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Ascent of Module Structures, Vanishing of Ext, and Extended Modules

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This paper is dedicated to Melvin Hochster on the occasion of his sixty-fifth birthday

Introduction

Suppose $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ are commutative Noetherian local rings and $\varphi\colon R \to S$ is a flat local homomorphism with the property that the induced homomorphism $R/\mathfrak{m} \to S/\mathfrak{m}S$ is bijective. We consider natural questions of ascent and descent of modules between R and S: (i) Given a finitely generated R-module M, when does M have an S-module structure that is compatible with the R-module structure via φ? (ii) Given a finitely generated S-module N, is there a finitely generated R-module M such that N is S-isomorphic to $S \otimes_R M$ or (iii) S-isomorphic to a direct summand of $S \otimes_R M$?

In Section 1 we make some general observations about homomorphisms $R \to S$ satisfying the condition $R/\mathfrak{m} = S/\mathfrak{m}S$. We show that if a compatible S-module structure exists, then it arises in an obvious way: The natural map $M \to S \otimes_R M$ is an isomorphism. (One example to keep in mind is that of a finite-length module M when $S = \hat{R}$, the $\mathfrak{m}$-adic completion.) Moreover, if $R \to S$ is flat, then M has a compatible S-module structure if and only if $S \otimes_R M$ is finitely generated as an R-module.

In Section 2 we prove, assuming that $R \to S$ is flat, that M has a compatible S-module structure if and only if $\operatorname{Ext}^i_R(S, M)$ is finitely generated as an R-module for $i = 1, \dots, \dim_R(M)$. We were motivated to investigate this implication because of the following result of Buchweitz and Flenner [BF] and Frankild and Sather-Wagstaff [FS-W2]: A finitely generated R-module M is $\mathfrak{m}$-adically complete if and only if $\operatorname{Ext}^i_R(\hat{R}, M) = 0$ for all $i \geq 1$. Theorem 2.5 summarizes the main results of the first two sections. Note that it subsumes the result of [BF; FS-W2], but our proof here is quite different.

In Section 3 we address questions (ii) and (iii) and show that (iii) always has an affirmative answer when S is the Henselization, but not necessarily when S is the $\mathfrak{m}$-adic completion.

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1. Ascent of Module Structures

Throughout this section, $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ are Noetherian local rings and $\varphi\colon R \to S$ is a local ring homomorphism. We consider the following condition on φ:

(†) (a) $\mathfrak{m}S = \mathfrak{n}$; and (b) $\varphi(R) + \mathfrak{n} = S$ (i.e., φ induces an isomorphism on residue fields).

This is equivalent to the following: The induced homomorphism $R/\mathfrak{m} \to S/\mathfrak{m}S$ is bijective. Familiar examples include the $\mathfrak{m}$-adic completion $R \to \hat{R}$, the Henselization $R \to R^{\mathrm{h}}$, and the natural map $R \twoheadrightarrow S = R/I$ for I a proper ideal of R.

From (a), it follows immediately that $\mathfrak{m}^t S = \mathfrak{n}^t$ for all t. Similarly, the next result shows that (b) carries over to powers (though here we need *both* (a) and (b), as is shown by the example $\mathbb{C}[[T^2, T^3]] \subseteq \mathbb{C}[[T]]$).

LEMMA 1.1. *If $\varphi\colon R \to S$ satisfies (†), then $\varphi(R) + \mathfrak{n}^t = S$ for each $t \geq 1$.*

Proof. By choosing a composition series, we see that every S-module of finite length has (the same) finite length as an R-module. In particular, $S/\mathfrak{n}^{t+1}$ has finite length and therefore is finitely generated as an R-module. We have

$$\frac{\varphi(R) + \mathfrak{n}^t}{\mathfrak{n}^{t+1}} + \mathfrak{m}\frac{S}{\mathfrak{n}^{t+1}} = \frac{\varphi(R) + \mathfrak{n}^t + \mathfrak{m}S}{\mathfrak{n}^{t+1}} = \frac{\varphi(R) + \mathfrak{n}}{\mathfrak{n}^{t+1}} = \frac{S}{\mathfrak{n}^{t+1}}.$$

Nakayama's lemma implies that $(\varphi(R) + \mathfrak{n}^t)/\mathfrak{n}^{t+1} = S/\mathfrak{n}^{t+1}$. □

The next result is an indispensable tool for several of our proofs.

PROPOSITION 1.2. *Assume $\varphi\colon R \to S$ satisfies (†). Let M and N be S-modules, with ${}_SN$ finitely generated. Then* $\operatorname{Hom}_R(M, N) = \operatorname{Hom}_S(M, N)$.

Proof. We'll show that $\operatorname{Hom}_R(M, N) \subseteq \operatorname{Hom}_S(M, N)$, since the reverse inclusion is obvious. Let $f \in \operatorname{Hom}_R(M, N)$. Given $x \in M$ and $s \in S$, we want to show that $f(sx) = sf(x)$. Since ${}_SN$ is finitely generated, it will suffice to show that $f(sx) - sf(x) \in \mathfrak{n}^t N$ for each $t \geq 1$.

Fix an integer $t \geq 1$, and note the following relations:

$$f(\mathfrak{n}^t M) = f(\mathfrak{m}^t M) \subseteq \mathfrak{m}^t N \subseteq \mathfrak{n}^t N.$$

Use Lemma 1.1 to choose an element $r \in R$ such that $\varphi(r) - s \in \mathfrak{n}^t$. Then we have

$$\begin{aligned} f(sx) - sf(x) &= f(sx) - f(rx) + rf(x) - sf(x) \\ &= f((s - \varphi(r))x) + (\varphi(r) - s)f(x). \end{aligned}$$

It follows that $f(sx) - sf(x)$ is in

$$f((s - \varphi(r))M) + (\varphi(r) - s)N \subseteq f(\mathfrak{n}^t M) + \mathfrak{n}^t N = \mathfrak{n}^t N. \qquad \square$$

COROLLARY 1.3. *Let $\varphi\colon R \to S$ be a local homomorphism satisfying (†), and let M be a finitely generated S-module. Then M is indecomposable as an R-module if and only if it is indecomposable as an S-module.*

Proof. We know that M is indecomposable as an R-module if and only if $\mathrm{End}_R(M)$ has no nontrivial idempotents, and similarly over S. The equality $\mathrm{End}_R(M) = \mathrm{End}_S(M)$ from Proposition 1.2 now yields the desired result. $\square$

For any ring homomorphism $\varphi\colon R \to S$, every S-module acquires an R-module structure via φ. We want to understand when the reverse holds: Given an R-module M, often assumed to be finitely generated, when does M have an S-module structure $(s, x) \mapsto s \circ x$ that is compatible with the R-module structure (i.e., $rx = \varphi(r) \circ x$ for $r \in R$ and $x \in M$)? When this happens, we will say simply that ${}_RM$ *has a compatible S-module structure*. We are particularly interested in the case where the S-module structure is *unique*.

LEMMA 1.4. *Assume $\varphi\colon R \to S$ satisfies* (†). *Let N be a finitely generated S-module, and let V be an R-submodule of N. Then ${}_RV$ has at most one compatible S-module structure. In detail, if V has an S-module structure $(s, v) \mapsto s \circ v$ that is compatible with the R-module structure on V inherited from the S-module structure $(s, n) \mapsto s \cdot n$ on N, then $s \circ v = s \cdot v$ for all $s \in S$ and $v \in V$.*

Proof. Let $s \in S$ and $v \in V$ be given. As before, we fix an integer $t \geq 1$ and choose $r \in R$ such that $\varphi(r) - s \in \mathfrak{n}^t$. Note the following relations:

$$\mathfrak{n}^t \circ V = (\mathfrak{m}^t S) \circ V = \mathfrak{m}^t \circ (S \circ V) = \mathfrak{m}^t \circ V = \mathfrak{m}^t \cdot V \subseteq \mathfrak{m}^t \cdot N = \mathfrak{n}^t \cdot N.$$

It follows that we have

$$\begin{aligned} s \circ v - s \cdot v &= s \circ v - r \circ v + r \cdot v - s \cdot v \\ &= (s - \varphi(r)) \circ v + (\varphi(r) - s) \cdot v \in \mathfrak{n}^t \circ V + \mathfrak{n}^t \cdot V \subseteq \mathfrak{n}^t \cdot N. \end{aligned}$$

Since t was chosen arbitrarily, we conclude that $s \circ v = s \cdot v$. $\square$

PROPOSITION 1.5. *Assume $\varphi\colon R \to S$ satisfies* (†). *Let M be an R-module* (*not necessarily finitely generated*) *that is an R-submodule of some finitely generated S-module N. Let $\mathcal{V}(M)$ be the set of R-submodules of M that have an S-module structure compatible with their R-module structure. Then $\mathcal{V}(M)$ is exactly the set of S-submodules of N that are contained in M. The set $\mathcal{V}(M)$ has a unique maximal element* $\mathrm{V}(M)$. *Moreover,* $\mathrm{V}(M) = \{x \in M \mid Sx \subseteq M\} = \{x \in N \mid Sx \subseteq M\}$.

Proof. The first assertion is clear from Lemma 1.4, so $\mathcal{V}(M)$ must be closed under sums. Since N is a Noetherian S-module, the other assertions follow easily. $\square$

Although $\mathrm{V}(M)$ is defined only when M can be embedded as an R-submodule of some finitely generated S-module N, its definition is intrinsic. Thus the submodule $\mathrm{V}(M)$ of M does not depend on the choice of the module N or on the choice of the R-embedding $M \hookrightarrow N$. (See Corollary 1.7 for another intrinsic characterization of $\mathrm{V}(M)$.)

PROPOSITION 1.6. *Assume $\varphi\colon R \to S$ satisfies* (†), *and let L be an S-module* (*not necessarily finitely generated*). *Let M be an R-submodule of some finitely generated S-module, and let* $\mathrm{V}(M)$ *be as in Proposition 1.5. Then the natural injection* $\mathrm{Hom}_R(L, \mathrm{V}(M)) \to \mathrm{Hom}_R(L, M)$ *is an isomorphism.*

Proof. Let $g \in \mathrm{Hom}_R(L, M)$, and let W be the image of g. We want to show that $W \subseteq \mathrm{V}(M)$. Let h be the composition $L \xrightarrow{g} M \hookrightarrow N$, where N is some finitely generated S-module containing M as an R-submodule. By Proposition 1.2, the map h is S-linear, so $W = h(L)$ is an S-submodule of N. Therefore we have $W \subseteq \mathrm{V}(M)$. $\square$

COROLLARY 1.7. *Assume $\varphi\colon R \to S$ satisfies (†). Let M be an R-submodule of a finitely generated S-module. The following natural maps are isomorphisms:*

$$\mathrm{V}(M) \xrightarrow{\cong} \mathrm{Hom}_S(S, \mathrm{V}(M)) \xrightarrow{=} \mathrm{Hom}_R(S, \mathrm{V}(M)) \xrightarrow{\cong} \mathrm{Hom}_R(S, M).$$

It follows that $\mathrm{V}(M)$ *is exactly the image of the natural map* $\varepsilon\colon \mathrm{Hom}_R(S, M) \to M$ *taking ψ to $\psi(1)$. In particular, if M is finitely generated as an R-module then so is* $\mathrm{Hom}_R(S, M)$.

The next result contains the first part of our answer to question (i) from the Introduction.

THEOREM 1.8. *Assume $\varphi\colon R \to S$ satisfies (†), and let M be a finitely generated R-module. The following conditions are equivalent.*

(1) *M has a compatible S-module structure.*
(2) *The natural map $\iota\colon M \to S \otimes_R M$ (taking x to $1 \otimes x$) is bijective.*
(3) *The natural map $\varepsilon\colon \mathrm{Hom}_R(S, M) \to M$ (taking ψ to $\psi(1)$) is bijective.*

If, in addition, φ is flat, then these conditions are equivalent to the following:

(4) *$S \otimes_R M$ is finitely generated as an R-module.*

Proof. The implications (2) $\Rightarrow$ (1), (3) $\Rightarrow$ (1), and (2) $\Rightarrow$ (4) are clear. Assume (1), and let $(s, x) \mapsto s \cdot x$ be a compatible S-module structure on M. To prove (2), we note that the module $S \otimes_R M$ has two compatible S-module structures: the one coming from multiplication in S and the one coming from the S-module structure on M. Moreover, with the first structure, $S \otimes_R M$ is finitely generated over S. By Lemma 1.4 the two S-module structures must be the same. In particular, for $s \in S$ and $x \in M$ we have $s \otimes x = s(1 \otimes x) = 1 \otimes (s \cdot x)$. Hence the multiplication map $\mu\colon S \otimes_R M \to M$ (taking $s \otimes x$ to $s \cdot x$) is the inverse of ι.

Still assuming (1), we prove (3). Since M is finitely generated as an S-module, Proposition 1.2 tells us that $\mathrm{Hom}_R(S, M) = \mathrm{Hom}_S(S, M)$. Therefore, the map $M \to \mathrm{Hom}_S(S, M)$ taking $x \in M$ to the map $s \mapsto s \cdot x$ is the inverse of ε.

(4) $\Rightarrow$ (2): Assume that φ is flat. By (4), the S-module $S \otimes_R S \otimes_R M$ is finitely generated for the S-action on the first variable; hence its two S-module structures (obtained by letting S act on each of the first two factors) are the same by Lemma 1.4. In particular, $s \otimes t \otimes x = st \otimes 1 \otimes x$ for $s, t \in S$ and $x \in M$. Therefore the map $S \otimes_R S \otimes_R M \to S \otimes_R M$ that takes $s \otimes t \otimes x$ to $st \otimes x$ is the inverse of $1 \otimes \iota\colon S \otimes_R M \to S \otimes_R S \otimes_R M$. By faithful flatness, ι is an isomorphism. $\square$

In light of Corollary 1.7, we see that the conditions in Theorem 1.8 are not equivalent to $\mathrm{Hom}_R(S, M)$ being finitely generated as an R-module, even when φ is flat.

In the next section, we will show that the "right" condition is that $\mathrm{Ext}_R^i(S, M)$ be finitely generated for $i = 1, \dots, \dim_R(M)$.

Next we revisit Theorem 1.8 from a slightly different perspective.

THEOREM 1.9. *Let $\varphi\colon R \to S$ be a flat local homomorphism satisfying* (†), *and let M be a finitely generated S-module. The following conditions are equivalent.*

(1) *M is finitely generated as an R-module.*
(2) *The natural map $\iota_M\colon M \to S \otimes_R M$ (taking x to $1 \otimes x$) is bijective.*
(3) *$S \otimes_R M$ is finitely generated as an R-module.*

In particular, if S has a faithful module that is finitely generated as an R-module, then φ is an isomorphism.

Proof. The implication (1) $\Rightarrow$ (2) is in Theorem 1.8. Suppose (2) holds. The R-module $S \otimes_R M$ has two S-module structures and, by (2), is finitely generated with respect to the S-action on the second factor. By Lemma 1.4, the two structures agree, and $S \otimes_R M$ is finitely generated with respect to the S-action on the first factor. By faithfully flat descent, M is finitely generated over R. Using (2) again, we obtain (3).

If (3) holds, then $S \otimes_R M$ is a fortiori finitely generated for the action of S on the first factor. Again using faithfully flat descent, we get (1).

To prove the last statement, suppose N is a faithful S-module that is finitely generated as an R-module. Let $x_1, \dots, x_t$ generate N as an S-module, and define $\alpha\colon S \to N^t$ by $1 \mapsto (x_1, \dots, x_t)$. The kernel of α is the intersection of the annihilators of the x_i, and this intersection is (0) because N is faithful. Hence S embeds in N^t and thus is finitely generated as an R-module. Now we put $M = S$ in (2) and observe that $\varphi \otimes_R S\colon R \otimes_R S \to S \otimes_R S$ is the composition $R \otimes_R S \xrightarrow{\cong} S \xrightarrow{\iota_S} S \otimes_R S$. Therefore, $\varphi \otimes_R S$ is an isomorphism and so, by faithful flatness, φ must be an isomorphism. □

PROPOSITION 1.10. *Assume $\varphi\colon R \to S$ satisfies* (†). *The following conditions are equivalent.*

(1) *R has a compatible S-module structure.*
(2) *φ is an R-split monomorphism.*
(3) *S is a free R-module.*
(4) *φ is a bijection.*

Proof. The implication (4) $\Rightarrow$ (3) is clear.

(1) $\Rightarrow$ (4): From Theorem 1.8 we conclude that the map $\iota\colon R \to S \otimes_R R$ is bijective, and it follows that φ is the composition of two bijections: $R \xrightarrow{\iota} S \otimes_R R \to S$.

(2) $\Rightarrow$ (1): Let $\pi : S \to R$ be an R-homomorphism such that $\pi\varphi = 1_R$. The composition $\varphi\pi : S \to S$ is S-linear by Proposition 1.2, so $\varphi(R) = \varphi\pi(S)$ is an S-module and then (1) follows.

(3) $\Rightarrow$ (2): Let B be a basis for S as an R-module. Write $1 = \sum_{i=1}^n r_i b_i$, where the r_i are in R and the b_i are distinct elements of B. If each r_i were in $\mathfrak{m}$ then we would have $1 \in \mathfrak{m}S = \mathfrak{n}$—contradiction. Thus we may assume that r_1 is a unit of

R. Let $\pi: S \to R$ be the R-homomorphism taking b_1 to r_1^{-1} and $b \in B - \{b_1\}$ to 0. Then $\pi\varphi = 1_R$, and we have (2). □

Now we focus on flat homomorphisms satisfying (†). (In this context *every* finitely generated R-module can be embedded in a finitely generated S-module—namely, $S \otimes_R M$—and so $\mathrm{V}(M)$ is always defined.) Every finite-length R-module has a compatible S-module structure. (This follows from Lemma 1.12 by induction on the length, since $R/\mathfrak{m} = S/\mathfrak{m}S$.) There are other examples, as follows.

EXAMPLE 1.11. Let R be a local ring and P a nonmaximal prime ideal such that R/P is $\mathfrak{m}$-adically complete (e.g., $R = (\mathbb{C}[X]_{(X)})[[Y]]$ and $P = (X)$). Then R/P has a compatible $\hat{R}$-module structure. Indeed, the map $R/P \to \hat{R}/P\hat{R}$ is bijective.

As we shall see in Theorem 1.13, the behavior of prime ideals tells the whole story.

The following lemma is clear from the five-lemma and Theorem 1.8(2).

LEMMA 1.12. *Let* $\varphi: R \to S$ *be a flat local homomorphism satisfying* (†), *and let*

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of finitely generated R-modules. Then M has a compatible S-module structure if and only if M′ and M″ have compatible S-module structures.

THEOREM 1.13. *Let* $\varphi: R \to S$ *be a flat local homomorphism satisfying* (†), *and let M be a finitely generated R-module. The following conditions are equivalent.*

(1) *M has a compatible S-module structure.*
(2) $S = R + PS$ (*equivalently, R/P has a compatible S-module structure*) *for every* $P \in \mathrm{Min}_R(M)$.
(3) $S = R + PS$ (*equivalently, R/P has a compatible S-module structure*) *for every* $P \in \mathrm{Supp}_R(M)$.

Proof. The condition $S = R + PS$ just says that the injection $R/P \hookrightarrow S \otimes_R (R/P)$ is an isomorphism; now Theorem 1.8 justifies the parenthetical comments. If (1) holds and $P \in \mathrm{Min}_R(M)$, then there is an injection $R/P \hookrightarrow M$ and so Lemma 1.12 with $M' = R/P$ yields (2). Assume (2). Given $P \in \mathrm{Supp}_R(M)$, we have $P \supseteq Q$ for some $Q \in \mathrm{Min}_R(M)$. Then $R/Q \twoheadrightarrow R/P$, and (3) follows from Lemma 1.12. Assuming (3), choose a prime filtration $M = M_0 \subset \cdots \subset M_t$ with $M_i/M_{i-1} \cong R/P_{i-1}$ for $P_i \in \mathrm{Spec}(R)$, $i = 1, \dots, t$. Then $P_i \in \mathrm{Supp}_R(M)$ for each i, and now (1) follows from Lemma 1.12. □

Let $\varphi: (R, \mathfrak{m}, k) \hookrightarrow (S, \mathfrak{n}, l)$ be a flat local homomorphism. Recall that φ is *separable* if the "diagonal" morphism $S \otimes_R S \to S$ (taking $a \otimes b$ to ab) splits as $(S \otimes_R S)$-modules (cf. [DI]). If also φ is essentially of finite type, then φ is said to be an *étale* extension of R (cf. [I]). An étale extension φ is a *pointed étale neighborhood* of R if $k = l$. It is easy to see that $\mathfrak{m}S = \mathfrak{n}$ whenever φ is an étale extension; thus, pointed étale neighborhoods satisfy condition (†). The R-isomorphism classes of pointed étale neighborhoods form a direct system, and the Henselization $R \to R^{\mathrm{h}}$ is the direct limit of them.

COROLLARY 1.14. *Let R be a local ring and M a finitely generated R-module. The following conditions are equivalent.*

(1) *M admits an R^{h}-module structure that is compatible with its R-module structure via the natural inclusion $R \to R^{\mathrm{h}}$.*
(2) *For each $P \in \mathrm{Supp}_R(M)$, the ring R/P is Henselian.*
(3) *For each $P \in \mathrm{Min}_R(M)$, the ring R/P is Henselian.*
(4) *The ring $R/\mathrm{Ann}_R(M)$ is Henselian.*

COROLLARY 1.15. *Let R be a local ring. The following conditions are equivalent.*

(1) *R is Henselian.*
(2) *For each $P \in \mathrm{Spec}(R)$, the ring R/P is Henselian.*
(3) *For each $P \in \mathrm{Min}(R)$, the ring R/P is Henselian.*

2. Vanishing of Ext

Our goal in this section is to add a fifth condition equivalent to the conditions in Theorem 1.8—namely, that $\mathrm{Ext}^i_R(S, M) = 0$ for $i > 0$. Here $R \to S$ is a flat local homomorphism satisfying (†) and M is a finitely generated R-module. Moreover, we will obtain a sixth equivalent condition: that $\mathrm{Ext}^i_R(S, M)$ is finitely generated over R for $i = 1, \ldots, \dim_R(M)$. Because our proof uses complexes, we will review the basics here.

2.1. NOTATION AND CONVENTIONS. An *R-complex* is a sequence of R-module homomorphisms

$$X = \cdots \xrightarrow{\partial^X_{n+1}} X_n \xrightarrow{\partial^X_n} X_{n-1} \xrightarrow{\partial^X_{n-1}} \cdots$$

such that $\partial^X_{n-1}\partial^X_n = 0$ for each integer n; the nth *homology module* of X is $\mathrm{H}_n(X) := \mathrm{Ker}(\partial^X_n)/\mathrm{Im}(\partial^X_{n+1})$. A complex X is *bounded* if $X_n = 0$ for $|n| \gg 0$, *bounded above* if $X_n = 0$ for $n \gg 0$, and *homologically finite* if its total homology module $\mathrm{H}(X) = \bigoplus_n \mathrm{H}_n(X)$ is a finitely generated R-module.

Let X, Y be R-complexes. The *Hom complex* $\mathrm{Hom}_R(X, Y)$ is the R-complex defined as

$$\mathrm{Hom}_R(X, Y)_n = \prod_p \mathrm{Hom}_R(X_p, Y_{p+n})$$

with nth differential $\partial_n^{\mathrm{Hom}_R(X,Y)}$ given by

$$\{f_p\} \mapsto \{\partial^Y_{p+n} f_p - (-1)^n f_{p-1}\partial^X_p\}.$$

A *morphism* $X \to Y$ is an element $f = \{f_p\} \in \mathrm{Hom}_R(X, Y)_0$ such that $\partial^Y_p f_p = f_{p-1}\partial^X_p$ for all p—that is, an element of $\mathrm{Ker}(\partial_0^{\mathrm{Hom}_R(X,Y)})$.

A morphism of complexes $\alpha \colon X \to Y$ induces homomorphisms on homology modules $\mathrm{H}_n(\alpha) \colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$, and α is a *quasi-isomorphism* when each $\mathrm{H}_n(\alpha)$ is bijective. The symbol "$\simeq$" indicates a quasi-isomorphism.

2.2. BASE CHANGE. Let $\varphi\colon R \to S$ be a flat homomorphism. For any R-complex X, the flatness of φ provides natural S-module isomorphisms

$$\mathrm{H}_i(S \otimes_R X) \cong S \otimes_R \mathrm{H}_i(X)$$

for each integer i.

2.3. A CONNECTION WITH CONDITION (†). Let $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a flat local ring homomorphism, and write $\bar{\varphi}\colon k \to S/\mathfrak{m}S$ for the induced ring homomorphism. Let $X \not\simeq 0$ be an R-complex such that each homology module $\mathrm{H}_i(X)$ is a finite-dimensional k-vector space, and let r_i denote the vector-space dimension of $\mathrm{H}_i(X)$. (In our applications we will consider the case $X = K^R$, the Koszul complex on a minimal system of generators for $\mathfrak{m}$; by [BrH, (1.6.5)], the homology $\mathrm{H}(K^R)$ is annihilated by $\mathfrak{m}$ and so each $\mathrm{H}_i(K^R)$ is a finite-dimensional k-vector space. Note that $K^R \not\simeq 0$ since $\mathrm{H}_0(K^R) \cong k$.) Define $\omega\colon X \to S \otimes_R X$ by the commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\omega"] \arrow[dr, "\cong"'] & & S \otimes_R X \\ & R \otimes_R X, \arrow[ur, "\varphi \otimes_R X"'] & \end{tikzcd}$$

where the southeast arrow represents the standard isomorphism. We have a commutative diagram of k-linear homomorphisms

$$\begin{tikzcd} \mathrm{H}_i(X) \arrow[r, "\mathrm{H}_i(\omega)"] \arrow[d, "\cong"'] & \mathrm{H}_i(S \otimes_R X) \arrow[r, "\cong"] & S \otimes_R \mathrm{H}_i(X) \arrow[r, "\cong"] & S \otimes_R k^{(r_i)} \arrow[d, "\cong"] \\ k^{(r_i)} \arrow[rrr, "\bar{\varphi}^{(r_i)}"] & & & (S/\mathfrak{m}S)^{(r_i)}. \end{tikzcd}$$

Consequently, the morphism ω is a quasi-isomorphism if and only if $\bar{\varphi}$ is an isomorphism—that is, if and only if $\varphi\colon R \to S$ satisfies condition (†).

The following result is contained in [FS-W1, (5.3)].

PROPOSITION 2.4. *Let X and Y be R-complexes such that $\mathrm{H}_n(X)$ and $\mathrm{H}_n(Y)$ are finitely generated R-modules for each n. Let $\alpha\colon X \to Y$ be a morphism. Assume P is a bounded complex of finitely generated projective R-modules such that $P \not\simeq 0$ and $\mathrm{Hom}_R(P, \alpha)$ is a quasi-isomorphism. Then α is a quasi-isomorphism.*

We can now put the finishing touch on Theorem 1.8.

MAIN THEOREM 2.5. *Let $\varphi\colon R \to S$ be a ring homomorphism satisfying (†), and let M be a finitely generated R-module. The following conditions are equivalent.*

(1) *M has a compatible S-module structure.*
(2) *The natural map $\iota\colon M \to S \otimes_R M$ (taking x to $1 \otimes x$) is bijective.*
(3) *The natural map $\varepsilon\colon \mathrm{Hom}_R(S, M) \to M$ (taking ψ to $\psi(1)$) is bijective.*

If, in addition, φ is flat, then these conditions are equivalent to the following.

(4) $S \otimes_R M$ *is finitely generated as an* R*-module.*

(5) $\operatorname{Ext}^i_R(S, M)$ *is a finitely generated* R*-module for* $i = 1, \dots, \dim_R(M)$.

(6) $\operatorname{Ext}^i_R(S, M) = 0$ *for all* $i > 0$.

Proof. The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) are in Theorem 1.8, as is (3) $\Leftrightarrow$ (4) when φ is flat. The implication (6) $\Rightarrow$ (5) is trivial, so it remains to assume that φ is flat and prove (5) $\Rightarrow$ (3) and (1) $\Rightarrow$ (6).

(5) $\Rightarrow$ (3): Assume $\operatorname{Ext}^i_R(S, M)$ is finitely generated over R for $i = 1, \dots, \dim_R(M)$. We first show that $\operatorname{Ext}^i_R(S, M) = 0$ for each $i > \dim_R(M)$. Let P be an R-projective resolution of S, and set $R' = R/\operatorname{Ann}_R(M)$. The fact that M is an R'-module yields the first isomorphism in the following sequence:

$$\operatorname{Hom}_R(P, M) \cong \operatorname{Hom}_R(P, \operatorname{Hom}_{R'}(R', M)) \cong \operatorname{Hom}_{R'}(P \otimes_R R', M). \tag{2.5.1}$$

The second isomorphism is Hom-tensor adjointness. Of course we have isomorphisms $\mathrm{H}_n(P \otimes_R R') \cong \operatorname{Tor}^R_n(S, R')$, so the flatness of φ yields $\mathrm{H}_n(P \otimes_R R') = 0$ for $n > 0$. Hence the complex $P \otimes_R R'$ is an R'-projective resolution of $S' := S \otimes_R R'$. Since S' is flat over R', we have $\operatorname{pd}_{R'}(S') \leq \dim(R')$ by a result of Raynaud and Gruson [RG, Seconde Partie, Thm. (3.2.6)] and Jensen [J, Prop. 6]. Consequently, $\operatorname{Ext}^n_{R'}(S', M) = 0$ for each $n > \dim(R') = \dim_R(M)$. This yields the vanishing in the next sequence for $n > \dim_R(M)$:

$$\begin{aligned} \operatorname{Ext}^n_R(S, M) &\cong \mathrm{H}_{-n}(\operatorname{Hom}_R(P, M)) \\ &\cong \mathrm{H}_{-n}(\operatorname{Hom}_{R'}(P \otimes_R R', M)) \cong \operatorname{Ext}^n_{R'}(S', M) = 0. \end{aligned}$$

The first isomorphism is by definition; the second one is from (2.5.1); and the third one is from the fact, already noted, that $P \otimes_R R'$ is an R'-projective resolution of $S' = S \otimes_R R'$.

Let I be an R-injective resolution of M. From Corollary 1.7 it follows that $\operatorname{Hom}_R(S, M)$ is a finitely generated R-module. Since $\operatorname{Ext}^n_R(S, M) = 0$ for $i > \dim_R(M)$ and since $\operatorname{Ext}^n_R(S, M)$ is finitely generated over R for $1 \leq n \leq \dim_R(M)$, it follows that the complex $\operatorname{Hom}_R(S, I)$ is homologically finite over R.

Consider the evaluation morphism $\alpha\colon \operatorname{Hom}_R(S, I) \to I$ given by $f \mapsto f(1)$. To verify condition (3), it suffices to show that α is a quasi-isomorphism. Indeed, assume for the rest of this paragraph that α is a quasi-isomorphism. It is then straightforward to show that the map $\mathrm{H}_0(\alpha)\colon \mathrm{H}_0(\operatorname{Hom}_R(S, I)) \to \mathrm{H}_0(I)$ is equivalent to the evaluation map $\varepsilon\colon \operatorname{Hom}_R(S, M) \to M$. The quasi-isomorphism assumption implies that ε is an isomorphism, so condition (3) must hold.

We now show that α is a quasi-isomorphism. Let $\mathbf{x} = x_1, \dots, x_m$ be a minimal generating sequence for $\mathfrak{m}$. The flatness of φ together with the condition $\mathfrak{m}S = \mathfrak{n}$ implies that $\varphi(\mathbf{x}) = \varphi(x_1), \dots, \varphi(x_m)$ is a minimal generating sequence for $\mathfrak{n}$. Let $K^R = K^R(\mathbf{x})$ and $K^S = K^S(\varphi(\mathbf{x}))$ denote the respective Koszul complexes, and note that we have $\operatorname{rank}_R(K^R_i) = \operatorname{rank}_S(K^S_i) = r := \binom{m}{i}$. Let $e_{i,1}, \dots, e_{i,r}$ be an R-basis for K^R_i, and let $f_{i,1}, \dots, f_{i,r}$ be the naturally corresponding S-basis for K^S_i. The construction yields a natural isomorphism of S-complexes

$\beta\colon K^R \otimes_R S \to K^S$ taking $e_{i,j} \otimes 1$ to $f_{i,j}$. On the other hand, let $K^\varphi\colon K^R \to K^S$ be given by $e_{i,j} \mapsto f_{i,j}$. By Section 2.3, the flatness of φ and condition (†) work together to show that K^φ is a quasi-isomorphism.

The source and target of the morphism $\alpha\colon \mathrm{Hom}_R(S, I) \to I$ are both homologically finite R-complexes, so it suffices to verify that the induced morphism

$$\mathrm{Hom}_R(K^R, \alpha)\colon \mathrm{Hom}_R(K^R, \mathrm{Hom}_R(S, I)) \to \mathrm{Hom}_R(K^R, I)$$

is a quasi-isomorphism; see Proposition 2.4. This isomorphism is verified by the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_R(K^R \otimes_R S, I) \arrow[d, "(*)"', "\cong"] & \mathrm{Hom}_R(K^S, I) \arrow[l, "\mathrm{Hom}_R(\beta, I)"', "\simeq"] \arrow[d, "\mathrm{Hom}(K^\varphi, I)"', "\simeq"] \\
\mathrm{Hom}_R(K^R, \mathrm{Hom}_R(S, I)) \arrow[r, "\mathrm{Hom}_R(K^R, \alpha)"] & \mathrm{Hom}_R(K^R, I),
\end{tikzcd}$$

where the isomorphism $(*)$ is Hom-tensor adjointness. The morphism $\mathrm{Hom}_R(\beta, I)$ is a quasi-isomorphism because I is a bounded-above complex of injective R-modules and β is a quasi-isomorphism (see e.g. the proof of [Wei, (2.7.6)]). The same reasoning shows that $\mathrm{Hom}(K^\varphi, I)$ is a quasi-isomorphism. From the commutativity of the diagram, it follows that $\mathrm{Hom}_R(K^R, \alpha)$ is a quasi-isomorphism as well.

(1) $\Rightarrow$ (6): Assume that M admits an S-module structure that is compatible with its R-module structure via φ. Since M is finitely generated over R, it admits a filtration by R-submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that $M_i/M_{i-1} \cong R/P_i$ for each $i = 1, \dots, n$, where $P_i \in \mathrm{Supp}_R(M)$. We prove the implication by induction on n.

When $n = 1$, we have $M \cong R/P$ for some $P \in \mathrm{Spec}(R)$. The implication (1) $\Rightarrow$ (2) yields an isomorphism $M \cong R/P \cong S \otimes_R R/P$. If Q is an R-projective resolution of S, then the flatness of S implies that $Q \otimes_R R/P$ is an R/P-projective resolution of $S \otimes_R R/P \cong R/P$. Using the isomorphisms

$$\begin{aligned}
\mathrm{Hom}_R(Q, R/P) &\cong \mathrm{Hom}_R(Q, \mathrm{Hom}_{R/P}(R/P, R/P)) \\
&\cong \mathrm{Hom}_{R/P}(Q \otimes_R R/P, R/P)
\end{aligned}$$

we conclude that

$$\mathrm{Ext}^i_R(S, M) = \mathrm{Ext}^i_R(S, R/P) \cong \mathrm{Ext}^i_{R/P}(R/P, R/P) = 0$$

for $i \neq 0$.

Now assume that $n > 1$ and that the implication holds for each R-module M' admitting a prime filtration with fewer than n inclusions. Because of the exact sequence

$$0 \to M_1 \to M \to M/M_1 \to 0,$$

Lemma 1.12 implies that M_1 and M/M_1 admit S-module structures that are compatible with their R-module structures via φ. The induction hypothesis implies

$\operatorname{Ext}^i_R(S, M_1) = 0 = \operatorname{Ext}^i_R(S, M/M_1)$ for all $i > 0$. Thus, the long exact sequence in Ext coming from the displayed sequence implies $\operatorname{Ext}^i_R(S, M) = 0$ for all $i > 0$. □

COROLLARY 2.6. *Let R be a local ring and $\mathfrak{a} \subset R$ an ideal.*

(1) *The $\mathfrak{a}$-adic completion $\hat{R}^{\mathfrak{a}}$ is R-projective if and only if R is $\mathfrak{a}$-adically complete.*
(2) *The Henselization R^{h} is R-projective if and only if R is Henselian.*
(3) *If $R \to R'$ is a pointed étale neighborhood and if R' is R-projective, then $R = R'$.*

Proof. Suppose $S := \hat{R}^{\mathfrak{a}}$ is R-projective. Putting $M = R$ in Theorem 2.5 and using Proposition 1.10, we see that $R = S$. This proves (1), and the proofs of (2) and (3) are essentially the same. □

We conclude this section with several examples showing the necessity of the hypotheses of Theorem 2.5 with respect to the implications (5) $\Rightarrow$ (1) and (6) $\Rightarrow$ (1). The examples depend on the following addendum to Proposition 1.10, in which we no longer assume condition (†).

PROPOSITION 2.7. *Let $\varphi\colon A \to B$ be an arbitrary homomorphism of commutative rings. The following conditions are equivalent*:

(1) *The A-module A has a B-module structure $(b, a) \mapsto b \circ a$ such that*
$$a_1 a_2 = \varphi(a_1) \circ a_2 \quad \textit{for all } a_1, a_2 \in A. \tag{a}$$
(2) *A is a ring retract of B; that is, there is a ring homomorphism $\psi\colon B \to A$ such that*
$$\psi\varphi(a) = a \quad \textit{for each } a \in A. \tag{b}$$

These conditions imply that φ is an A-split injection.

Proof. Assuming (1), we define a function $\psi\colon B \to A$ by putting $\psi(b) := b \circ 1_A$ for each $b \in B$. Condition (b) follows immediately from (a). Also, given $a \in A$ and $b \in B$, we have $\psi(ab) = \psi(\varphi(a)b) = (\varphi(a)b) \circ 1_A = \varphi(a) \circ (b \circ 1_A)$ by associativity of the B-module structure. Condition (a) implies $\varphi(a) \circ (b \circ 1_A) = a(b \circ 1_A) = a\psi(b)$, so ψ is A-linear. This shows that φ is an A-split injection.

Still assuming (1), let $b_1, b_2 \in B$. By associativity of the B-module structure, we have
$$\psi(b_1 b_2) = (b_1 b_2) \circ 1_A = b_1 \circ (b_2 \circ 1_A) = b_1 \circ \psi(b_2). \tag{c}$$
On the other hand, the A-linearity of ψ yields $\psi(b_1)\psi(b_2) = \psi(b_1\varphi(\psi(b_2)))$. By (c), this implies
$$\psi(b_1\varphi(\psi(b_2))) = b_1 \circ \psi\varphi\psi(b_2) = b_1 \circ \psi(b_2).$$
Thus $\psi(b_1)\psi(b_2) = b_1 \circ \psi(b_2)$ and so (c) implies that ψ is a ring homomorphism.

For the converse, assume (2) and set $b \circ a := \psi(b\varphi(a))$ for all $a \in A$ and $b \in B$. One checks readily the equalities $(b_1 b_2) \circ a = a\psi(b_1 b_2) = b_1 \circ (b_2 \circ a)$ for

$b_i \in B$ and $a \in A$. Thus we have defined a legitimate B-module structure on A. The verification of (a) is easy and is left to the reader. □

The following example shows why we need to assume that the induced map between the residue fields of R and S is an isomorphism in the implications (5) ⇒ (1) and (6) ⇒ (1) of Theorem 2.5.

EXAMPLE 2.8. Let $\varphi\colon K \to L$ be a proper field extension. Then φ is a flat local homomorphism and $\mathfrak{m}_K L = \mathfrak{m}_L$ (but the induced map $K/\mathfrak{m}_K \to L/\mathfrak{m}_L$ is not an isomorphism). If we take $M = R$, then conditions (5) and (6) of Theorem 2.5 are satisfied but (1) is not. Indeed, suppose (1) holds. Proposition 2.7 provides a field homomorphism $\psi\colon L \to K$ such that $\psi\varphi$ is the identity map on K. Since ψ is necessarily injective, it follows that ψ and φ are reciprocal isomorphisms—a contradiction.

The next example shows the necessity of the condition $\mathfrak{m}S = \mathfrak{n}$ for the implications (5) ⇒ (1) and (6) ⇒ (1) in Theorem 2.5.

EXAMPLE 2.9. Let k be a field and $p \geq 2$ an integer. Set $R = k[[X^p]]$ and $S = k[[X]]$, and let $\varphi\colon R \to S$ be the inclusion map. Again, we put $M = R$. Then φ is a local homomorphism inducing an isomorphism on residue fields (but $\mathfrak{m}_R S \neq \mathfrak{m}_S$). Since S is a free R-module (with basis $\{1, X, \dots, X^{p-1}\}$), conditions (5) and (6) are satisfied. Suppose, by way of contradiction, that (1) is satisfied. Using Proposition 2.7, we get a ring homomorphism $\psi\colon S \to R$ such that $\psi\varphi$ is the identity map on R. Putting $z := \psi(X)$, we see that $X^p = \psi(z^p) \in \mathfrak{m}_R^p$, an obvious contradiction.

Similarly, let R be a regular local ring of characteristic $p > 0$. Take $S = R$ and assume that R is F-finite, that is, that the Frobenius endomorphism $\varphi\colon R \to S$ is module-finite. (This holds, for example, if R is a power series ring over a perfect field.) As an R-module, S is flat by [K] and therefore free. Thus conditions (5) and (6) hold. Assume $k := R/\mathfrak{m}_R$ is perfect and that $\dim(R) > 0$. Then φ induces an isomorphism on residue fields, and essentially the same argument as before shows that condition (1) fails.

The next two examples will show why we need φ to be flat for the implications (5) ⇒ (1) and (6) ⇒ (1), respectively. Note that the homomorphism φ satisfies (†) in both examples and has finite flat dimension in Example 2.10.

EXAMPLE 2.10. Let R be a local ring with $\operatorname{depth}(R) \geq 1$, and fix an R-regular element $x \in \mathfrak{m}$. We consider the natural surjection $\varphi\colon R \to R/(x)$. It is straightforward to show that $\operatorname{Ext}^1_R(R/(x), R) \cong R/(x)$ and $\operatorname{Ext}^n_R(R/(x), R) = 0$ when $n \neq 1$. In particular, each $\operatorname{Ext}^n_R(R/(x), R)$ is finitely generated over R. Suppose (1) holds, and let $\psi\colon S \to R$ be the retraction promised by Proposition 2.7. Then $x = \psi\varphi(x) = 0$, a contradiction.

EXAMPLE 2.11. Let R be a local Artinian Gorenstein ring with residue field $k \neq R$, and consider the natural surjection $\varphi\colon R \to k$. Because R is self-injective, we have $\operatorname{Ext}^n_R(k, R) = 0$ when $n \neq 0$. Thus conditions (5) and (6) of Theorem 2.5 hold. As in Example 2.10, we see easily that (1) fails.

3. Extended Modules

Let $\varphi\colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local homomorphism. Given a finitely generated S-module N, we say that N is *extended* provided there is an R-module M such that $S \otimes_R M \cong N$ as S-modules. By faithfully flat descent, such a module M (if it exists) is unique up to R-isomorphism and is necessarily finitely generated.

We begin with a "two out of three" principle, which is well known when $S = \hat{R}$. The proof in general seems to require a different approach from the proof in that special case. The following notation will be used in the proof: given a ring A and A-modules U and V, we write $U \mid_A V$ to indicate that U is isomorphic to a direct summand of V.

Proposition 3.1. *Let $\varphi\colon R \to S$ be a flat local homomorphism. Let N_1 and N_2 be finitely generated S-modules, and put $N = N_1 \oplus N_2$. If two of the modules N_1, N_2, N are extended, then so is the third.*

Proof. We begin with a claim: If M_1 and M are finitely generated R-modules and if $S \otimes_R M_1 \mid_S S \otimes_R M$, then $M_1 \mid_R M$. To prove the claim, write $S \otimes_R M \cong (S \otimes_R M_1) \oplus U$. We assume, temporarily, that R is Artinian. By [W1, (1.2)] we know, at least, that there is some positive integer r such that $M_1 \mid_R M^{(r)}$ (a suitable direct sum of copies of M). Write $M_1 \cong \bigoplus_{i=1}^s V_i$, where each V_i is indecomposable. We proceed by induction on s. Since $V_1 \mid_R M^{(r)}$, the Krull–Remak–Schmidt theorem (for finite-length modules) implies that $V_1 \mid_R M$; say, $V_1 \oplus W \cong M$. This takes care of the base case $s = 1$. For the inductive step, assume $s > 1$ and set $W_1 = \bigoplus_{i=2}^s V_i$. We have $V_1 \oplus W \cong M$ and $M_1 \cong V_1 \oplus W_1$, so

$$\begin{aligned}(S \otimes_R V_1) \oplus (S \otimes_R W) &\cong S \otimes_R M \cong (S \otimes_R M_1) \oplus U \\ &\cong (S \otimes_R V_1) \oplus (S \otimes_R W_1) \oplus U.\end{aligned}$$

Direct-sum cancellation [Ev] implies $(S \otimes_R W) \cong (S \otimes_R W_1) \oplus U$. The inductive hypothesis, applied to the pair W_1, W, now implies that $W_1 \mid_R W$; therefore, $M_1 \mid_R M$. This completes the proof of the claim when R is Artinian.

In the general case, let t be an arbitrary positive integer and consider the flat local homomorphism $R/\mathfrak{m}^t \to S/\mathfrak{m}^t S$. By the Artinian case, $M_1/\mathfrak{m}^t M_1 \mid_{R/\mathfrak{m}^t} M/\mathfrak{m}^t M$. Now we apply [Gu, Cor. 2] to conclude that $M_1 \mid_R M$, as desired.

Having proved our claim, we now complete the proof of the proposition. If N_1 and N_2 are extended then clearly N is extended. Assuming N_1 and N are extended, we will prove that N_2 is extended. (The third possibility will then follow by symmetry.) Let $N_1 \cong S \otimes_R M_1$ and $N \cong S \otimes_R M$. Thus $S \otimes_R M_1 \mid_S S \otimes_R M$, and by the claim there is an R-module M_2 such that $M_1 \oplus M_2 \cong M$. Now $N_1 \oplus (S \otimes_R M_2) \cong S \otimes_R M \cong N_1 \oplus N_2$ and so, by direct-sum cancellation [Ev], we have $S \otimes_R M_2 \cong N_2$. $\square$

There is a "two out of three" principle for short exact sequences as well, though some restrictions apply. Variations on this theme have been used in the literature (e.g., in [CPST; LO; Wes]).

PROPOSITION 3.2. *Let $\varphi\colon R \to S$ be a flat local homomorphism satisfying* (†), *and consider an exact sequence of finitely generated S-modules* $0 \to N_1 \to N \to N_2 \to 0$.

(1) *Assume that N_1 and N_2 are extended. If $\operatorname{Ext}^1_S(N_2, N_1)$ is finitely generated as an R-module, then N is extended.*
(2) *Assume that N and N_2 are extended. If $\operatorname{Hom}_S(N, N_2)$ is finitely generated as an R-module, then N_1 is extended.*
(3) *Assume that N_1 and N are extended. If $\operatorname{Hom}_S(N_1, N)$ is finitely generated as an R-module, then N_2 is extended.*

Proof. For (1), let $N_i = S \otimes_R M_i$ where the M_i are finitely generated R-modules. We have natural homomorphisms $\operatorname{Ext}^1_R(M_2, M_1) \xrightarrow{\alpha} S \otimes_R \operatorname{Ext}^1_R(M_2, M_1) \xrightarrow{\beta} \operatorname{Ext}^1_S(N_2, N_1)$. The map β is an isomorphism because φ is flat, M_2 is finitely generated, and R is Noetherian. Therefore, $S \otimes_R \operatorname{Ext}^1_R(M_2, M_1)$ is finitely generated as an R-module, and now Theorem 1.8 ((4) $\Rightarrow$ (2)) states that α is an isomorphism. This means that the given exact sequence of S-modules is isomorphic to $S \otimes_R \mathbf{M}$ for some exact sequence of R-modules $\mathbf{M} = (0 \to M_1 \to M \to M_2 \to 0)$. Clearly, this implies $S \otimes_R M \cong N$.

To prove (2), let $N \cong S \otimes_R M$ and $N_2 \cong S \otimes_R M_2$, where M and M_2 are finitely generated R-modules. Essentially the same proof as in (1), but with Hom in place of Ext, shows that the given homomorphism $N \to N_2$ comes from a homomorphism $f\colon M \to M_2$. Then $M_1 \cong S \otimes_R \operatorname{Ker}(f)$.

For (3), let $N_1 \cong S \otimes_R M_1$ and $N \cong S \otimes_R M$; we deduce that the given homomorphism $N_1 \to N$ comes from a homomorphism $g\colon M_1 \to M$. Then $N_2 \cong S \otimes_R \operatorname{Coker}(g)$. □

Here is a simple application of Proposition 3.2(1) (see [LO; W2] for more general results).

PROPOSITION 3.3. *Let $(R, \mathfrak{m})$ be a one-dimensional local ring whose $\mathfrak{m}$-adic completion $S = \hat{R}$ is a domain. Then every finitely generated S-module is extended.*

Proof. Given a finitely generated S-module N, let $\{x_1, \dots, x_n\}$ be a maximal S-linearly independent subset of N. The submodule F generated by the x_i is free and hence extended; the quotient module N/F is torsion and hence of finite length. Therefore, N/F is extended. Since $\operatorname{Ext}^1_S(N/F, F)$ has finite length, it follows by Proposition 3.2 that the module N is extended. □

Notice that Proposition 3.2(1) applies also when N_2 is free on the punctured spectrum, for in this case $\operatorname{Ext}^1_S(N_2, N_1)$ has finite length over S and thus is finitely generated as an R-module. A more subtle condition that forces $\operatorname{Ext}^1_S(N_2, N_1)$ to have finite length is that there are only finitely many isomorphism classes of modules X fitting into a short exact sequence $0 \to N_1 \to X \to N_2 \to 0$ (cf. [CPST, (4.1)]).

Of course, not every module over the completion—or over the Henselization—is extended. Suppose, for example, that $R = \mathbb{C}[X, Y]_{(X,Y)}/(Y^2 - X^3 - X^2)$, the local ring of a node. Then R is a domain, but $\hat{R} \cong \mathbb{C}[[U, V]]/(UV))$, which has two minimal prime ideals $P = (U)$ and $Q = (V)$. Since R is a domain, any extended $\hat{R}$-module N must have the property that N_P and N_Q have the same vector-space dimension (over $\hat{R}_P$ and $\hat{R}_Q$, respectively). Hence the $\hat{R}$-module $\hat{R}/P$ is not extended. (This behavior was the basis for the first example of failure of the Krull–Remak–Schmidt theorem for finitely generated modules over local rings. See the example due to R. G. Swan in [Ev]; the idea is developed further in [W2].) The module $\hat{R}/P$ is free on the punctured spectrum and therefore, by Elkik's theorem [E], is extended from the Henselization R^{h}. With $\hat{R}/P \cong \hat{R} \otimes_{R^{\mathrm{h}}} V$, we see that the R^{h}-module V is not extended from R.

Next, we turn to the question of whether every finitely generated module over S is a direct summand of a finitely generated extended module. This weaker property is often useful in questions concerning ascent of finite representation type (cf. [W1, Lemma 2.1]). Although the next result is not explicitly stated in [W1], the main ideas of the proof occur there. Note that we do not require that $R/\mathfrak{m} = S/\mathfrak{n}$.

THEOREM 3.4. *Let $\varphi\colon R \to S$ be a flat local homomorphism, and assume that S is separable over R (i.e., the diagonal map $S \otimes_R S \to S$ splits as $S \otimes_R S$-modules). Then every finitely generated S-module is a direct summand of a finitely generated extended module.*

Proof. Given a finitely generated S-module N, we apply $- \otimes_S N$ to the diagonal map and obtain a split surjection of S-modules $\pi : S \otimes_R N \twoheadrightarrow N$, where the S-module structure on $S \otimes_R N$ comes from the S-action on S, not on N. Thus we have a split injection of S-modules $j\colon N \to S \otimes_R N$. Now write N as a direct union of finitely generated R-modules M_i. The flatness of φ implies that $S \otimes_R N$ is a direct union of the modules $S \otimes_R M_i$. The finitely generated S-module $j(N)$ must be contained in some $S \otimes_R M_i$. Since $j(N)$ is a direct summand of $S \otimes_R N$, it must be a direct summand of the smaller module $S \otimes_R M_i$. □

COROLLARY 3.5. *Let $R \to R^{\mathrm{h}}$ be the Henselization of the local ring R. Then every finitely generated R^{h}-module is a direct summand of a finitely generated extended module.*

Proof. Let N be a finitely generated R^{h}-module. Since $R \to R^{\mathrm{h}}$ is a direct limit of étale neighborhoods $R \to S_i$, it follows that N is extended from some S_i. Now apply Theorem 3.4 to the extension $R \to S_i$. □

The analogous result can fail for the completion; this is illustrated in the following example.

EXAMPLE 3.6. Let $(R, \mathfrak{m})$ be a countable local ring of dimension ≥ 2. Then R has only countably many isomorphism classes of finitely generated modules. Using the Krull–Remak–Schmidt theorem over the $\mathfrak{m}$-adic completion $\hat{R}$, we see that only countably many isomorphism classes of indecomposable $\hat{R}$-modules occur in direct-sum decompositions of finitely generated extended modules. We claim, on the other hand, that $\hat{R}$ has uncountably many isomorphism classes of finitely generated indecomposable modules. To see this, we recall that $\hat{R}$, being complete, has countable prime avoidance (see [ShV]). By Krull's principal ideal theorem, the maximal ideal of $\hat{R}$ is the union of the height-1 prime ideals. It follows that $\hat{R}$ must have uncountably many height-1 primes P and that the $\hat{R}$-modules $\hat{R}/P$ are pairwise nonisomorphic and indecomposable.

If $\varphi\colon (R,\mathfrak{m},k) \to (S,\mathfrak{n},l)$ is flat and satisfies (†), then we know that every finite-length S-module is extended. We close with an example showing that the condition $k = l$ cannot be deleted, even for a module-finite étale extension of Artinian local rings.

Example 3.7. Let $R = \mathbb{R}[X,Y]/(X,Y)^2$ and $S = \mathbb{C} \otimes_{\mathbb{R}} R = \mathbb{C}[X,Y]/(X,Y)^2$. We claim that, for $c \in \mathbb{C}$, the module $N := S/(X + cY)$ is extended (if and) only if $c \in \mathbb{R}$. The minimal presentation of N is $S \xrightarrow{X+cY} S \to N \to 0$. If N were extended, the 1×1 matrix $X + cY$ would be equivalent to a matrix over R. In other words, we would have $X + cY = u(r + sX + tY)$ for some unit u of S and suitable elements $r,s,t \in \mathbb{R}$. Writing $u = a + bX + dY$, with $a,b,d \in \mathbb{C}$ and $a \neq 0$, we see, by comparing coefficients of 1, X, and Y, that $c = t/s \in \mathbb{R}$.

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