$\begin{aligned} \mathrm{Dom}(H) = \Bigl\{ \bigl(\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\bigr) \Bigm| f_1 \in H^1(\mathbf{R}),\ f_2 \in H^1(\mathbf{R} \setminus \Gamma), \\ f_2(x+0) - f_2(x-0) = -i\beta f_1(x) \text{ for } x \in \Gamma \Bigr\}. \end{aligned}$$

The operator H is self-adjoint, and the spectrum of H has the band structure. The purpose of this paper is to establish a relationship between the asymptotic behavior of the spectral gaps of H and the number-theoretical properties of parameters involved in H.

In order to formulate our main result, we describe basic spectral properties of the operator H. Toward this end, we first introduce the discriminant of H, which plays the most fundamental role in the analysis of the spectrum of H (cf. [8; 11; 14; 16, Sec. XIII]). For a parameter $\lambda \in \mathbf{R}$, let $M(\lambda, x) \in M_2(\mathbf{C})$ stand for the solution to the equations

$$\begin{cases} \bigl(-i\sigma_1 \frac{d}{dx} + m\sigma_3\bigr) Y(x) = \lambda Y(x), & x \in \mathbf{R} \setminus \Gamma, \\ Y(x+0) = \bigl(\begin{smallmatrix} 1 & 0 \\ -i\beta & 1 \end{smallmatrix}\bigr) Y(x-0), & x \in \Gamma, \end{cases}$$

subject to the initial condition

$$Y(+0) = I,$$

where I is the 2×2 identity matrix. We call $M(\lambda, x)$ the *monodromy matrix* of H. The discriminant of H is defined as

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$$d(\lambda) = \operatorname{Tr} M(\lambda, 2\pi + 0).$$

We have $\lambda \in \sigma(H)$ if and only if $|d(\lambda)| \leq 2$. The basic spectral properties of H are summarized as follows, which we demonstrate in Section 2.

PROPOSITION 1.1. (i) *There exists a unique pair of real sequences* $\{\lambda_j^+\}_{j=-\infty}^{\infty}$ *and* $\{\lambda_j^-\}_{j=-\infty}^{\infty}$ *such that*:

(a) $\{\lambda_j^+\}_{j=-\infty}^{\infty}$ *gives all the zeros of* $d(\cdot) - 2$ *repeated according to multiplicity, while* $\{\lambda_j^-\}_{j=-\infty}^{\infty}$ *gives all the zeros of* $d(\cdot) + 2$ *repeated according to multiplicity*;

(b) $\lambda_{2j}^- \leq \lambda_{2j+1}^- < \lambda_{2j+1}^+ \leq \lambda_{2j+2}^+ < \lambda_{2j+2}^-$ *for all* $j \in \mathbf{Z}$; *and*

(c)
$$\begin{array}{ll} -m = \lambda_{-1}^+ < \lambda_0^+ & \text{if } \beta > -2\pi m, \\ -m = \lambda_{-1}^+ = \lambda_0^+ & \text{if } \beta = -2\pi m, \\ \lambda_{-1}^+ < \lambda_0^+ = -m & \text{if } \beta < -2\pi m. \end{array}$$

(ii) *The spectrum of* H *is expressed as*

$$\sigma(H) = \bigcup_{j=-\infty}^{\infty} B_j,$$

where

$$B_j = \begin{cases} [\lambda_j^+, \lambda_j^-] & \text{for } j \text{ even}, \\ [\lambda_j^-, \lambda_j^+] & \text{for } j \text{ odd}. \end{cases}$$

We call the closed interval B_j the jth *band* of $\sigma(H)$. Let G_j be the open interval between B_j and B_{j+1}:

$$G_j = \begin{cases} (\lambda_j^-, \lambda_{j+1}^-) & \text{for } j \text{ even}, \\ (\lambda_j^+, \lambda_{j+1}^+) & \text{for } j \text{ odd}. \end{cases} \tag{1.1}$$

We call G_j the jth *gap* of $\sigma(H)$. We denote by $|G_j|$ the length of G_j.

We also introduce some notation. Let

$$\tau = 2\pi - \kappa,$$

$$\kappa_0 = \frac{\tau}{\kappa},$$

$$\theta = \frac{1 - \kappa_0}{\pi} \tan^{-1}\left(\frac{2}{\beta}\right),$$

$$X = \frac{2(\tau - \kappa) m\beta}{\pi^2(\beta^2 + 4)},$$

$$W = \frac{2\pi^2 |\beta| \sqrt{4 + \beta^2}}{4\pi^2 + \beta^2 \kappa\tau},$$

where $\tan^{-1}(\cdot)$ stands for the inverse function of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \ni y \mapsto \tan y \in \mathbf{R}$. Henceforth, we assume that κ_0 is irrational. Throughout this paper we employ the following convention for the sake of brevity. A sentence that contains either $\pm$

or $\mp$ is meant to express two sentences, one for the upper signs and the other for the lower signs. For example, $a \pm b < \mp c$ means two inequalities $a + b < -c$ and $a - b < c$. For a real number x, let $\|x\|$ stand for the difference, taken positively, between x and the nearest integer; that is, $\|x\| = \min\{|x - n| \mid n \in \mathbf{Z}\}$. For $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ and $\gamma \in \mathbf{R}$, we define

$$M_\pm(\alpha, \gamma) = \liminf_{q \to \infty} q\|\pm q\alpha + \gamma\| \quad \text{and}$$

$$M(\alpha, \gamma) = \min\{M_+(\alpha, \gamma), M_-(\alpha, \gamma)\},$$

where q runs through $\mathbf{N}$. These are called *approximation constants* in the theory of Diophantine approximation. We also define

$$A_\pm(\alpha, \gamma) = \{q(q\alpha + \gamma + p) \mid \pm q \in \mathbf{N},\ p \in \mathbf{Z}\}.$$

By $A'_\pm(\alpha, \gamma)$ we designate the set of the accumulation points of $A_\pm(\alpha, \gamma)$. Let $A'(\alpha, \gamma) = A'_+(\alpha, \gamma) \cup A'_-(\alpha, \gamma)$. The sets $A'_\pm(\alpha, \gamma)$ and $A'(\alpha, \gamma)$ are closely related with the approximation constants; we have

$$\min\{|x| \mid x \in A'_\pm(\alpha, \gamma)\} = M_\pm(\alpha, \gamma), \tag{1.2}$$

$$\min\{|x| \mid x \in A'(\alpha, \gamma)\} = M(\alpha, \gamma). \tag{1.3}$$

Our main result is the following theorem, which we prove in Section 3.

THEOREM 1.2.

$$\left\{\lim_{k \to \infty} f_k \;\middle|\; \{f_k\}_{k=1}^\infty \textit{ is a convergent subsequence of } \{j|G_{\pm j}|\}_{j=1}^\infty\right\} = \{W|y + X| \mid y \in A'_\pm(\kappa_0, \theta)\}. \tag{1.4$\pm$}$$

We also obtain the following result, which gives the aforementioned relationship more instantly.

COROLLARY 1.3. *If $m = 0$, then*

$$\liminf_{j \to \infty} j|G_{\pm j}| = WM_\pm(\kappa_0, \theta).$$

We note that $M_\pm(\alpha, \gamma)$ and $M(\alpha, \gamma)$ are called *inhomogeneous* approximation constants when γ is not of the form $\gamma = q\alpha + p$ for integers q, p. We also note that if $\gamma = q\alpha + p$ for some integers q, p, then the approximation constants reduce to the homogeneous one: $M_+(\alpha, \gamma) = M_-(\alpha, \gamma) = M(\alpha, 0)$. Because (1.2) and (1.3) as well as Corollary 1.3 generally involve the inhomogeneous approximation constants, it is useful to recall here their basic properties.

REMARK 1.4. (a) The most fundamental result in the theory of inhomogeneous Diophantine approximation is the Minkowski theorem, which states that if $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ and if $\gamma \in \mathbf{R}$ is not of the shape $\gamma = q\alpha + p$ for integers q, p, then $M(\alpha, \gamma) \le \frac{1}{4}$.

(b) Grace [9] has proved that the Minkowski theorem is optimal in the following sense: there exist an irrational α_0 and a γ_0 not of the form $q\alpha_0 + p$ for integers q, p such that $M(\alpha_0, \gamma_0) = \frac{1}{4}$.

(c) Cassels [4] has shown that if $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ and if $\gamma \in \mathbf{R}$ is not of the form $q\alpha + p$ for integers q, p, then $M_+(\alpha, \gamma) \le \frac{4}{11}$ *except* when α and γ have the forms

$$\alpha = \frac{A\omega + B}{C\omega + D}, \qquad (AD - BC)\gamma = \frac{-3\omega - 7 + 14E + 14F\omega}{14|C\omega + D|},$$

where A, B, C, D, E, and F are integers with $AD - BC \in \{1, -1\}$ and $\omega = \sqrt{7}$. Moreover, Cassels proved that, in this exceptional case, $M_+(\alpha, \gamma) = \frac{27}{28\sqrt{7}}$ $\left(> \frac{4}{11} > \frac{1}{4}\right)$.

For further results in the inhomogeneous approximation theory, we refer to [5, Chap. III; 7; 12; 17, Chap. IV, Sec. 9; 19] and the references therein. For the basic results in the homogeneous approximation theory, we consult [5; 13; 17].

Turning our attention to the Diophantine matters involved in Theorem 1.2, we establish the following two theorems.

THEOREM 1.5. *Let $\{a_i\}_{i=1}^{\infty}$ be a sequence of positive even integers that satisfies $a_i \to \infty$ as $i \to \infty$. Put*

$$\alpha = \frac{1}{a_1+}\frac{1}{a_2+}\frac{1}{a_3+}\cdots$$

and $\gamma = \frac{1}{2}(\alpha - 1)$. Then

$$A'_{\pm}(\alpha, \gamma) = \left\{ \frac{2j - 1}{4} \;\middle|\; j \in \mathbf{Z} \right\}.$$

THEOREM 1.6. *Suppose that α is a real quadratic irrationality—that is, suppose there exist integers a, b, and c such that $a\alpha^2 + b\alpha + c = 0$, $a \ne 0$, $d := b^2 - 4ac > 0$, and d is not a square. Let p, q, and r be integers such that $p \ge 1$. Put $\gamma = \frac{q\alpha + r}{p}$ and*

$$F = \{k \in \mathbf{Z} \setminus \{0\} \mid \text{there exists } (x, y) \in \mathbf{Z}^2 \text{ such that } ax^2 + bxy + cy^2 = k, \\ x \equiv -r \pmod{p}, \text{ and } y \equiv q \pmod{p}\}.$$

Then

$$A'(\alpha, \gamma) = \left\{ \pm \frac{k}{p^2\sqrt{d}} \;\middle|\; k \in F \right\} \quad \text{for } \alpha = \frac{-b \mp \sqrt{d}}{2a}.$$

We note that the pair α, γ in Theorem 1.5 has been utilized in the proof of the optimality of the Minkowski theorem; see [5, Chap. III, Thm. IIB] and [17, Chap. IV, Sec. 9, Thm. 2]. Theorem 1.6 gives a representation of $A'(\alpha, \gamma)$ in the case where both α and γ belong to a quadratic number field, which is useful for the computation of $A'(\alpha, \gamma)$ (see Example 5.3). We prove Theorems 1.5 and 1.6 in Sections 4 and 5, respectively.

One of the motivations for our work here stems from [21], where we discussed the Schrödinger operator formally expressed as

$$L = -\frac{d^2}{dx^2} + \sum_{l=-\infty}^{\infty} (\beta_1 \delta'(x - \kappa - 2\pi l) + \beta_2 \delta'(x - 2\pi l)) \text{ in } L^2(\mathbf{R})$$

and obtained a connection between the asymptotic nature of its spectrum and the theory of *homogeneous* Diophantine approximation. We emphasize that the spectrum of the Dirac operator H has a richer structure than that of the Schrödinger operator L, in the sense that Theorem 1.2 relates the asymptotic behavior of the spectrum of H to either the inhomogeneous or homogeneous approximation theory according as the shape of θ.

We also stress that the proof of the main result in this paper is entirely different from that in [21]. The proof in [21] relies heavily on the fact that the discriminant of L, denoted by $D(\lambda)$, has a simple leading term $\beta_1\beta_2\lambda\sin\kappa\sqrt{\lambda}\sin\tau\sqrt{\lambda}$, which enabled us to approximate the zeros of $D(\lambda)-2$ and $D(\lambda)+2$ by those of $\sin\kappa\sqrt{\lambda}\sin\tau\sqrt{\lambda}$. In the case of the Dirac operator H, however, the situation is considerably complicated because the discriminant of the operator H does not admit such a simple approximation, since all the terms in $d(\lambda)$ have an equal magnitude; see (2.1). We eliminate this difficulty by using a new geometric argument. In order to introduce it, we sketch the proof of Theorem 1.2. Developing the old ideas of Meissner [15] and Hochstadt [10], we obtain a geometric characterization of the gaps in Lemma 3.6. Utilizing this characterization, we reduce the problem to the analysis of a geometric relation between a straight line L and a quasi-lattice $\{V_{q,p}\}$; see Lemmas 3.8 and 3.10. Combining these results with the precise asymptotic expansion of the points $V_{q,p}$ (see Lemmas 3.3 and 3.4), we complete the proof.

The relativistic δ-interaction was first introduced by Gesztesy and Šeba [8]. Among other things they studied the Dirac operator in $(L^2(\mathbf{R}))^2$ of the form

$$(Df)(x) = -ic\sigma_1\frac{d}{dx}f(x) + \frac{c^2}{2}\sigma_3 f(x), \quad x\in\mathbf{R}\setminus a\mathbf{Z},$$

$$\begin{aligned}\mathrm{Dom}(D) = \Big\{\tbinom{f_1}{f_2} \;\Big|\; & f_1\in H^1(\mathbf{R}),\ f_2\in H^1(\mathbf{R}\setminus a\mathbf{Z}),\\ & f_2(x+0)-f_2(x-0) = -(i\beta/c)f_1(x) \text{ for } x\in a\mathbf{Z}\Big\},\end{aligned}$$

where $a, c>0$ and $\beta\in\mathbf{R}\setminus\{0\}$, and proved that the length of the jth gap of $\sigma(D)$ admits the asymptotic expansion of the form

$$\frac{2c}{a}\tan^{-1}\frac{|\beta|}{2c} + \mathcal{O}(j^{-1}) \quad \text{as } j\to\pm\infty. \tag{1.5}$$

Our work here is motivated by [8] as well as [21]. In contrast to (1.5), our results involve number-theoretical objects. We note that there are many works concerning relativistic point interactions; see [1; 2; 3; 8; 11; 18] and the references therein.

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2. Proof of Proposition 1.1

Let

$$f_\pm(\lambda) = \sqrt{\lambda\pm m}.$$

By a direct calculation, we get

$$\begin{aligned} d(\lambda) &= 2\cos\tau f_+(\lambda) f_-(\lambda)\cos\kappa f_+(\lambda) f_-(\lambda) \\ &\quad + 2\beta\frac{f_+(\lambda)}{f_-(\lambda)}\sin\tau f_+(\lambda) f_-(\lambda)\cos\kappa f_+(\lambda) f_-(\lambda) \\ &\quad + 2\beta\frac{f_+(\lambda)}{f_-(\lambda)}\cos\tau f_+(\lambda) f_-(\lambda)\sin\kappa f_+(\lambda) f_-(\lambda) \\ &\quad - \left(2-\beta^2\left(\frac{f_+(\lambda)}{f_-(\lambda)}\right)^2\right)\sin\tau f_+(\lambda) f_-(\lambda)\sin\kappa f_+(\lambda) f_-(\lambda). \end{aligned} \tag{2.1}$$

We have $d(-m) = 2$ and $d'(-m) = 4\pi(2\pi m + \beta)$. Furthermore, we infer that if $d'(-m) = 0$ then

$$d''(-m) = -4\pi^2 + \frac{4m^2\pi^2}{3}(3\kappa\tau - 4\pi^2) \le -4\pi^2 - \frac{4m^2\pi^4}{3} < 0.$$

Using these relations and the standard argument employed in the proof of [16, Thm. XIII.89(e)] (see also [8; 11; 20]), we have the implication.

3. Proof of Theorem 1.2

First, we reduce the function $d(\lambda)-2$. We define $x(\lambda) = f_+(\lambda) f_-(\lambda)$ and $r(\lambda) = f_+(\lambda)/f_-(\lambda)$. By a straightforward computation we obtain

$$\begin{aligned} d(\lambda) - 2 &= -4\cos^2\frac{\tau}{2}x(\lambda)\cos^2\frac{\kappa}{2}x(\lambda)\Big(\tan\frac{\tau}{2}x(\lambda) + \tan\frac{\kappa}{2}x(\lambda) - \beta r(\lambda)\Big) \\ &\quad \times \Big(\tan\frac{\tau}{2}x(\lambda) + \tan\frac{\kappa}{2}x(\lambda) + \beta r(\lambda)\tan\frac{\tau}{2}x(\lambda)\tan\frac{\kappa}{2}x(\lambda)\Big), \end{aligned} \tag{3.1}$$

provided that $\cos\frac{\tau}{2}x(\lambda)\cos\frac{\kappa}{2}x(\lambda) \ne 0$. Let μ stand for the inverse function of $[m,\infty) \ni \lambda \mapsto \frac{\kappa}{2}x(\lambda) \in [0,\infty)$; that is,

$$\mu(y) = \sqrt{\left(\tfrac{2}{\kappa}y\right)^2 + m^2}, \quad y \in [0,\infty).$$

Let $\tilde{d}(y) = d(\mu(y))$ and $h(y) = r(\mu(y))$. It then follows that

$$\begin{aligned} \tilde{d}(y) - 2 &= -4\cos^2\kappa_0 y\cos^2 y(\tan\kappa_0 y + \tan y - \beta h(y)) \\ &\quad \times ((1+\beta h(y)\tan y)\tan\kappa_0 y + \tan y) \end{aligned} \tag{3.2}$$

if $\cos\kappa_0 y\cos y \ne 0$. We define

$$Z_1 = \left\{-\tfrac{\pi}{2} + n\pi \;\middle|\; n \in \mathbf{N}\right\},$$

$$Z_2 = \{y \in [0,\infty) \setminus Z_1 \mid 1 + \beta h(y)\tan y = 0\};$$

$$F_1(y) = \begin{cases} \tan^{-1}(\tan y - \beta h(y)), & y \in [0,\infty) \setminus Z_1, \\ \frac{\pi}{2}, & y \in Z_1, \end{cases} \tag{3.3}$$

$$F_2(y) = \begin{cases} \tan^{-1}\frac{\tan y}{1+\beta h(y)\tan y}, & y \in [0,\infty) \setminus (Z_1 \cup Z_2), \\ \tan^{-1}\frac{1}{\beta h(y)}, & y \in Z_1, \\ \frac{\pi}{2}, & y \in Z_2. \end{cases} \tag{3.4}$$

We obtain the following result.

LEMMA 3.1. *We have $\tilde{d}(y) - 2 = 0$ if and only if either $-\kappa_0 y \equiv F_1(y) \pmod{\pi}$ or $-\kappa_0 y \equiv F_2(y) \pmod{\pi}$.*

Proof. In the case where $\cos\kappa_0 y \cos y \neq 0$, we have the claim by (3.1). Using the continuity of $\tilde{d}$ and (3.1), we obtain

$$\tilde{d}(y) - 2 = \begin{cases} -4\cos\kappa_0 y(\beta h(y)\sin\kappa_0 y + \cos\kappa_0 y) & \text{if } \cos y = 0, \\ -4\cos y(\cos y + \beta h(y)\sin y) & \text{if } \cos\kappa_0 y = 0. \end{cases}$$

This combined with $\beta h(y) \neq 0$ yields the claim for $\cos\kappa_0 y\cos y = 0$. □

Next, we reduce the function $d(\lambda) + 2$. We have

$$\begin{aligned}\tilde{d}(y) + 2 = 4\sin^2\kappa_0 y\cos^2 y(\cot\kappa_0 y - \tan y + \beta h(y)) \\ \times((1 + \beta h(y)\tan y)\cot\kappa_0 y - \tan y)\end{aligned}$$

provided that $\sin\kappa_0 y\cos y \neq 0$. Combining this with $\cot s = -\tan\left(s + \frac{\pi}{2}\right)$, we obtain the following implication.

LEMMA 3.2. *We have $\tilde{d}(y) + 2 = 0$ if and only if either $-\kappa_0 y \equiv F_1(y) + \frac{\pi}{2} \pmod{\pi}$ or $-\kappa_0 y \equiv F_2(y) + \frac{\pi}{2} \pmod{\pi}$.*

The next step is to analyze the intersection points of the curves $x = F_1(y)$ and $x = F_2(y)$ in the yx-plane. Toward this end, we investigate the roots of $F_1(y) = F_2(y)$, or

$$\tan y = \tfrac{1}{2}\left(\beta h(y) \pm \sqrt{\beta^2 h(y)^2 + 4}\right). \tag{3.5±}$$

We define

$$\xi^{\pm} = \tan^{-1}\left(\frac{\beta \pm \sqrt{\beta^2 + 4}}{2}\right).$$

LEMMA 3.3. *There exists an $n_0 \in \mathbf{N}$ such that, for any integer $n \geq n_0$, equation (3.5±) has a unique root $\xi_n^{\pm}$ on the interval $\left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right)$. Moreover, we have*

$$\xi_n^{\pm} = \xi^{\pm} + n\pi + \frac{\kappa m\beta}{2\pi(\beta^2 + 4)} \cdot \frac{1}{n} + o\left(\frac{1}{n}\right) \quad \textit{as } n \to \infty. \tag{3.6±}$$

Proof. Let $K_{\pm}(y)$ stand for the right-hand side of (3.5±). Put $L_{\pm}(y) = \tan y - K_{\pm}(y)$. A straightforward calculation gives

$$h(y) = 1 + \frac{m\kappa}{2} \cdot \frac{1}{y} + \mathcal{O}\left(\frac{1}{y^2}\right), \tag{3.7}$$

$$h'(y) = \mathcal{O}\left(\frac{1}{y^2}\right) \tag{3.8}$$

as $y \to \infty$, so that

$$K'_{\pm}(y) = \frac{1}{2}\beta h'(y) \pm \frac{\beta^2 h(y) h'(y)}{2\sqrt{\beta^2 h(y)^2 + 4}} = \mathcal{O}\left(\frac{1}{y^2}\right) \quad \text{as } y \to \infty.$$

Combining this with $L'_\pm(y) = 1 + \tan^2 y - K'_\pm(y)$, we infer that there exists an $n_0 \in \mathbf{N}$ such that $L'_\pm(y) \geq \frac{1}{2}$ on $(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi)$ for any integer $n \geq n_0$. For each integer n, we have $L_\pm(y) \to -\infty$ as $y \downarrow -\frac{\pi}{2} + n\pi$ while $L_\pm(y) \to \infty$ as $y \uparrow -\frac{\pi}{2} + n\pi$. So, we get the first assertion.

Next, we prove (3.6+). By (3.5+) we have

$$\tan \xi_n^+ - \tan \xi^+ = \frac{1}{2}\left\{\beta(h(\xi_n^+) - 1) + \sqrt{\beta^2+4}\left(\sqrt{1 + \frac{\beta^2}{\beta^2+4}(h(\xi_n^+)^2 - 1)} - 1\right)\right\}. \tag{3.9}$$

Combining this with the first assertion, (3.7), and $\sqrt{1+x} = 1 + \mathcal{O}(x)$ as $x \to 0$, we deduce that $\tan \xi_n^+ - \tan \xi^+ = \mathcal{O}\left(\frac{1}{n}\right)$ as $n \to \infty$; from this it follows that

$$|\xi_n^+ - (\xi^+ + n\pi)| \leq |\tan \xi_n^+ - \tan(\xi^+ + n\pi)| = \mathcal{O}\left(\frac{1}{n}\right). \tag{3.10}$$

We define $C = \kappa m\beta/\{2\pi(\beta^2+4)\}$ and $f_n = \xi_n^+ - \left(\xi^+ + n\pi + \frac{C}{n}\right)$. We infer by (3.10) that $f_n = \mathcal{O}\left(\frac{1}{n}\right)$, whereupon

$$\begin{aligned}\text{LHS of (3.9)} &= \tan\left(\xi^+ + \frac{C}{n} + f_n\right) - \tan \xi^+ \\ &= (1 + \tan^2 \xi^+)\left(\frac{C}{n} + f_n\right) + o\left(\frac{1}{n}\right).\end{aligned} \tag{3.11}$$

Using the first claim, (3.7), and $\sqrt{1+x} - 1 = \frac{1}{2}x + o(x)$ as $x \to 0$, we have

$$\begin{aligned}\text{RHS of (3.9)} &= \frac{1}{2}\beta \cdot \frac{m\kappa}{2\xi_n^+} + \frac{\beta^2}{2\sqrt{\beta^2+4}} \cdot \frac{m\kappa}{2\xi_n^+} + o\left(\frac{1}{n}\right) \\ &= \left(\beta + \frac{\beta^2}{\sqrt{\beta^2+4}}\right)\frac{\kappa m}{4\pi} \cdot \frac{1}{n} + o\left(\frac{1}{n}\right).\end{aligned} \tag{3.12}$$

By (3.11) and (3.12) we obtain

$$f_n = \left(\frac{1}{1+\tan^2 \xi^+}\left(\beta + \frac{\beta^2}{\sqrt{\beta^2+4}}\right)\frac{\kappa m}{4\pi} - C\right)\frac{1}{n} + o\left(\frac{1}{n}\right). \tag{3.13}$$

Substituting $\tan \xi^+ = \left(\beta + \sqrt{\beta^2+4}\right)/2$ and $C = \kappa m\beta/\{2\pi(\beta^2+4)\}$ for (3.13), we have $f_n = o\left(\frac{1}{n}\right)$, from which (3.6+) follows. In a similar fashion, we get (3.6−). □

In the next lemma, we analyze the x-coodinates of the intersection points.

LEMMA 3.4.

$$F_1(\xi_n^\pm) = -\xi^\mp - \frac{\kappa m\beta}{2\pi(\beta^2+4)} \cdot \frac{1}{n} + o\left(\frac{1}{n}\right) \quad \textit{as } n \to \infty. \tag{3.14$\pm$}$$

Proof. First, we prove (3.14+). By (3.7) and Lemma 3.3 we have

$$\begin{aligned}
&\tan\xi_n^+ - \beta h(\xi_n^+)\\
&\quad = \tan\xi^+ + (1+\tan^2\xi^+)(\xi_n^+ - n\pi - \xi^+) - \beta - \frac{\kappa m\beta}{2\pi}\cdot\frac{1}{n} + o\left(\frac{1}{n}\right)\\
&\quad = \tan\xi^+ - \beta - \frac{\kappa m\beta}{2\pi}\left(-\frac{1+\tan^2\xi^+}{\beta^2+4}+1\right)\frac{1}{n} + o\left(\frac{1}{n}\right)\\
&\quad = \tan\xi^+ - \beta + \frac{\kappa m\beta\tan\xi^-}{2\pi\sqrt{\beta^2+4}}\cdot\frac{1}{n} + o\left(\frac{1}{n}\right).
\end{aligned}$$

Hence, we obtain

$$F_1(\xi_n^+) = \tan^{-1}(\tan\xi^+ - \beta) + \frac{1}{1+(\tan\xi^+ - \beta)^2}\cdot\frac{\kappa m\beta\tan\xi^-}{2\pi\sqrt{\beta^2+4}}\cdot\frac{1}{n} + o\left(\frac{1}{n}\right),$$

from which (3.14+) follows. Likewise, we get (3.14−). □

Next, we investigate the asymptotic behavior of $(F_1)'(y)$ and $(F_2)'(y)$ as $y\to\infty$. We introduce a parameter $c>0$ and put $J^\pm(c,n) = [\xi_n^\pm - \frac{c}{n}, \xi_n^\pm + \frac{c}{n}]$.

LEMMA 3.5.

$$(F_1)'(y) = \frac{1+\tan^2 y}{1+(\tan y - \beta)^2} + \mathcal{O}\left(\frac{1}{y}\right) \quad \textit{as}\ \ y\to\infty, \tag{3.15}$$

$$(F_2)'(y) = \frac{1+\tan^2 y}{(1+\beta\tan y)^2+\tan^2 y} + \mathcal{O}\left(\frac{1}{y}\right) \quad \textit{as}\ \ y\to\infty, \tag{3.16}$$

$$\max_{y\in J^\pm(c,n)}\left|(F_1)'(y) - \frac{2+\beta^2\pm\beta\sqrt{\beta^2+4}}{2}\right| = \mathcal{O}\left(\frac{1}{n}\right) \quad \textit{as}\ \ n\to\infty, \tag{3.17}$$

$$\max_{y\in J^\pm(c,n)}\left|(F_2)'(y) - \frac{2}{2+\beta^2\pm\beta\sqrt{\beta^2+4}}\right| = \mathcal{O}\left(\frac{1}{n}\right) \quad \textit{as}\ \ n\to\infty. \tag{3.18}$$

Proof. We have

$$(F_2)'(y) = \frac{1+\tan^2 y - \beta h'(y)\tan^2 y}{(1+\beta h(y)\tan y)^2+\tan^2 y}.$$

Combining this with (3.7) and (3.8), we get (3.16). Similarly, we obtain (3.15). It follows by (3.16) and Lemma 3.3 that

$$\max_{y\in J^\pm(c,n)}\left|(F_2)'(y) - \frac{1+\tan^2\xi^\pm}{(1+\beta\tan\xi^\pm)^2+\tan^2\xi^\pm}\right| = \mathcal{O}\left(\frac{1}{n}\right). \tag{3.19}$$

Substituting $\tan\xi^\pm = \left(\beta\pm\sqrt{\beta^2+4}\right)/2$ for (3.19), we arrive at (3.18). Likewise, we obtain (3.17). □

Next, we give a geometric characterization of the gaps. We introduce some functions needed for that purpose. For integers $n\geq n_0$, we put

$$\xi_{2n-1} = \xi_n^- \quad \text{and} \quad \xi_{2n} = \xi_n^+.$$

For integers p, q with $q \geq 2n_0 - 1$, we define $\tilde{F}_{1,q,p} \colon [\xi_q, \xi_{q+1}] \to \mathbf{R}$ as follows. When q is odd, we put

$$\tilde{F}_{1,q,p}(y) = F_1(y) + \tfrac{\pi}{2} p, \quad y \in [\xi_q, \xi_{q+1}].$$

We set

$$\tilde{F}_{1,q,p}(y) = \begin{cases} F_1(y) + \frac{\pi}{2}(p-1), & y \in \left[\xi_q, \frac{q}{2}\pi + \frac{\pi}{2}\right], \\ F_1(y) + \frac{\pi}{2}(p+1), & y \in \left(\frac{q}{2}\pi + \frac{\pi}{2}, \xi_{q+1}\right] \end{cases}$$

when q is even. Since

$$\tan y - \beta h(y) \to \mp\infty \quad \text{as } y \to \tfrac{q}{2}\pi + \tfrac{\pi}{2} \pm 0 \text{ for } q \text{ even},$$

the function $\tilde{F}_{1,q,p}$ is continuous on $[\xi_q, \xi_{q+1}]$. Arguing as in the proof of Lemma 3.3, we see that there exists an integer $n_1 \geq n_0$ such that, for any integer $n \geq n_1$, the equation $1 + \beta h(y) \tan y = 0$ admits a unique root y_n in $\left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right)$. Furthermore, we obtain

$$y_n = n\pi + \tan^{-1}\left(-\frac{1}{\beta}\right) + o(1) \quad \text{as } n \to \infty.$$

This together with Lemma 3.3 yields that there exists an integer $n_2 \geq n_1$ such that $y_n < \xi_n^-$ if $\beta > 0$ and $n \geq n_2$ while $\xi_n^+ < y_n$ if $\beta < 0$ and $n \geq n_2$. We define $y_{2n-2}^* = y_n$ if $\beta > 0$ and $n \geq n_2$, with $y_{2n}^* = y_n$ if $\beta < 0$ and $n \geq n_2$. For integers p, q with $q \geq 2n_2$, we define $\tilde{F}_{2,q,p} \colon [\xi_q, \xi_{q+1}] \to \mathbf{R}$ as follows. If q is odd, we put

$$\tilde{F}_{2,q,p}(y) = F_2(y) + \tfrac{\pi}{2} p, \quad y \in [\xi_q, \xi_{q+1}].$$

If q is even, we define

$$\tilde{F}_{2,q,p}(y) = \begin{cases} F_2(y) + \frac{\pi}{2}(p-1), & y \in [\xi_q, y_q^*], \\ F_2(y) + \frac{\pi}{2}(p+1), & y \in (y_q^*, \xi_{q+1}]. \end{cases}$$

Because

$$\frac{\tan y}{1 + \beta h(y) \tan y} \to \mp\infty \quad \text{as } y \to y_q^* \pm 0 \text{ for } q \text{ even},$$

the function $\tilde{F}_{2,q,p}$ is continuous on $[\xi_q, \xi_{q+1}]$.

We also introduce some geometric objects. For $j = 1, 2$, we denote by $C_{j,q,p}$ the curve $x = \tilde{F}_{j,q,p}(y)$, $y \in [\xi_q, \xi_{q+1}]$. We infer by the first assertion of Lemma 3.3 that

$$\tilde{F}_{1,q,p}(y) = \tilde{F}_{2,q,p}(y) \quad \text{if and only if } y \in \{\xi_q, \xi_{q+1}\}. \tag{3.20}$$

It follows from Lemma 3.5 that there exist $C > 0$ and $n_3 \in \mathbf{N}$ with $n_3 \geq 2n_2$ such that

$$(\tilde{F}_{j,q,p})'(y) \geq C \text{ on } [\xi_q, \xi_{q+1}] \quad \text{for any } q \geq n_3, \ p \in \mathbf{Z}, \text{ and } j = 1, 2.$$

Let $V_{q,p}$ stand for the point $(\xi_q, \tilde{F}_{1,q,p}(\xi_q))$. By $B_{q,p}$ we designate the region bounded by the parallelogram with vertices $V_{q,p}$, $V_{q+1,p}$, $V_{q+1,p+1}$, and $V_{q,p+1}$. We note that

$$V_{q+1,p+1} = \tilde{F}_{1,q,p}(\xi_{q+1}).$$

For $P, Q \in \mathbf{R}^2$, we denote by PQ the line segment whose endpoints are P and Q; that is, $PQ = \{tQ + (1-t)P \mid t \in [0,1]\}$. We also note that the slope of the segment $V_{q,p}V_{q+1,p}$, which we denote by s_q, is independent of p. By Lemmas 3.3 and 3.4 and since $\xi^+ - \xi^- = \frac{\pi}{2}$, we have $s_q \to 0$ as $q \to \infty$. Hence there exists an integer $N \geq n_3$ such that $|s_q| < \min\{\kappa_0, C\}$ for any integer $q \geq N$. Accordingly, we have

$$(y, \tilde{F}_{j,q,p}(y)) \in B_{q,p} \ \text{ on } (\xi_q, \xi_{q+1})$$

for integers q, p with $q \geq N$ and for $j = 1, 2$. Let

$$\tilde{B}_{q,p} = \overline{B_{q,p}} \setminus (V_{q,p+1}V_{q+1,p+1} \cup V_{q+1,p}V_{q+1,p+1}).$$

By L we designate the line $x = -\kappa_0 y$. Let η stand for the projection $\mathbf{R}^2 \ni (y, x) \mapsto y \in \mathbf{R}$. The geometric characterization is stated as follows.

LEMMA 3.6. *Assume that $\tilde{B}_{q,p} \cap L \neq \emptyset$ and $q \geq N$. Then the following claims hold.*

(i) *For $j = 1, 2$, the curve $C_{j,q,p}$ and the line L admit a unique intersection point $W_{j,q,p}$.*

(ii) *By $I_{q,p}$ we designate the closed interval whose endpoints are $\eta(W_{1,q,p})$ and $\eta(W_{2,q,p})$. We have $\eta(W_{1,q,p}) = \eta(W_{2,q,p})$ if and only if $V_{q,p} \in L$. If $V_{q,p} \notin L$, then $|\tilde{d}(y)| \geq 2$ on $I_{q,p}$ while $|\tilde{d}(y)| < 2$ on $\eta(\tilde{B}_{q,p} \cap L) \setminus I_{q,p}$.*

Proof. (i) Since $\tilde{B}_{q,p} \cap L \neq \emptyset$ and $|s_q| < \kappa_0$, we have $\tilde{F}_{j,q,p}(\xi_q) \leq -\kappa_0 \xi_q$ and $\tilde{F}_{j,q,p}(\xi_{q+1}) > -\kappa_0 \xi_{q+1}$, so that the curve $C_{j,q,p}$ and the line L possess an intersection point. Since $x = \tilde{F}_{j,q,p}(y)$ is strictly increasing and since $x = -\kappa_0 y$ is strictly decreasing, such an intersection point is unique.

(ii) The first assertion follows from (3.20). Let us prove the second one. We observe that $\tilde{F}_{j,q,p}(y) \equiv F_j(y) \pmod{\pi}$ on $[\xi_q, \xi_{q+1}]$ if $q - p$ is odd, while $\tilde{F}_{j,q,p}(y) \equiv F_j(y) + \frac{\pi}{2} \pmod{\pi}$ on $[\xi_q, \xi_{q+1}]$ if $q - p$ is even. First, we consider the case where $q - p$ is odd. Combining our observation with Lemma 3.1, we get $\tilde{d}(\eta(W_{1,q,p})) = \tilde{d}(\eta(W_{2,q,p})) = 2$. Furthermore, we infer by Lemma 3.2 that the function $\tilde{d}(y) + 2$ admits no zero in $\eta(\tilde{B}_{q,p} \cap L)$. These statements together with Proposition 1.1 imply the second claim. Similarly, we get the second assertion in the case where $q - p$ is even. □

Let M be the greatest integer for which $L \cap \tilde{B}_{N,M} \neq \emptyset$. If $G_j \neq \emptyset$, we put $\tilde{G}_j = \overline{G_j}$; otherwise we set $\tilde{G}_j = B_j \cap B_{j+1}$. We see by Lemma 3.6 that there exists a unique integer K for which $\mu(I_{N,M}) = \tilde{G}_K$. Let us prove the following implication.

LEMMA 3.7. *Suppose that $\tilde{B}_{q,p} \cap L \neq \emptyset$ and $q \geq N$. Then*

$$\mu(I_{q,p}) = \tilde{G}_{K+(q-N)-(p-M)}.$$

Proof. We put $\Omega = \{(i, j) \in \mathbf{Z}^2 \mid N \leq i \leq q, p \leq j \leq M\}$ and

$$\tilde{L} = L \cap \bigcup_{(i,j) \in \Omega} \tilde{B}_{i,j}. \tag{3.21}$$

We observe the interval $\tilde{L}$ in two ways. Let $\Omega' = \{(i,j) \in \Omega \mid \tilde{B}_{i,j} \cap L \neq \emptyset\}$. The intervals $\tilde{B}_{i,j} \cap L$, $(i,j) \in \Omega'$, form a subdivision of the interval $\tilde{L}$. This combined with Lemma 3.6 yields

$$\mu(I_{q,p}) = \tilde{G}_{K+\#\Omega'-1}, \tag{3.22}$$

where $\#\Omega'$ stands for the number of the elements of Ω'. On the other hand, we see from (3.21) that such a subdivision is also obtained by dividing $\tilde{L}$ by the segments $V_{i,p}V_{i,M+1}$ $(N+1 \le i \le q)$ and the broken lines

$$\bigcup_{s=N}^{q} V_{s,j}V_{s+1,j} \quad (p+1 \le j \le M).$$

Thus, we get

$$\#\Omega' = q - N - p + M + 1.$$

Combining this with (3.22), we get the assertion. □

We define

$$a = \begin{cases} \dfrac{2+\beta^2+\beta\sqrt{\beta^2+4}}{2} & \text{if } \beta > 0, \\ \dfrac{2+\beta^2-\beta\sqrt{\beta^2+4}}{2} & \text{if } \beta < 0. \end{cases}$$

We note that $a > 1$. We introduce a parameter $\alpha > 0$. The following lemma plays the most important role in proving Theorem 1.2.

LEMMA 3.8. (i) *There exist $\tilde{N} \in \mathbf{N}$ and $C_\alpha > 0$ such that if $q \ge \tilde{N}$, $\tilde{B}_{q,p} \cap L \neq \emptyset$, and* $\operatorname{dist}(V_{q,p}, L) \le \alpha/q$, *then*

$$\begin{aligned} &\left(\frac{a^2-1}{(a+\kappa_0)(1+a\kappa_0)} - \frac{C_\alpha}{q}\right)|\kappa_0\xi_q + \tilde{F}_{1,q,p}(\xi_q)| \\ &\quad \le |\eta(W_{1,q,p}) - \eta(W_{2,q,p})| \\ &\quad \le \left(\frac{a^2-1}{(a+\kappa_0)(1+a\kappa_0)} + \frac{C_\alpha}{q}\right)|\kappa_0\xi_q + \tilde{F}_{1,q,p}(\xi_q)|. \end{aligned}$$

(ii) *There exist $\hat{N} \in \mathbf{N}$ and $C_\alpha > 0$ such that if $q \ge \hat{N}$, $\tilde{B}_{q,p} \cap L \neq \emptyset$, and* $\operatorname{dist}(V_{q+1,p+1}, L) \le \alpha/q$, *then*

$$\begin{aligned} &\left(\frac{a^2-1}{(a+\kappa_0)(1+a\kappa_0)} - \frac{C_\alpha}{q}\right)|\kappa_0\xi_{q+1} + \tilde{F}_{1,q,p}(\xi_{q+1})| \\ &\quad \le |\eta(W_{1,q,p}) - \eta(W_{2,q,p})| \\ &\quad \le \left(\frac{a^2-1}{(a+\kappa_0)(1+a\kappa_0)} + \frac{C_\alpha}{q}\right)|\kappa_0\xi_{q+1} + \tilde{F}_{1,q,p}(\xi_{q+1})|. \end{aligned}$$

In the proof of this lemma, we use the following elementary assertion.

PROPOSITION 3.9. *Let $c > 1$ and $d \in \mathbf{R}$. We designate the lines $x = cy$, $x = \frac{1}{c}y$, and $x = -\kappa_0 y + d$ in the yx-plane by l_1, l_2, and n, respectively. Let D be the distance between the line n and the origin. For $j = 1, 2$, we denote by p_j the y-coordinate of the intersection point of l_j and n. It then holds that*

$$|p_1 - p_2| = \frac{D(c^2-1)\sqrt{1+\kappa_0^2}}{(c+\kappa_0)(1+c\kappa_0)}.$$

Proof of Lemma 3.8. First, we prove (i) in the case where $\beta > 0$. We put

$$J_q = \left[\xi_q, \xi_q + \frac{\sqrt{1+\kappa_0^2}}{\kappa_0} \cdot \frac{\alpha}{q}\right].$$

For $i = 1, 2$, we define

$$a_{i,q}^+ = \max\{(F_i)'(y) \mid y \in J_q\},$$
$$a_{i,q}^- = \min\{(F_i)'(y) \mid y \in J_q\}.$$

We also define $j = j(q) = 1$ and $k = k(q) = 2$ for q even, while $j = j(q) = 2$ and $k = k(q) = 1$ for q odd. It follows by Lemma 3.5 that there exists an integer $\tilde{N} \geq N$ such that if $q \geq \tilde{N}$, then $a_{k,q}^- \leq a_{k,q}^+ < 1 < a_{j,q}^- \leq a_{j,q}^+$. We define $b_q = \max\{a_{j,q}^+, 1/a_{k,q}^-\}$ and $c_q = \min\{a_{j,q}^-, 1/a_{k,q}^+\}$. For $q \geq \tilde{N}$ and $y \in J_q$, we have

$$\tilde{F}_{1,q,p}(\xi_q) + c_q(y - \xi_q) \leq \tilde{F}_{j,q,p}(y) \leq \tilde{F}_{1,q,p}(\xi_q) + b_q(y - \xi_q),$$
$$\tilde{F}_{1,q,p}(\xi_q) + \frac{1}{b_q}(y - \xi_q) \leq \tilde{F}_{k,q,p}(y) \leq \tilde{F}_{1,q,p}(\xi_q) + \frac{1}{c_q}(y - \xi_q).$$

Combining these inequalities with Proposition 3.9, we get

$$\begin{aligned}\frac{(c_q^2-1)\sqrt{1+\kappa_0^2}}{(c_q+\kappa_0)(1+c_q\kappa_0)} \operatorname{dist}(V_{q,p}, L) &\leq |\eta(W_{1,q,p}) - \eta(W_{2,q,p})| \\ &\leq \frac{(b_q^2-1)\sqrt{1+\kappa_0^2}}{(b_q+\kappa_0)(1+b_q\kappa_0)} \operatorname{dist}(V_{q,p}, L) \quad (3.23)\end{aligned}$$

provided $q \geq \tilde{N}$, $\overline{B_{q,p}} \cap L \neq \emptyset$, and $\operatorname{dist}(V_{q,p}, L) \leq \alpha/q$. On the other hand, we infer by Lemma 3.5 that $b_n = a + \mathcal{O}\left(\frac{1}{n}\right)$ and $c_n = a + \mathcal{O}\left(\frac{1}{n}\right)$ as $n \to \infty$. Combining these with (3.23), we get (i) in the case where $\beta > 0$. In a similar fashion, we obtain (i) in the case where $\beta < 0$. An analogous argument also gives (ii). □

We pick a μ for which $1 < \mu < a$. Let us prove the following claim.

LEMMA 3.10. *There exist $J > 0$, $K > 0$, and $N' \in \mathbf{N}$ such that the following statements hold.*

(i) *If $q \geq N'$, $\tilde{B}_{q,p} \cap L \neq \emptyset$, and $\operatorname{dist}(\{V_{q,p}, V_{q+1,p+1}\}, L) \leq J$, then*

$$\begin{aligned}&|\eta(W_{1,q,p}) - \eta(W_{2,q,p})| \\ &\qquad \geq \frac{(\mu^2-1)\sqrt{1+\kappa_0^2}}{(\mu+\kappa_0)(1+\mu\kappa_0)} \operatorname{dist}(\{V_{q,p}, V_{q+1,p+1}\}, L). \quad (3.24)\end{aligned}$$

(ii) *If $q \geq N'$, $\tilde{B}_{q,p} \cap L \neq \emptyset$, and $\operatorname{dist}(\{V_{q,p}, V_{q+1,p+1}\}, L) \geq J$, then*

$$|\eta(W_{1,q,p}) - \eta(W_{2,q,p})| \geq K. \quad (3.25)$$

Proof. First, we consider the case where $\beta > 0$. It follows by Lemma 3.5 that there exist an $\eta > 0$ and an $n_4 \in \mathbf{N}$ such that, for any integer $q \geq n_4$ and for any integer p,

$$\min\{(\tilde{F}_{j,q,p})'(y) \mid y \in [\xi_q, \xi_q + 2\eta]\} \geq \mu,$$

$$\max\{(\tilde{F}_{k,q,p})'(y) \mid y \in [\xi_q, \xi_q + 2\eta]\} \leq \frac{1}{\mu},$$

$$\min\{(\tilde{F}_{k,q,p})'(y) \mid y \in [\xi_{q+1} - 2\eta, \xi_{q+1}]\} \geq \mu,$$

$$\max\{(\tilde{F}_{j,q,p})'(y) \mid y \in [\xi_{q+1} - 2\eta, \xi_{q+1}]\} \leq \frac{1}{\mu},$$

where $j = j(q) = 1$ and $k = k(q) = 2$ for q even, while $j = j(q) = 2$ and $k = k(q) = 1$ for q odd. We designate the lines $x = \mu y$, $x = \frac{1}{\mu} y$, and $x = -\kappa_0 y + \kappa_0 \eta + \frac{\eta}{\mu}$ by l_1, l_2, and l_3, respectively. For $i = 1, 2$, let s_i be the y-coordinate of the intersection point of l_i and l_3. We note that $s_2 = \eta$. Let J stand for the distance between l_3 and the origin. A discussion similar to that in the proof of Lemma 3.8 then gives (3.24), provided $\tilde{B}_{q,p} \cap L \neq \emptyset$, $\operatorname{dist}(\{V_{q,p}, V_{q+1,p+1}\}, L) \leq J$, and q is sufficiently large.

We note that there exists a unique continuous function $H_1(y)$ on $\left[\xi^+, \xi^+ + \frac{\pi}{2}\right]$ such that $H_1(\xi^+) = -\xi^-$ and $\tan H_1(y) = \tan y - \beta$ for $y \in \left[\xi^+, \xi^+ + \frac{\pi}{2}\right] \setminus \left\{\frac{\pi}{2}\right\}$. We also note that there exists a unique continuous function $H_2(y)$ on $\left[\xi^+, \xi^+ + \frac{\pi}{2}\right]$ such that $H_2(\xi^+) = -\xi^-$ and

$$\tan H_2(y) = \frac{\tan y}{1 + \beta \tan y} \quad \text{for } y \in \left[\xi^+, \xi^+ + \tfrac{\pi}{2}\right] \setminus \left\{\tfrac{\pi}{2}, \tan^{-1}\left(-\tfrac{1}{\beta}\right) + \pi\right\}.$$

We have

$$p := \min\left\{H_1(y) - H_2(y) \;\middle|\; y \in \left[\xi^+ + \tfrac{s_1}{2}, \xi^+ + \tfrac{\pi}{2} - \tfrac{s_1}{2}\right]\right\} > 0.$$

We define $H_1^*(y) = H_1(y) - \frac{p}{4}$ and $H_2^*(y) = H_2(y) + \frac{p}{4}$ for $y \in \left[\xi^+ + \frac{s_1}{2}, \xi^+ + \frac{\pi}{2} - \frac{s_1}{2}\right]$. Let

$$I = \left[\kappa_0\left(\xi^+ + \tfrac{s_1}{2}\right) + H_1^*\left(\xi^+ + \tfrac{s_1}{2}\right), \kappa_0\left(\xi^+ + \tfrac{\pi}{2} - \tfrac{s_1}{2}\right) + H_2^*\left(\xi^+ + \tfrac{\pi}{2} - \tfrac{s_1}{2}\right)\right].$$

For $t \in I$, we designate the line $x = -\kappa_0 y + t$ by $l(t)$. For $i = 1, 2$, we denote by $S_i(t)$ the y-coordinate of the intersection point of the line $l(t)$ and the curve $x = H_i^*(y)$. We define $K = \min\{S_2(t) - S_1(t) \mid t \in I\}$. We have $K > 0$. Integrating both sides of (3.15) and those of (3.16) on $[\xi_q, t]$ for $t \in [\xi_q + s_1, \xi_{q+1} - s_1]$, we infer that there exists an $n_5 \in \mathbf{N}$ such that, for any even integer $q \geq n_5$,

$$\tilde{F}_{1,q,p}(y) \geq H_1^*\left(y - \frac{q}{2}\pi\right) + \frac{\pi(p-1)}{2} \quad \text{on } [\xi_q + s_1, \xi_{q+1} - s_1],$$

$$\tilde{F}_{2,q,p}(y) \leq H_2^*\left(y - \frac{q}{2}\pi\right) + \frac{\pi(p-1)}{2} \quad \text{on } [\xi_q + s_1, \xi_{q+1} - s_1].$$

Therefore, we have (3.25) provided that $\tilde{B}_{q,p} \cap L \neq \emptyset$, $\operatorname{dist}(\{V_{q,p}, V_{q+1,p+1}\}, L) \geq J$, and q is sufficiently large and even. Likewise, we obtain (3.25) if $\tilde{B}_{q,p} \cap L \neq \emptyset$, $\operatorname{dist}(\{V_{q,p}, V_{q+1,p+1}\}, L) \geq J$, and q is sufficiently large and odd. Thus we obtain the assertion of the lemma in the case where $\beta > 0$. In a similar fashion, we obtain the claim in the case where $\beta < 0$. □

We are now in a position to complete the proof of Theorem 1.2. We put

$$\Lambda = \left\{ \lim_{k\to\infty} f_k \;\middle|\; \{f_k\}_{k=1}^{\infty} \text{ is a convergent subsequence of } \{j|G_j|\}_{j=1}^{\infty} \right\},$$

$$\hat{\theta} = \frac{2}{\pi}(\kappa_0 - 1)\xi^+.$$

Let us first prove the inclusion

$$\{W|y+X| \mid y \in A'_+(\kappa_0,\hat{\theta})\} \subset \Lambda. \tag{3.26}$$

We pick a $y \in A'_+(\kappa_0,\hat{\theta})$ arbitrarily. Then there exists a sequence $\{(q(n), p(n))\}_{n=1}^{\infty}$ in $\mathbf{N}\times\mathbf{Z}$ such that $q(n)\to\infty$ and $q(n)(\kappa_0 q(n) + \hat{\theta} + p(n)) \to y$ as $n\to\infty$. This combined with Lemmas 3.3 and 3.4 and $\xi^+ - \xi^- = \frac{\pi}{2}$ yields

$$\lim_{n\to\infty} q(n)(\kappa_0\xi_{q(n)} + \tilde{F}_{1,q(n),p(n)}(\xi_{q(n)})) = \frac{\pi}{2}(y+X). \tag{3.27}$$

We pick an $\alpha > 0$ for which $\left|\frac{\pi}{2}(y+X)\right| < \alpha\sqrt{1+\kappa_0^2}$. It follows by (3.27) that $\operatorname{dist}(V_{q(n),p(n)}, L) \le \alpha/q(n)$ for sufficiently large n. Moreover, we get $\tilde{B}_{q(n),p(n)} \cap L \ne \emptyset$ in the case where $\kappa_0\xi_{q(n)} + \tilde{F}_{1,q(n),p(n)}(\xi_{q(n)}) \le 0$; otherwise we get $\tilde{B}_{q(n)-1,p(n)-1} \cap L \ne \emptyset$. Using (3.27) and Lemmas 3.7 and 3.8, we obtain

$$\lim_{n\to\infty} q(n)|\mu^{-1}(G_{K+(q(n)-N)-(p(n)-M)})| = \frac{a^2-1}{(a+\kappa_0)(1+a\kappa_0)}\cdot\frac{\pi}{2}|y+X|,$$

so that

$$\lim_{n\to\infty}(K+(q(n)-N)-(p(n)-M))|G_{K+(q(n)-N)-(p(n)-M)}| = W|y+X|.$$

Thus, we get the inclusion (3.26).

Next, we prove the reverse inclusion of (3.26). Let $\{j(k)\}_{k=1}^{\infty}$ be a subsequence of $\{j\}_{j=1}^{\infty}$ for which $\{j(k)|G_{j(k)}|\}_{k=1}^{\infty}$ converges. Let $(q(k), p(k)) \in \mathbf{N}\times\mathbf{Z}$ be such that $\mu(I_{q(k),p(k)}) = \tilde{G}_{j(k)}$. We have $q(k)\to\infty$ as $k\to\infty$. Since $j(k) \ge q(k)$ for sufficiently large k, the sequence $\{q(k)|I_{q(k),p(k)}|\}_{k=1}^{\infty}$ is bounded. This together with Lemma 3.10 implies that

$$\{q(k)\operatorname{dist}(\{V_{q(k),p(k)}, V_{q(k)+1,p(k)+1}\}, L)\}_{k=1}^{\infty}$$

is a bounded sequence, so that there exists a subsequence $\{k(l)\}_{l=1}^{\infty}$ of $\{k\}_{k=1}^{\infty}$ that satisfies at least one of the following:

(i) the sequence $\{q(k(l))(\kappa_0\xi_{q(k(l))} + \tilde{F}_{1,q(k(l)),p(k(l))}(\xi_{q(k(l))}))\}_{l=1}^{\infty}$ converges;

(ii) the sequence $\{(q(k(l))+1)(\kappa_0\xi_{q(k(l))+1} + \tilde{F}_{1,q(k(l)),p(k(l))}(\xi_{q(k(l))+1}))\}_{l=1}^{\infty}$ converges.

Let us consider case (i). Because of Lemmas 3.3 and 3.4, we see that the sequence $\{q(k(l))(\kappa_0 q(k(l)) + \hat{\theta} + p(k(l)))\}_{l=1}^{\infty}$ converges. Designating its limit by y_0, we infer, as in the previous discussion, that

$$\lim_{l\to\infty} j(k(l))|G_{j(k(l))}| = W|y_0 + X|,$$

whence

$$\lim_{k\to\infty} j(k)|G_{j(k)}| \in \{W|y + X| \mid y \in A'_+(\kappa_0, \hat{\theta})\}. \tag{3.28}$$

Similarly, we get (3.28) in case (ii). Therefore, we have the reverse inclusion of (3.26). Inasmuch as $2\xi^+ \equiv -\tan^{-1}\left(\frac{2}{\beta}\right) \pmod{\pi}$, we have $A'_+(\kappa_0, \hat{\theta}) = A'_+(\kappa_0, \theta)$. Hence, we arrive at (1.4+). In a similar fashion, we obtain (1.4−). This completes the proof of Theorem 1.2.

4. Proof of Theorem 1.5

We use basic results in the theory of regular continued fractions (cf. [5; 13, Vol. I; 17]). Let $a_0 = 0$. We define $\{p_i\}_{i=-2}^{\infty}$ and $\{q_i\}_{i=-2}^{\infty}$ by the recurrence relations

$$p_k = a_k p_{k-1} + p_{k-2} \quad (k \ge 0), \qquad p_{-2} = 0, \quad p_{-1} = 1;$$
$$q_k = a_k q_{k-1} + q_{k-2} \quad (k \ge 0), \qquad q_{-2} = 1, \quad q_{-1} = 0.$$

It holds for $k \ge 1$ that

$$\frac{1}{a_1+}\,\frac{1}{a_2+}\cdots\frac{1}{a_k} = \frac{p_k}{q_k}.$$

For $k \ge 0$, the integers q_{2k-1} and p_{2k-2} are even, while q_{2k-2} and p_{2k-1} are odd. Let

$$D_k = q_k\alpha - p_k$$

for $k \ge -1$. We note that $D_{-1} = -1$, $D_0 = \alpha$, and $D_{k+1} = a_{k+1}D_k + D_{k-1}$ for $k \ge 0$. We also define $\xi_n = q_{n-1}/q_n$ and $\zeta_n = -D_{n-2}/D_{n-1}$. It then holds that

$$\xi_n = \frac{1}{a_n+}\,\frac{1}{a_{n-1}+}\cdots\frac{1}{a_1},$$
$$\zeta_n = a_n + \frac{1}{a_{n+1}+}\,\frac{1}{a_{n+2}+}\cdots.$$

Since $(-1)^n q_{n+1}D_n = (1 + \xi_{n+1}\zeta_{n+2}^{-1})^{-1}$ (with $0 < \xi_{n+1} < a_{n+1}^{-1}$ and $0 < \zeta_{n+2}^{-1} < a_{n+2}^{-1}$) and since $a_n \to \infty$ as $n \to \infty$, we arrive at

$$(-1)^n q_{n+1}D_n \to 1 \quad \text{as} \quad n \to \infty. \tag{4.1}$$

Moreover, we note that $\frac{1}{2} < (-1)^n q_{n+1}D_n < 1$ for all $n \in \mathbf{N}$.

We also use the Sós theory for the inhomogeneous Diophantine approximation (see [17; 19]). We have

$$-\gamma = \sum_{j=0}^{\infty} \frac{1}{2} a_{j+1} D_j,$$

which is called the *α-expansion* of $-\gamma$. For integers k, r with $k \ge 1$ and $1 \le r \le \frac{1}{2}a_{k+1} + 1$, we define

$$Q_{k,r} = \sum_{j=0}^{k-1} \frac{1}{2} a_{j+1} q_j + (r-1) q_k,$$

$$P_{k,r} = \sum_{j=0}^{k-1} \frac{1}{2} a_{j+1} p_j + (r-1) p_k.$$

The numbers $Q_{k,r}$ are called the *adjacent multiples* in [19]. We note that

$$Q_{k,r} = \left(r - \frac{1}{2}\right) q_k + \frac{1}{2} q_{k-1} - \frac{1}{2},$$
$$P_{k,r} = \left(r - \frac{1}{2}\right) p_k + \frac{1}{2} p_{k-1} - \frac{1}{2}.$$

We also note that if $r = \frac{1}{2} a_{k+1} + 1$, then $Q_{k,r} = Q_{k+1,1}$. Put

$$L = \left\{ \frac{2j-1}{4} \;\middle|\; j \in \mathbf{Z} \right\}.$$

Let us prove the following implication.

Lemma 4.1.

$$A'_+(\alpha, \gamma) \supset L.$$

Proof. We have

$$\begin{aligned} &Q_{k,r}(Q_{k,r}\alpha - P_{k,r} + \gamma) \\ &\quad = \left\{ \left(r - \frac{1}{2}\right) q_k + \frac{1}{2} q_{k-1} - \frac{1}{2} \right\} \left\{ \frac{1}{2} D_{k-1} + \left(r - \frac{1}{2}\right) D_k \right\} \\ &\quad = \frac{1}{2} q_k D_{k-1} \left(r - \frac{1}{2} + \frac{1}{2} \xi_k - \frac{1}{2q_k} \right) (1 - (2r-1)\zeta_{k+1}^{-1}). \end{aligned}$$

This combined with (4.1) implies that

$$(-1)^{k-1} Q_{k,r}(Q_{k,r}\alpha - P_{k,r} + \gamma) \to \frac{1}{2}\left(r - \frac{1}{2}\right) \quad \text{as } k \to \infty, \tag{4.2}$$

from which we obtain the assertion. □

Next, we demonstrate the reverse inclusion. We pick $M > 1$ arbitrarily. We choose a positive integer p such that $p > 8M$. We also pick an integer $k_0 \geq 4$ such that $a_k > 8M$ for all $k \geq k_0$. Let $k \geq k_0$ and $1 \leq r \leq a_{k+1}/2$. We pick an integer x satisfying $Q_{k,r} < x < Q_{k,r+1}$. Since $0 < Q_{k,r+1} - x < q_k$, the Ostrowski representation theorem (see [17, Chap. II, Sec. 4]) implies that there exist integers $c_1, c_2, \ldots, c_k$ such that

$$Q_{k,r+1} - x = \sum_{j=0}^{k-1} c_{j+1} q_j,$$

$(c_1, c_2, \ldots, c_k) \neq (0, 0, \ldots, 0)$, $0 \leq c_j \leq a_j$ for $j \geq 2$, $0 \leq c_1 < a_1$, and $c_j = 0$ if $c_{j+1} = a_{j+1}$. Let n be the least positive integer such that $c_n \neq 0$. It follows that

$$\|\alpha x+\gamma\| = \left\| \sum_{j=n-1}^{k-1} c_{j+1}D_j + \left(\frac{1}{2}a_{k+1} - r\right)D_k - \frac{1}{2}D_k - \frac{1}{2}D_{k+1} \right\|.$$

We define

$$S(x) = \sum_{j=n-1}^{k-1} c_{j+1}D_j + \left(\frac{1}{2}a_{k+1} - r\right)D_k - \frac{1}{2}D_k - \frac{1}{2}D_{k+1}.$$

In the following Lemmas 4.2, 4.3, and 4.4 we analyze the shape of x such that $x|S(x)| < M$.

LEMMA 4.2. *If $x|S(x)| < M$, then $n \geq k-1$.*

Proof. Seeking a contradiction, we suppose that $n \leq k-2$. In the case where n is odd, we have

$$\begin{aligned} S(x) &\geq c_n D_{n-1} + (a_{n+1} - 1)D_n + \sum_{j=1}^{\infty} a_{n+2j+1}D_{n+2j} - \frac{1}{2}D_k - \frac{1}{2}D_{k+1} \\ &= (c_n - 1)D_{n-1} - D_n - \frac{1}{2}D_k - \frac{1}{2}D_{k+1} \\ &\geq -D_n - \frac{1}{2}D_k - \frac{1}{2}D_{k+1} \\ &\geq |D_{k-2}| - \frac{1}{2}|D_k| \\ &> \frac{1}{2}|D_{k-2}|, \end{aligned}$$

since $D_{2j} > 0$ and $D_{2j-1} < 0$ for $j \geq 0$ and since $c_{n+1} \leq a_{n+1} - 1$. Likewise, we obtain $-S(x) > \frac{1}{2}|D_{k-2}|$ in the case where n is even. In each of the cases, we get

$$x|S(x)| > \tfrac{1}{2}a_k q_{k-1} \cdot \tfrac{1}{2}|D_{k-2}| > \tfrac{1}{8}a_k > M,$$

which is a contradiction. Thus, the assertion follows. □

LEMMA 4.3. *If $x|S(x)| < M$ and $n = k-1$, then $c_{k-1} = 1$, $c_k > a_k - p$, and $r < p$.*

Proof. First, we prove that $c_{k-1} = 1$ by contradiction. Suppose that $c_{k-1} \geq 2$. Then

$$\begin{aligned} |S(x)| &= \left|c_{k-1}D_{k-2} + c_k D_{k-1} + \left(\tfrac{1}{2}a_{k+1} - r - \tfrac{1}{2}\right)D_k - \tfrac{1}{2}D_{k+1}\right| \\ &> \left|2D_{k-2} + (a_k - 1)D_{k-1} - \tfrac{1}{2}D_k\right| \\ &= \left|D_{k-2} - D_{k-1} + \tfrac{1}{2}D_k\right| \\ &> |D_{k-2}|, \end{aligned}$$

so that

$$x|S(x)| > \tfrac{1}{2}q_k|D_{k-2}| > \tfrac{1}{2}a_k q_{k-1}|D_{k-2}| > \tfrac{1}{4}a_k > M,$$

which is a contradiction. So, we get $c_{k-1} = 1$.

Next, we show that $c_k > a_k - p$. Seeking a contradiction, we suppose that $c_k \leq a_k - p$. Then

$$\begin{aligned} x|S(x)| &> \left|D_{k-2} + (a_k - p)D_{k-1} - \tfrac{1}{2}D_k\right|\left(r - \tfrac{1}{2}\right)q_k \\ &= \left|-pD_{k-1} + \tfrac{1}{2}D_k\right|\left(r - \tfrac{1}{2}\right)q_k \\ &> p|D_{k-1}|q_k\left(r - \tfrac{1}{2}\right) \\ &> M, \end{aligned}$$

which is a contradiction. We therefore have $c_k > a_k - p$. Likewise, we get $r < p$. □

An argument parallel to that in the proof of Lemma 4.3 gives the following result.

LEMMA 4.4. *If $x|S(x)| < M$ and $n = k$, then $c_k < p$ and $r < p$.*

We are now ready to complete the proof of Theorem 1.5. For $1 \leq t < p$ and $1 \leq s < p$, we have

$$\lim_{j\to\infty}(Q_{j,s+1} - tq_{j-1})\|(Q_{j,s+1} - tq_{j-1})\alpha + \gamma\| = \left(\frac{1}{2} + s\right)\left(t - \frac{1}{2}\right)$$

and

$$\begin{aligned} \lim_{j\to\infty}(Q_{j,s+1} - q_{j-2} - (a_j - t)q_{j-1})\|(Q_{j,s+1} - q_{j-2} - (a_j - t)q_{j-1})\alpha + \gamma\| \\ = \left(t + \frac{1}{2}\right)\left(s - \frac{1}{2}\right). \end{aligned}$$

By a method similar to that in the proof of [17, Chap. II, Sec. 4, Thm. 1], we have either $\|\alpha x + \gamma\| \geq \frac{1}{2}D_2$ or $\|\alpha x + \gamma\| = |S(x)|$. In the former case, we have $x\|\alpha x + \gamma\| \geq 4^{-1}q_k D_2 > 8^{-1}a_k > M$. Combining these with (4.2) and Lemmas 4.2–4.4, we obtain $A'_+(\alpha, \gamma) \cap (-M, M) \subset L$. Since this inclusion holds for any $M > 0$, we get $A'_+(\alpha, \gamma) \subset L$. This together with Lemma 4.1 implies that $A'_+(\alpha, \gamma) = L$. Since

$$q(\alpha q + \gamma + p) = \frac{q}{q+1}(-q-1)\{\alpha(-q-1) + \gamma + (-p+1)\}$$

for $q \in \mathbf{N}$ and $p \in \mathbf{Z}$, we have $A'_-(\alpha, \gamma) = A'_+(\alpha, \gamma)$. This completes the proof of Theorem 1.5.

5. Proof of Theorem 1.6

We give a proof of Theorem 1.6 for $\alpha = \frac{-b-\sqrt{d}}{2a}$ only, because the proof for the other case proceeds similarly. We define $f(x, y) = ax^2 + bxy + cy^2$ and $\alpha' = \frac{-b+\sqrt{d}}{2a}$. First, we prove the following assertion.

LEMMA 5.1.

$$A'(\alpha, \gamma) \subset \left\{ \frac{k}{p^2\sqrt{d}} \;\middle|\; k \in F \right\}. \tag{5.1}$$

Proof. We pick a $w \in A'(\alpha, \gamma)$ arbitrarily. Then there exists a sequence

$$\{(y_n, x_n)\}_{n=1}^{\infty} \subset (\mathbf{Z} \setminus \{0\}) \times \mathbf{Z}$$

for which

$$y_n(\alpha y_n + \gamma - x_n) \to w \quad \text{and} \quad |y_n| \to \infty$$

as $n \to \infty$. We note that

$$\alpha y_n + \gamma - x_n = \tfrac{1}{p}\{(py_n + q)\alpha - px_n + r\}.$$

Let

$$s_n = a\{(py_n + q)\alpha' - px_n + r\}\{(py_n + q)\alpha - px_n + r\}.$$

Noticing that $x_n/y_n \to \alpha$ as $n \to \infty$, we get

$$\begin{aligned}\lim_{n\to\infty} s_n &= \lim_{n\to\infty} a\left\{\left(p + \frac{q}{y_n}\right)\alpha' - p\frac{x_n}{y_n} + \frac{r}{y_n}\right\} y_n\{(py_n + q)\alpha - px_n + r\}\\ &= p^2\sqrt{d}w.\end{aligned}$$

On the other hand, it holds for all $n \in \mathbf{N}$ that $s_n = f(px_n - r, py_n + q) \in \mathbf{Z}$. Hence, there exists an $m \in \mathbf{N}$ such that $s_n = s_m$ for all integers $n \geq m$. Therefore, we get $p^2\sqrt{d}w = s_m$, whence $w \in \left\{k/\left(p^2\sqrt{d}\right) \mid k \in F\right\}$. We thus obtain the assertion. □

By (X_1, Y_1) we denote the fundamental solution of the Pell equation $X^2 - dY^2 = 1$ (cf. [13, Vol. I, Chap. 8, Sec. 2]). For $n \in \mathbf{Z} \setminus \{1\}$, we define $(X_n, Y_n) \in \mathbf{Z}^2$ by the equation

$$X_n + Y_n\sqrt{d} = \left(X_1 + Y_1\sqrt{d}\right)^n. \tag{5.2}$$

We recall that the sequences $\{(X_n, Y_n)\}_{n=-\infty}^{\infty}$ and $\{(-X_n, -Y_n)\}_{n=-\infty}^{\infty}$ provide all the solutions of $X^2 - dY^2 = 1$. The following lemma plays a key role in proving Theorem 1.6.

LEMMA 5.2. *There exists an $s \in \mathbf{N}$ such that $X_{sj} \equiv 1 \pmod p$ and $Y_{sj} \equiv 0 \pmod p$ for all $j \in \mathbf{Z}$.*

Proof. It follows by the pigeonhole principle that there exist $t \in \mathbf{Z}$ and $s \in \mathbf{N}$ such that $X_t \equiv X_{t+s} \pmod p$ and $Y_t \equiv Y_{t+s} \pmod p$. On the other hand, we infer by (5.2) that, for all $n \in \mathbf{Z}$,

$$X_{n+1} = X_1X_n + dY_1Y_n,$$
$$Y_{n+1} = Y_1X_n + X_1Y_n.$$

Thus, we have $X_j \equiv X_{j+s} \pmod p$ and $Y_j \equiv Y_{j+s} \pmod p$ for all $j \in \mathbf{Z}$. We get $(X_0, Y_0) = (1, 0)$ and so, a fortiori, $X_0 \equiv 1 \pmod p$ and $Y_0 \equiv 0 \pmod p$. The conclusion then follows. □

We are now in a position to complete the proof of Theorem 1.6. It suffices to prove the reverse inclusion of (5.1). We achieve this by combining Lemma 5.2 with the standard technique employed in [13, Vol. I, Thm. 8-10]. We pick a $k \in F$ arbitrarily. Then there exists $(t, u) \in \mathbf{Z}^2$ such that

$$f(t, u) = k, \quad t \equiv -r \pmod p, \quad u \equiv q \pmod p.$$

We define

$$x'_n = X_{ns} - bY_{ns}, \qquad y'_n = 2aY_{ns}$$

for $n \in \mathbf{Z}$. It then holds that $f(x'_n, y'_n) = a$. For $n \in \mathbf{Z}$, we also define $(x_n, y_n) \in \mathbf{Z}^2$ by the equation

$$2ax_n + by_n + y_n\sqrt{d} = \frac{1}{2a}\left(2at + bu + u\sqrt{d}\right)\left(2ax'_n + by'_n - y'_n\sqrt{d}\right) \tag{5.3}$$

or

$$x_n = tx'_n + (bt + cu)(y'_n/a),$$
$$y_n = ux'_n - ty'_n. \tag{5.4}$$

Using $f(x'_n, y'_n) = a$, $f(t,u) = k$, and (5.3), we have $f(x_n, y_n) = k$. By Lemma 5.2 we get $x'_n \equiv 1 \pmod p$ and $y'_n/a \equiv 0 \pmod p$. These combined with (5.4) imply that

$$x_n \equiv t \equiv -r \pmod p \quad \text{and} \quad y_n \equiv u \equiv q \pmod p.$$

Since $f(x_n, y_n) = a(x_n - \alpha y_n)(x_n - \alpha' y_n) = k$, we obtain

$$y_n(\alpha y_n - x_n) = \frac{k y_n}{a(\alpha' y_n - x_n)}. \tag{5.5}$$

By (5.2) we have $X_n - Y_n\sqrt{d} \to 0$ as $n \to \infty$, so that $2ax'_n + by'_n - y'_n\sqrt{d} \to 0$ as $n \to \infty$. This combined with (5.3) implies that $\alpha y_n - x_n \to 0$ as $n \to \infty$; from this, together with (5.5), we get

$$y_n(\alpha y_n - x_n) \to \frac{k}{\sqrt{d}} \quad \text{as } n \to \infty.$$

Writing $\hat{y}_n = \frac{1}{p}(y_n - q)$ and $\hat{x}_n = \frac{1}{p}(x_n + r)$, we have $(\hat{x}_n, \hat{y}_n) \in \mathbf{Z}^2$ and

$$\hat{y}_n(\alpha \hat{y}_n + \gamma - \hat{x}_n) \to \frac{k}{p^2\sqrt{d}} \quad \text{as } n \to \infty,$$

so that $k/\big(p^2\sqrt{d}\big) \in A'(\alpha, \gamma)$. Therefore, we obtain the reverse inclusion of (5.1). This completes the proof of Theorem 1.6.

EXAMPLE 5.3. Let us consider the case where $\alpha = 1/\sqrt{2}$ and $\gamma = 1/2$. We take $a = 2$, $b = 0$, $c = -1$, $p = 2$, $q = 0$, and $r = 1$. It then follows by Theorem 1.6 that

$$A'_+\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right) = A'_-\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right) = \left\{ \frac{k}{8\sqrt{2}} \,\middle|\, k \in F \right\}.$$

The ideal theory in quadratic number fields (cf. [6]) implies that F is the set of numbers of the form

$$\varepsilon \cdot 2s^2 \prod_{j=1}^{l} p_j^{\alpha_j},$$

where $\varepsilon = 1$ or -1, s is an odd integer, l is a positive integer, $\alpha_j \in \{0,1\}$ for $1 \le j \le l$, p_j is prime for $1 \le j \le l$, and, for each $j \in \{1, 2, \ldots, l\}$, either $p_j \equiv 1 \pmod 8$ or $p_j \equiv -1 \pmod 8$. This implies, in particular, that

$$M_+\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right) = M_-\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right) = \frac{1}{4\sqrt{2}}.$$

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