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Adjoints of Ideals

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Adjoint ideals and multiplier ideals have recently emerged as a fundamental tool in commutative algebra and algebraic geometry. In characteristic 0 they may be defined using resolution of singularities. In positive prime characteristic p, Hara and Yoshida [4] introduced the analogue of multiplier ideals as generalized test ideals for a tight closure theory. For all characteristics, even mixed, Lipman gave the following definition.

Definition 0.1. Let R be a regular domain and I an ideal in R. Then the *adjoint* $\operatorname{adj} I$ of I is defined as

$$\operatorname{adj} I = \bigcap_{v} \{r \in R \mid v(r) \geq v(I) - v(J_{R_v/R})\},$$

where the intersection varies over all valuations v on the field of fractions K of R that are nonnegative on R and for which the corresponding valuation ring R_v is a localization of a finitely generated R-algebra. Here $J_{R_v/R}$ denotes the Jacobian ideal of R_v over R.

By our assumption on v, each valuation in the definition of $\operatorname{adj} I$ is Noetherian.

Many valuations v have the same valuation ring R_v; any two such valuations are positive real multiples of each other and are called *equivalent*. In Definition 0.1 we need only use one v from each equivalence class. In the sequel, we will always choose *normalized* valuations—that is, the integer-valued valuation v such that, for all $r \in R$, $v(r)$ equals the nonnegative integer n satisfying that rR_v equals the nth power of the maximal ideal of R_v.

Lipman proved that, for any ideal I in R and any $x \in R$, $\operatorname{adj}(xI) = x \operatorname{adj}(I)$. In particular, $\operatorname{adj}(xR) = (x)$.

A crucial and powerful property is the subadditivity of adjoints: $\operatorname{adj}(IJ) \subseteq \operatorname{adj}(I)\operatorname{adj}(J)$. This was proved in characteristic 0 by Demailly, Ein, and Lazarsfeld [3] and for generalized test ideals in characteristic p by Hara and Yoshida [4, Thm. 6.10]. A simpler proof in characteristic p can be found in [1, Lemma 2.10]. A version of subadditivity formula on singular varieties was proved by Takagi in [21]. But subadditivity of adjoints is unknown in general. We prove it for generalized monomial ideals in Section 4 and for ideals in two-dimensional regular

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domains in Section 5. The case of subadditivity of adjoints for ordinary monomial ideals can be deduced from Howald's work [5] using toric resolutions, and the two-dimensional case has been proved by Takagi and Watanabe [22] using multiplier ideals. The case for generalized monomial ideals proved here is new.

One aspect of proving subadditivity and computability of adjoints is whether there exist only finitely many valuations $v_1, \dots, v_m$ such that, for all n,

$$\operatorname{adj}(I^n) = \bigcap_{i=1}^{m} \{r \in R \mid v_i(r) \geq v_i(I^n) - v_i(J_{R_{v_i}/R})\}.$$

We prove in Section 4 that Rees valuations suffice for the generalized monomial ideals. We also give an example (the first example in Section 5) showing that Rees valuations do not suffice in general. In Section 5 we give a general criterion for when the adjoint of an ideal is determined by its Rees valuations. A corollary is that Rees valuations suffice for ideals in two-dimensional regular domains. The first three sections develop the background on generalized monomial ideals.

We refer the reader to the article by Smith and Thompson [19] and to Järvilehto's thesis [8] for results on what divisors (i.e., valuations) are needed to compute the multiplier ideals with rational coefficients. In general, Rees valuations do not suffice there, even in dimension 2.

1. Generalized Regular System of Parameters

DEFINITION 1.1. Let R be a regular domain. Elements $x_1, \dots, x_d$ in R are called a *generalized regular system of parameters* if $x_1, \dots, x_d$ is a permutable regular sequence in R such that, for every $i_1, \dots, i_s \in \{1, \dots, d\}$, $R/(x_{i_1}, \dots, x_{i_s})$ is a regular domain.

REMARK. Any part of a generalized regular system of parameters is again a generalized regular system of parameters.

For example, when R is regular local, an arbitrary regular system of parameters (or a part thereof) is a generalized regular system of parameters. And if R is a polynomial ring over a field, then the variables are a generalized regular system of parameters.

Let $\mathfrak{p}$ be any prime ideal containing the generalized regular system of parameters $x_1, \dots, x_d$. Because $R/(x_1, \dots, x_d)$ is regular, so is $R_{\mathfrak{p}}/(x_1, \dots, x_d)_{\mathfrak{p}}$, whence $x_1, \dots, x_d$ is part of a (usual) regular system of parameters in $R_{\mathfrak{p}}$.

LEMMA 1.2. *Let R be a regular domain and let $x_1, \dots, x_d$ be a generalized regular system of parameters. Then, for any normalized valuation v as in Definition 0.1, $v(J_{R_v/R}) \geq v(x_1 \cdots x_d) - 1$.*

Proof. By possibly taking a subset of the x_i, we can state without loss of generality that all $v(x_i)$ are positive. Let $\mathfrak{p}$ be the contraction of the maximal ideal of R_v to R. If we localize $\mathfrak{p}$, then $x_1, x_2, \dots, x_d$ are a part of a regular system of parameters (see comment preceding the lemma). We may possibly extend the x_i to a full regular system of parameters, so we may assume that $\mathfrak{p} = (x_1, \ldots, x_d)$ is the unique maximal ideal in R. We may also assume that $v(x_1) \geq v(x_2) \geq \cdots \geq v(x_d) \geq 1$.

If $d = 0$ then the lemma holds trivially. If $d = 1$ then v is the $\mathfrak{p}$-adic valuation, in which case $R_v = R = J_{R_v/R}$. Since v is normalized, it follows that $v(x_1) = 1$ and the lemma holds again. Now let $d > 1$ and let $S = R\big[\frac{x_1}{x_d}, \ldots, \frac{x_{d-1}}{x_d}\big]$. Then S is a regular ring contained in R_v and $\frac{x_1}{x_d}, \ldots, \frac{x_{d-1}}{x_d}, x_d$ constitute a generalized regular system of parameters in S. Clearly $J_{S/R}$ equals $x_d^{d-1}S$. By induction on $\sum_i v(x_i)$ we conclude that $v(J_{R_v/S}) \geq v\big(\frac{x_1}{x_d} \cdots \frac{x_{d-1}}{x_d} x_d\big) - 1$, and so, by Lipman and Sathaye [14, p. 201], $v(J_{R_v/R}) = v(J_{S/R}) + v(J_{R_v/S}) \geq v(x_d^{d-1}) + v\big(\frac{x_1}{x_d} \cdots \frac{x_{d-1}}{x_d} x_d\big) - 1 = v(x_1 \cdots x_d) - 1$. □

Though in general not as nicely behaved as variables in a polynomial or power series ring, generalized regular systems of parameters come close to them in many aspects. One property of interest is the following.

Proposition 1.3. *Let R be a regular domain and let $x_1, \ldots, x_d$ be a generalized regular system of parameters of R. Furthermore, let $s \leq d$, let $\mathfrak{p} = (x_1, \ldots, x_s)$, and let f be a nonzero element of R. Then there exist monomials $m_1, \ldots, m_t$ in $x_1, \ldots, x_s$ and elements $h, g_1, \ldots, g_t \in R \setminus \mathfrak{p}$ such that*

$$h \cdot f = \sum_{i=1}^{t} g_i \cdot m_i.$$

Proof. Clearly we may assume that R is local with maximal ideal $\mathfrak{p}$, and we then prove the proposition with $h = 1$. First we reduce to the case of complete local rings. Let $\hat{R}$ be the completion of R, and observe that $x_1, \ldots, x_s$ is a regular system of parameters of $\hat{R}$. Suppose we know the result for $\hat{R}$ and that $x_1, \ldots, x_s \in \hat{R}$. Write $f = \sum_{i=1}^{t} h_i m_i$ for some $h_i \in \hat{R}$ with $h_i \notin \mathfrak{p}$. Clearly we may assume that none of the monomials is a multiple of another one. Let $I = (m_1, \ldots, m_t) \subseteq R$. Since $f \in I\hat{R} \cap R = I$ (by faithful flatness), we may write $f = \sum_{i=1}^{t} g_i m_i$ with $g_i \in R$; in $\hat{R}$ we obtain $\sum_{i=1}^{t} (g_i - h_i) m_i = 0$ and hence conclude from [9, Sec. 5] that $g_i - h_i \in \mathfrak{p}\hat{R}$, implying that $g_i \notin \mathfrak{p}$. Thus it suffices to prove the proposition for the case of R complete local with maximal ideal $\mathfrak{p} = (x_1, \ldots, x_s)$.

Assume now that R is complete and let $f \in R$. Assume $f \in \mathfrak{p}^{n_1} \setminus \mathfrak{p}^{n_1+1}$. Then

$$f = \sum_{i=1}^{t_1} a_{1i} m_{1i} + f_2,$$

with some (unique) monomials m_{1i} of degree n_1 in $x_1, \ldots, x_s$, some $a_{1i} \notin \mathfrak{p}$ (unique modulo $\mathfrak{p}$), and some $f_2 \in \mathfrak{p}^{n_1+1}$. Let $M_1 = (m_{11}, \ldots, m_{1t_1})$. If $f_2 = 0$ then we are done. Otherwise, we write

$$f_2 = \sum_{i=1}^{t_2} a_{2i} m_{2i} + f_3,$$

with some $a_{2i} \notin \mathfrak{p}$, some monomials m_{2i} of degree n_2 in $x_1, \ldots, x_s$, and some $f_3 \in \mathfrak{p}^{n_2+1}$. Set $M_2 = M_1 + (m_{21}, \ldots, m_{2t_2})$ and continue. In this way we derive an ascending chain $M_1 \subseteq M_2 \subseteq \cdots$ of monomial ideals that must stabilize eventually, $M_\rho = M_{\rho+1} = \cdots =: M_\infty$. Let

$$M_\infty = (m_1, \ldots, m_t),$$

with each m_i a monomial of degree d_i in $x_1, \ldots, x_s$. We may assume that none of these monomials divides any of the other ones and that all m_i appear in a presentation of some f_j as before. Then in each step we may write

$$f_l = \sum_{i=1}^{t} n_{li} m_i + f_{l+1},$$

where $n_{li} \in \mathfrak{p}^{n_l - d_i}$ and where, if l is the smallest integer such that m_i appears with a nontrivial coefficient in the expansion of f_l, then $n_{li} \notin \mathfrak{p}$. Hence

$$f = \sum c_{li} m_i + f_{l+1},$$

with some $c_{li} \notin \mathfrak{p}$ (or $c_{li} = 0$) and with $c_{l+1,i} - c_{li} \in \mathfrak{p}^{n_l - d_i}$ (and $f_{l+1} \in \mathfrak{p}^{n_{l+1}+1}$). Because R is complete, this converges and we obtain

$$f = \sum c_i m_i$$

with some $c_i \notin \mathfrak{p}$. □

2. Integral Closures of (General) Monomial Ideals

Monomial ideals typically denote ideals in a polynomial ring or in a power series ring over a field that are generated by monomials in the variables. Such ideals have many good properties; in particular, their integral closures and multiplier ideals are known to be monomial as well. The result just stated on multiplier ideals for the standard monomial ideals is due to Howald [5]. In this section we consider generalized monomial ideals and present their integral closures. For alternate proofs on the integral closure of generalized monomial ideals see Kiyek and Stückrad [10].

We define monomial ideals more generally as follows.

DEFINITION 2.1. Let R be a regular domain, and let $x_1, \ldots, x_d$ in R be a generalized regular system of parameters. By a *monomial ideal* (*in* $x_1, \ldots, x_d$) we mean an ideal in R generated by monomials in $x_1, \ldots, x_d$.

As in the usual monomial ideal case, we can also define the Newton polyhedron.

DEFINITION 2.2. Let R and $x_1, \ldots, x_d$ be as before, and let I be an ideal generated by monomials $\underline{x}^{\underline{a}_1}, \ldots, \underline{x}^{\underline{a}_s}$. Then the *Newton polyhedron of* I (relative to $x_1, \ldots, x_d$) is the set

$$\mathrm{NP}(I) = \left\{ \underline{e} \in \mathbb{Q}^d_{\geq 0} \mid \underline{e} \geq \textstyle\sum_i c_i \underline{a}_i \text{ for some } c_i \in \mathbb{Q}_{\geq 0},\ \sum_i c_i = 1 \right\}.$$

Note that $\mathrm{NP}(I)$ is an unbounded closed convex set in $\mathbb{Q}^d_{\geq 0}$. We use $\mathrm{NP}^\circ(I)$ to denote the interior of $\mathrm{NP}(I)$.

THEOREM 2.3. *Let R be a regular domain and let $x_1, \dots, x_d$ be a generalized regular system of parameters. Let I be an ideal generated by monomials in $x_1, \dots, x_d$. Then the integral closure $\overline{I^n}$ of I^n equals*

$$\overline{I^n} = (\{\underline{x}^{\underline{e}} \mid \underline{e} \in n \cdot \mathrm{NP}(I) \cap \mathbb{N}^d\}),$$

so it is generated by monomials.

Proof. Since $\mathrm{NP}(I^n) = n \cdot \mathrm{NP}(I)$, we may assume that $n = 1$. Write $I = (\underline{x}^{\underline{a}_1}, \dots, \underline{x}^{\underline{a}_s})$, and let $\alpha = \underline{x}^{\underline{e}}$ be such that $\underline{e} \in \mathrm{NP}(I) \cap \mathbb{N}^d$. Then there exist $c_1, \dots, c_s \in \mathbb{Q}_{\geq 0}$ such that $\sum c_i = 1$ and $\underline{e} \geq \sum c_i \underline{a}_i$ (componentwise). Write $c_i = m_i/n$ for some $m_i \in \mathbb{N}$ and $n \in \mathbb{N}_{>0}$. Then

$$\alpha^n = x_1^{ne_1 - \sum m_i a_{i1}} \cdots x_d^{ne_d - \sum m_i a_{id}} (\underline{x}^{\underline{a}_1})^{m_1} \cdots (\underline{x}^{\underline{a}_s})^{m_s} \in I^{m_1 + \cdots + m_s} = I^n,$$

so that $\alpha \in \bar{I}$. It remains to prove the other inclusion.

Let S be the set of hyperplanes that bound $\mathrm{NP}(I)$ and are not coordinate hyperplanes. For each $H \in S$, if an equation for H is $h_1 X_1 + \cdots + h_d X_d = h$ with $h_i \in \mathbb{N}$ and $h \in \mathbb{N}_{>0}$, then define $I_H = \left(\underline{x}^{\underline{e}} \mid \underline{e} \in \mathbb{N}^d, \sum_i h_i e_i \geq h\right)$. Clearly $I \subseteq I_H$, $\mathrm{NP}(I_H) \subseteq \{e \in \mathbb{Q}^d_{\geq 0} \mid \sum h_i e_i \geq h\}$, and $\mathrm{NP}(I_H) \cap \mathbb{N}^d = \{e \in \mathbb{Q}^d_{\geq 0} \mid \sum h_i e_i \geq h\} \cap \mathbb{N}^d$. Suppose that the theorem is known for the (generalized) monomial ideals I_H. Then

$$\begin{aligned}
\bar{I} &\subseteq \bigcap_{H \in S} \overline{I_H} \\
&\subseteq \bigcap_{H \in S} \left(\left\{\underline{x}^{\underline{e}} \mid \underline{e} \in \mathbb{N},\ \textstyle\sum_i h_i e_i \geq h \text{ if } H =: \sum_i h_i X_i = h\right\}\right) \\
&= (\{\underline{x}^{\underline{e}} \mid \underline{e} \in \mathrm{NP}(I) \cap \mathbb{N}^d\}).
\end{aligned}$$

Hence it suffices to prove the theorem for I_H. As before, let $\sum_i h_i X_i = h$ define H. By possibly reindexing, we may assume that $h, h_1, \dots, h_t$ are positive integers and that $h_{t+1} = \cdots = h_d = 0$. As noted previously, it suffices to show that

$$\overline{I_H} = \left(\left\{\underline{x}^{\underline{e}} \mid e \in \mathbb{N}^d \text{ and } \textstyle\sum h_i e_i \geq h\right\}\right).$$

Let $Y_1, \dots, Y_t$ be variables over R and $R' = R[Y_1, \dots, Y_t]/(Y_1^{h_1} - x_1, \dots, Y_t^{h_t} - x_t)$. This is a free finitely generated R-module and $Y_1, \dots, Y_t$ is a regular sequence in R'. Set $\mathfrak{p} = (Y_1, \dots, Y_t)R'$. Then $R'/\mathfrak{p} = R/(x_1, \dots, x_t)$ is a regular domain and so $\mathfrak{p}$ is a prime ideal; thus, for any prime ideal $\mathfrak{q}$ in R' containing $\mathfrak{p}$, $R'_{\mathfrak{q}}$ is a regular local ring. By construction, $I_H R'$ is contained in $(Y_1, \dots, Y_t)^h = \mathfrak{p}^h$. Since $R'_{\mathfrak{p}}$ is a regular local ring, $\mathfrak{p}^h R'_{\mathfrak{p}}$ is integrally closed, and since R' is finitely generated over a locally formally equidimensional (regular) ring, $R'_{\mathfrak{q}}$ is locally formally equidimensional for every prime ideal $\mathfrak{q}$ containing $\mathfrak{p}$. By a theorem of Ratliff [17], because $\mathfrak{p}$ is generated by a regular sequence, the integral closure of $\mathfrak{p}^h R'_{\mathfrak{q}}$ has no embedded prime ideals. It follows that the integral closure of $\mathfrak{p}^h R'_{\mathfrak{q}}$ is $\mathfrak{p}^h R'_{\mathfrak{p}} \cap R'_{\mathfrak{q}}$. Because $R'_{\mathfrak{q}}$ is a regular domain and $\mathfrak{p}$ is generated by a regular sequence, we have $\mathfrak{p}^h R'_{\mathfrak{p}} \cap R'_{\mathfrak{q}} = \mathfrak{p}^h R'_{\mathfrak{q}}$. It follows that $\mathfrak{p}^h R'_{\mathfrak{p}} \cap R' = \mathfrak{p}^h$ is the integral closure of $\mathfrak{p}^h$. Hence $\overline{I_H} \subseteq \overline{\mathfrak{p}^h} \cap R = \mathfrak{p}^h \cap R$ and, by freeness of R' over R, the last ideal is exactly $(\underline{x}^{\underline{e}} \mid \underline{e} \in \mathbb{N}, \sum_i h_i e_i \geq h)$, which finishes the proof. $\square$

3. Rees Valuations of (General) Monomial Ideals

Recall that the Rees valuations of a nonzero ideal in a Noetherian domain form a unique minimal set $\mathcal{RV}(I)$ of finitely many normalized valuations such that, for all positive integers n, $\overline{I^n} = \{r \in R \mid v(r) \geq nv(I) \text{ for all } v \in \mathcal{RV}(I)\}$.

In an arbitrary Noetherian domain, for arbitrary ideals I and J we have that $\mathcal{RV}(I) \cup \mathcal{RV}(J) \subseteq \mathcal{RV}(IJ)$ and that equality holds in two-dimensional regular domains. (This has appeared in the literature in several places; see e.g. Muhly and Sakuma [16] or the Rees valuations chapter in the book by Swanson and Huneke [20].)

We will prove that the Rees valuations of an ideal generated by monomials in a regular system of parameters are especially nice.

DEFINITION 3.1. Let R be a regular domain, and let $x_1, \dots, x_d$ be a generalized regular system of parameters. A valuation v on the field of fractions of R is said to be *monomial* on $x_1, \dots, x_d$ if, for some $i_1, \dots, i_s \in \{1, \dots, d\}$ and for any polynomial $f = \sum c_\nu x_{i_1}^{\nu_{i_1}} \cdots x_{i_s}^{\nu_{i_s}} \in R$ with all c_ν either 0 or not in $(x_{i_1}, \dots, x_{i_s})$,

$$v(f) = \min\{v(\underline{x}^{\underline{\nu}}) \mid c_\nu \neq 0\}.$$

When the x_i are understood from the context, we say that v is *monomial*.

Observe that $v(f) = 0$ for any $f \notin (x_{i_1}, \dots, x_{i_s})$. In particular $v(x_j) = v(1) = 0$ if $j \notin \{i_1, \dots, i_s\}$.

PROPOSITION 3.2. *Let R be a regular domain, let $x_1, \dots, x_d$ be a generalized regular system of parameters, and let $a_1, \dots, a_d$ be nonnegative rational numbers not all of which are zero. Then there exists a unique valuation v on the field of fractions $K = Q(R)$ of R that is monomial on $x_1, \dots, x_d$ with $v(x_i) = a_i$ for all i.*

Proof. By reindexing we may assume that $a_1 > 0, \dots, a_s > 0$ and that $a_{s+1} = \cdots = a_d = 0$ for some $s > 0$; we may also assume that all a_i are integers.

The uniqueness of v is immediate from Proposition 1.3. To prove the existence we may replace R by $R_{\mathfrak{p}}$ (with $\mathfrak{p} = (x_1, \dots, x_s)$) and assume that R is local. Let R' be the regular local ring obtained by adjoining an (a_i)th root y_i of x_i to R ($i = 1, \dots, s$), and let $\mathfrak{n}$ be the maximal ideal of R'. Then the $\mathfrak{n}$-adic valuation w on $L = Q(R')$ is monomial in $y_1, \dots, y_s$ with $w(y_i) = 1$ for all i. The restriction $v := w|_K$ is a monomial valuation as desired. $\square$

COROLLARY 3.3. *Let R be a regular domain, and let $x_1, \dots, x_d$ be a generalized regular system of parameters. Let I be an ideal generated by monomials in $x_1, \dots, x_d$. Then all the Rees valuations of I are monomial in $x_1, \dots, x_d$. Furthermore, if $H_1, \dots, H_\rho$ are the noncoordinate hyperplanes bounding* $\mathrm{NP}(I)$, *then the H_j are in one-to-one correspondence with the Rees valuations v_j of I.*

Proof. The Newton polyhedron $\mathrm{NP}(I)$ of I is the intersection of finitely many half-spaces in $\mathbb{Q}^d$. Some of them are coordinate half-spaces $\{x_i \geq 0\}$, and each of the others is determined by a hyperplane H of the form $h_1 x_1 + \cdots + h_d x_d = h$, where $h_1, \dots, h_d, h$ are nonnegative integers, $h > 0$, and $\gcd(h_1, \dots, h_d, h) = 1$. This hyperplane corresponds to a valuation v_H that is monomial on $x_1, \dots, x_d$ and such that $v_H(x_i) = h_i$. By Theorem 2.3, the integral closure of I is determined by these v_H. Using $\mathrm{NP}(I^n) = n \cdot \mathrm{NP}(I)$, we see that the integral closure of I^n is also determined by these v_H. So each Rees valuation is one such v_H. Suppose that the set of Rees valuations is a proper subset of the set of all the v_H, and suppose that one such v_H is not needed in the computation of the integral closures of powers of I. Since the hyperplanes H were chosen to be nonredundant, omitting any one of them yields a point $(e_1, \dots, e_d) \in \mathbb{Q}^d_{\geq 0}$ that is on the unbounded side of all the hyperplanes bounding $\mathrm{NP}(I)$ other than H but is not on the unbounded side of H. Then there exist $m_1, \dots, m_d \in \mathbb{N}$ and $n > 0$ such that, for each i, $e_i = m_i/n$. By assumption we have $x_1^{m_1} \cdots x_d^{m_d} \in \overline{I^n}$, but $(m_1, \dots, m_n) \notin n \cdot \mathrm{NP}(I)$—a contradiction. $\square$

The following is a local version of [5, Lemma 1]. Howald's proof relies on the existence of a log resolution.

LEMMA 3.4. *Let R be a regular domain, and let $x_1, \dots, x_d$ be a generalized regular system of parameters. Let v be a discrete valuation that is monomial on $x_1, \dots, x_d$, is nonnegative on R, and has value group contained in $\mathbb{Z}$. Then*

$$v(J_{R_v}/R) = v(x_1 \cdots x_d) - \gcd(v(x_i) \mid i).$$

Proof. Since v is monomial in the x_i, the center of v on R is contained in $\mathfrak{m} = (x_1, \dots, x_d)$. By localizing, we may assume that $\mathfrak{m}$ is the only maximal ideal in R. Let $a_i = v(x_i)$. Without loss of generality, $a_1 \geq a_2 \geq \cdots \geq a_d$; let s be the largest integer such that $a_s > 0$. Because v is monomial, if $s = 0$ then $v = 0$ and the lemma holds trivially. Hence we assume that $s > 0$. If $s = 1$, then necessarily $a_1 = \gcd(v(x_i) \mid i)$, and v is a_1 times the (x_1)-grading. Then $R_v = R_{(x_1)}$, $J_{R_v/R} = R_v$, and $v(J_{R_v/R}) = 0 = v(x_1 \cdots x_d) - \gcd(v(x_i) \mid i)$. So the lemma holds in the case $s = 1$. We proceed by induction on $\sum_i a_i$. We may assume that $s > 1$. Let $S = R\left[\frac{x_1}{x_s}, \dots, \frac{x_{s-1}}{x_s}\right]$. Then S is a regular ring contained in R_v, and $\frac{x_1}{x_s}, \dots, \frac{x_{s-1}}{x_s}, x_s, \dots, x_d$ is a generalized regular system of parameters. For these elements in S, v is still a monomial valuation, their v-values are nonnegative integers, and the total sum of their v-values is strictly smaller than $\sum_i a_i$. Thus, by induction,

$$\begin{aligned} v(J_{R_v/S}) &= v\left(\frac{x_1}{x_s}\cdots\frac{x_{s-1}}{x_s}x_s\cdots x_d\right) \\ &\quad - \gcd\left(v\left(\frac{x_1}{x_s}\right),\dots,v\left(\frac{x_{s-1}}{x_s}\right),v(x_s),\dots,v(x_d)\right) \\ &= v(x_1\cdots x_d) - (s-1)v(x_s) - \gcd(v(x_1),\dots,v(x_d)). \end{aligned}$$

Since $R \subseteq S \subseteq R_v$ are all finitely generated algebras over R that are regular rings and have the same field of fractions, it follows by Lipman and Sathaye [14, p. 201] that $J_{R_v/R} = J_{S/R}J_{R_v/S}$. Clearly $J_{S/R}$ equals x_s^{s-1}, whence

$$\begin{aligned} v(J_{R_v/R}) &= v(x_s^{s-1}) + v(x_1\cdots x_d) - (s-1)v(x_s) - \gcd(v(x_1),\dots,v(x_d)) \\ &= v(x_1\cdots x_m) - \gcd(v(x_1),\dots,v(x_d)). \end{aligned}$$ □

4. Adjoints of (General) Monomial Ideals

A proof similar to the proof of Theorem 2.3 shows that the adjoint of a (general) monomial ideal is monomial. This generalizes Howald's result [5].

THEOREM 4.1. *Let R be a regular domain, and let $x_1,\dots,x_d$ be a generalized regular system of parameters. Let I be an ideal generated by monomials in $x_1,\dots,x_d$. Then, for all $n \ge 1$,*

$$\begin{aligned} \operatorname{adj}(I^n) &= \bigcap_v (\{\underline{x}^{\underline{e}} \mid v(\underline{x}^{\underline{e}}) \ge v(I^n) - v(x_1\cdots x_d) + 1\}) \\ &= \bigcap_v (\{\underline{x}^{\underline{e}} \mid v(\underline{x}^{\underline{e}}) \ge v(I^n) - v(J_{R_v/R})\}) \\ &= (\{\underline{x}^{\underline{e}} \mid \underline{e} \in \mathbb{N}^d \text{ and } e + (1,\dots,1) \in \mathrm{NP}^\circ(I^n)\}) \end{aligned}$$

as v varies over the (normalized) Rees valuations of I. In particular, the adjoint is also generated by monomials.

Proof. Since I^n is monomial and since the Rees valuations of I^n are contained in the set of Rees valuations of I, it suffices to prove the theorem for $n = 1$. By Corollary 3.3 and Lemma 3.4, the second and third equalities hold. It therefore suffices to prove that adj I equals the other three expressions (when $n = 1$).

First we prove that $\underline{x}^{\underline{e}} \in \operatorname{adj}(I)$ whenever $\underline{e} \in \mathbb{N}^d$ with $\underline{e} + (1,\dots,1) \in \mathrm{NP}^\circ(I)$). Let v be a valuation as in the definition of $\operatorname{adj}(I)$. Since $(x_1\cdots x_d\underline{x}^{\underline{e}})^n \in \overline{I^{n+1}}$ for some positive integer n, it follows that $v(x_1\cdots x_d\underline{x}^{\underline{e}}) > v(I)$. Because v is normalized, $v(\underline{x}^{\underline{e}}) \ge v(I) - v(x_1\cdots x_d) + 1$. By Lemma 1.2, $v(J_{R_v/R}) \ge v(x_1\cdots x_d) - 1$, so that $v(\underline{x}^{\underline{e}}) \ge v(I) - v(J_{R_v/R})$. Because v was arbitrary, this proves that $(\underline{x}^{\underline{e}} \mid \underline{e} \in \mathbb{N}^d, \underline{e} + (1,\dots,1) \in \mathrm{NP}^\circ(I)) \subset \operatorname{adj} I$. It remains to prove the other inclusion.

Let S be the set of bounding hyperplanes of $\mathrm{NP}(I)$ that are not coordinate hyperplanes. For each $H \in S$, if an equation for H is $h_1X_1 + \cdots + h_dX_d = h$ with $h_i \in \mathbb{N}$ and $h \in \mathbb{N}_{>0}$, define $I_H = (\underline{x}^{\underline{e}} \mid \underline{e} \in \mathbb{N}^d, \sum_i h_ie_i \ge h)$. By the definition of Newton polyhedrons, $I \subseteq I_H$.

After possibly reindexing, we claim without loss of generality that $h_1, \dots, h_t > 0$ and $h_{t+1} = \dots = h_d = 0$. By Proposition 3.2 there exists a monomial valuation v_H on $Q(R)$ defined by $v_H(x_i) = h_i$. By construction, $v_H(I) \geq v_H(I_H) \geq h$ (even equalities hold), and $\operatorname{adj}(I_H) \subseteq \{r \in R \mid v_H(r) \geq v_H(I_H) - v_H(J_{R_{v_H}/R})\}$. By the properties of v_H, the last ideal is generated by monomials in the x_i. By Lemma 3.4, $v_H(J_{R_{v_H}/R}) = v_H(x_1 \cdots x_d) - 1$, so that

$$\begin{aligned}\operatorname{adj}(I_H) &\subseteq (\{\underline{x}^{\underline{e}} \mid \underline{e} \in \mathbb{N}^d,\ v_H(\underline{x}^{\underline{e}}) > v_H(I_H) - v_H(x_1 \cdots x_d)\}) \\ &\subseteq \left(\left\{\underline{x}^{\underline{e}} \mid \underline{e} \in \mathbb{N}^d,\ \textstyle\sum_i h_i(e_i+1) > v_H(I_H)\right\}\right) \\ &\subseteq \left(\left\{\underline{x}^{\underline{e}} \mid \underline{e} \in \mathbb{N}^d,\ \textstyle\sum_i h_i(e_i+1) > h\right\}\right),\end{aligned}$$

whence

$$\begin{aligned}\operatorname{adj} I &\subseteq \bigcap_{H \in S} \operatorname{adj}(I_H) \\ &\subseteq \bigcap_{H \in S} \left(\left\{\underline{x}^{\underline{e}} \mid \underline{e} \in \mathbb{N},\ \textstyle\sum_i h_i(e_i+1) > h \text{ if } H =: \sum_i h_i X_i = h\right\}\right) \\ &= (\{\underline{x}^{\underline{e}} \mid \underline{e} \in \mathbb{N}^d,\ \underline{e} + (1, \dots, 1) \in \mathrm{NP}^\circ(I)\}). \qquad \square\end{aligned}$$

Theorem 4.1 allows to address the subadditivity problem for monomial ideals as follows.

COROLLARY 4.2. *Let $I, J \subseteq R$ be ideals generated by monomials in the generalized regular system of parameters $x_1, \dots, x_d$. Then*

$$\operatorname{adj}(IJ) \subseteq \operatorname{adj}(I) \cdot \operatorname{adj}(J).$$

Proof. Let $\underline{x}^{\underline{a}} \in \operatorname{adj}(IJ)$ be a monomial. By Theorem 4.1, $\underline{a} + (1, \dots, 1) \in \mathrm{NP}^\circ(I \cdot J)$. Since $\mathrm{NP}^\circ(IJ) \subseteq \mathrm{NP}^\circ(I) + \mathrm{NP}^\circ(J)$, there exist $\underline{b} \in \mathrm{NP}^\circ(I)$ and $\underline{c} \in \mathrm{NP}^\circ(J)$ with $\underline{a} + (1, \dots, 1) = \underline{b} + \underline{c}$. Set $\underline{f} = (f_1, \dots, f_d)$ and $\underline{g} = (g_1, \dots, g_d)$ with $f_i = \lceil b_i \rceil - 1$ and $g_i = \lfloor c_i \rfloor$. Then $\underline{x}^{\underline{g}}$ and $\underline{x}^{\underline{f}}$ are monomials with $\underline{x}^{\underline{g}} \cdot \underline{x}^{\underline{f}} = \underline{x}^{\underline{a}}$; moreover,

$$\begin{aligned}\underline{f} + (1, \dots, 1) &\in \underline{b} + \mathbb{Q}^d_{\geq 0} \subseteq \mathrm{NP}^\circ(I), \\ \underline{g} + (1, \dots, 1) &\in \underline{c} + \mathbb{Q}^d_{\geq 0} \subseteq \mathrm{NP}^\circ(J),\end{aligned}$$

implying by Theorem 4.1 that $\underline{x}^{\underline{f}} \in \operatorname{adj}(I)$ and $\underline{x}^{\underline{g}} \in \operatorname{adj}(J)$. $\square$

From the proof of Theorem 4.1 it is clear that the Rees valuations of the adjoint depend on the Rees valuations of the original ideal. The number of Rees valuations of I need not be an upper bound on the number of Rees valuations of $\operatorname{adj}(I)$, and there is in general no overlap between the set of Rees valuations of I and the set of Rees valuations of $\operatorname{adj} I$.

EXAMPLE. Let R be a regular local ring with regular system of parameters x, y. Let I be the integral closure of (x^5, y^7). Then, by the structure theorem, I has only one Rees valuation and $I = (x^5, x^4y^2, x^3y^3, x^2y^5, xy^6, y^7)$. By Theorem 4.1 (or by [5] or [7]), $\operatorname{adj}(I) = (x^4, x^3y, x^2y^2, xy^4, y^5)$, which is not the integral closure of (x^4, y^5). Thus $\operatorname{adj}(I)$ has more than one Rees valuation. In fact, it has two Rees valuations, each of which is monomial and neither of which is equivalent to the Rees valuation of I:

$$v_1(x) = 1 = v_1(y), \quad v_1(\operatorname{adj}(I)) = 4;$$

$$v_2(x) = 3, \quad v_2(y) = 2, \quad v_2(\operatorname{adj}(I)) = 10.$$

Nevertheless, the one Rees valuation of I still determines the adjoints of all the powers of I.

5. Adjoints of Ideals and Rees Valuations

In this section we characterize those ideals I for which $\operatorname{adj}(I^n)$ is determined by the Rees valuations of I for all n. In the previous section we saw that this is true for monomial ideals. That the Rees valuations of an ideal I should, in general, play a crucial role in determining the adjoint of I is also implied by the following result.

PROPOSITION 5.1. *Let I be an ideal in a regular domain R, and let $\mathcal{V}$ be a finite set of valuations on the field of fractions of R such that, for all $n \in \mathbb{N}$,*

$$\operatorname{adj} I = \bigcap_{v \in \mathcal{V}} \{r \in R \mid v(r) \geq v(I) - v(J_{R_v/R})\}.$$

Then $\mathcal{V}$ contains the Rees valuations of I.

Proof. Assume that there exist some Rees valuations of I not contained in $\mathcal{V}$. By the defining property of Rees valuations, there exist a nonnegative integer n and an element $r \in R$ with

(1) $v(r) \geq n \cdot v(I)$ for all $v \in \mathcal{V}$ and
(2) $r \notin \overline{I^n}$.

Let w be a Rees valuation of I with $w(r) \leq n \cdot w(I) - 1$. Assume that I is l-generated and let $t \geq l \cdot w(I)$. Then

$$w(r^t) = t \cdot w(r) < (nt - l + 1)w(I)$$

and so

$$r^t \notin \overline{I^{nt-l+1}}.$$

On the other hand,

$$v(r^t) \geq nt \cdot v(I) \geq nt \cdot v(I) - v(J_{R_v/R}) \quad \text{for all } v \in \mathcal{V},$$

implying that

$$r^t \in \operatorname{adj}(I^{nt}) \subseteq \overline{I^{nt-l+1}}$$

by [12, (1.4.1)]—a contradiction. □

It is not true in general that the set of Rees valuations determines the adjoint of an arbitrary ideal, as the following example shows.

EXAMPLE. Let $(R, \mathfrak{m})$ be a d-dimensional regular local ring with $d > 2$, and let $\mathfrak{p}$ be a prime ideal in R of height $h \in \{2, \dots, d-1\}$ generated by a regular sequence. Then the $\mathfrak{p}$-adic valuation $v_\mathfrak{p}$ is the only Rees valuation of $\mathfrak{p}$. If $v_\mathfrak{p}$ defined $\operatorname{adj}(\mathfrak{p}^n)$ in the sense that

$$\operatorname{adj}(\mathfrak{p}^n) = \{r \in R \mid v_\mathfrak{p}(r) \geq n v_\mathfrak{p}(\mathfrak{p}) - v_\mathfrak{p}(J_{R_{v_\mathfrak{p}}/R})\} \quad \text{for all } n,$$

then, since $v_\mathfrak{p}(\mathfrak{p}) = 1$ and $J_{R_{v_\mathfrak{p}}/R} = \mathfrak{p}^{h-1} R_{v_\mathfrak{p}}$, it follows that

$$\operatorname{adj}(\mathfrak{p}^{h-1}) = \{r \in R \mid v_\mathfrak{p}(r) \geq 0\} = R.$$

However, if $\mathfrak{p}$ is generated by elements in $\mathfrak{m}^e$, where $e \geq d/(h-1)$, and if v denotes the $\mathfrak{m}$-adic valuation, then

$$\begin{aligned} \operatorname{adj}(\mathfrak{p}^{h-1}) &\subseteq \{r \in R \mid v(r) \geq v(\mathfrak{p}^{h-1}) - v(J_{R_v/R})\} \\ &\subseteq \{r \in R \mid v(r) \geq d - (d-1)\} \subseteq \mathfrak{m}, \end{aligned}$$

which is a contradiction. A concrete example of this is $R = k[[X, Y, Z]]$ with the prime ideal $\mathfrak{p} = (X^4 - Z^3, Y^3 - X^2 Z)$, which defines the monomial curve (t^9, t^{10}, t^{12}).

The following is a geometric reformulation of [18] (see also [2, 2.3] or [13, 1.4]).

REMARK. Let R be a regular domain, let $I \subseteq R$ be an ideal of R, and let $Y = \operatorname{Spec}(R)$. Let $P = R[IT]$, the Rees ring of I, and let $\bar{P}$ be its normalization and $\varphi \colon X = \operatorname{Proj}(\bar{P}) \to Y$ the induced scheme. Then X/Y is essentially of finite type by [14, p. 200] (see also [20, 9.2.3] for details). Thus φ is a projective birational morphism, X is a normal Noetherian scheme, and $I\mathcal{O}_X$ is an invertible ideal. Let $\mathfrak{P}_1, \dots, \mathfrak{P}_r$ be the irreducible components of the vanishing set $\mathfrak{V}(I\mathcal{O}_X)$ of $I\mathcal{O}_X$ (i.e., those points x of X of codimension 1 such that $I\mathcal{O}_{X,x}$ is a proper ideal of $\mathcal{O}_{X,x}$). Then $\mathcal{O}_{X,\mathfrak{P}_i}$ is a discrete valuation ring (with field of fractions $K = Q(R)$) and the corresponding valuations $v_1, \dots, v_r$ are exactly the Rees valuations of I.

If $(R, \mathfrak{m})$ is local and I is $\mathfrak{m}$-primary, the Rees valuations correspond to the irreducible components of the closed fibre $\varphi^{-1}(\mathfrak{m})$, which in this case is a $(\dim(R)-1)$-dimensional projective scheme (in general neither reduced nor irreducible).

Let $f \colon Z \to Y$ be birational and of finite type. Then the Jacobian ideal $\mathcal{J}_{Z/Y} \subseteq \mathcal{O}_Z$ is well-defined (being locally the zeroth Fitting ideal of the relative Kähler differentials). If, in addition, Z is normal, then

$$\omega_{Z/Y} := \mathcal{O}_Z : \mathcal{J}_{Z/Y} = \mathcal{H}om_Z(\mathcal{J}_{Z/Y}, \mathcal{O}_Z)$$

is a canonical dualizing sheaf for f with

$$\mathcal{O}_Z \subseteq \omega_{Z/Y} \subseteq \mathcal{M}_Z,$$

where $\mathcal{M}_Z$ denotes the constant sheaf of meromorphic functions on Z. If

$$g \colon Z' \to Z$$

is another birational morphism and if g is proper and Z' is normal as well, then

$$g_*\omega_{Z'/Y} \subseteq \omega_{Z/Y}$$

(cf. [14, 2.3] and [15, Sec. 4]).

THEOREM 5.2. *Let R be a regular domain, and let $I \subseteq R$ be a nontrivial ideal. Furthermore, let $Y = \operatorname{Spec}(R)$ and let $\varphi\colon X \to Y$ be the normalized blow-up of I. Then the following are equivalent.*

(1) $\operatorname{adj}(I^n) = \bigcap_{v \in \mathcal{RV}(I)} \{r \in R \mid v(r) \geq n \cdot v(I) - v(J_{R_v/R})\}$ *for all positive integers n.*

(2) *If Z is a normal scheme and $\pi : Z \to X$ is proper and birational, then*

$$\pi_*\omega_{Z/Y} = \omega_{X/Y}.$$

REMARK. If for Theorem 5.2(2) the scheme X is Cohen–Macaulay as well, then X has pseudo-rational singularities only [15, Sec. 4].

REMARK. For Theorem 5.2(1), the set $\mathcal{RV}(I)$ is the unique smallest set of valuations defining $\operatorname{adj}(I^n)$ in view of Proposition 5.1.

Proof of Theorem 5.2. If $f\colon Z \to Y$ is proper and birational with Z normal and $I\mathcal{O}_Z$ invertible, we set

$$\operatorname{adj}_Z(I^n) = H^0(Z, I^n\omega_{Z/Y}) \quad (\subseteq R).$$

Then $\operatorname{adj}(I^n) = \bigcap \operatorname{adj}_Z(I^n)$ by [12], where $f\colon Z \to Y$ varies over all such morphisms. By the universal properties of blow-up and normalization, any such f factors as

$$Z \xrightarrow{\pi} X \xrightarrow{\varphi} Y.$$

Since $\pi_*\omega_{Z/Y} \subseteq \omega_{X/Y}$ and since $I\mathcal{O}_Z$ is invertible, it follows from the projection formula that

$$\begin{aligned} H^0(Z, I^n\omega_{Z/Y}) &= H^0(X, \pi_* I^n\omega_{Z/Y}) \\ &= H^0(X, I^n\pi_*\omega_{Z/Y}) \\ &\subseteq H^0(X, I^n\omega_{X/Y}). \end{aligned}$$

Therefore,

$$\operatorname{adj}_Z(I^n) \subseteq \operatorname{adj}_X(I^n) \text{ for all positive integers } n \tag{$*$}$$

for any such $f\colon Z \to Y$.

Because $\omega_{X/Y}$ is reflexive (by [14, p. 203]) and $I\mathcal{O}_X$ is invertible, $I^n\omega_{X/Y}$ must be a reflexive coherent subsheaf of the sheaf of meromorphic functions of X. We therefore have

$$H^0(X, I^n\omega_{X/Y}) = \bigcap_{x \in X : \operatorname{ht}(x)=1} (I^n\omega_{X/Y})_x.$$

For $x \in X$ with $\varphi(x) \notin \mathfrak{V}(I)$, the set of primes containing I, we have

$$I\mathcal{O}_{X,x} = \mathcal{O}_{X,x},$$

$$\omega_{X/Y,x} = \mathcal{O}_{X,x},$$

since φ is an isomorphism away from $\mathfrak{V}(I)$. Those $x \in X$ with $\operatorname{ht}(x) = 1$ and $\varphi(x) \in \mathfrak{V}(I)$ correspond to the Rees valuations of I, and thus

$$\begin{aligned}
\operatorname{adj}_X(I^n) &= H^0(X, I^n\omega_{X/Y}) \\
&= \bigcap_{x \in X : \operatorname{ht}(x)=1} (I^n\omega_{X/Y})_x \\
&= \bigcap_{x \in X : \operatorname{ht}(x)=1,\, \varphi(x) \in \mathfrak{V}(I)} (I^n\omega_{X/Y})_x \cap \bigcap_{x \in X : \operatorname{ht}(x)=1,\, \varphi(x) \notin \mathfrak{V}(I)} \mathcal{O}_{X,x} \\
&\supseteq \bigcap_{v \in \mathcal{RV}(I)} \{r \in R \mid v(r) \geq n \cdot v(I) - v(J_{R_v/R})\},
\end{aligned}$$

where we have used that $\omega_{R_v/R}$ is an invertible fractional ideal with inverse $J_{R_v/R}$. Since $\pi_*\omega_{X/Y} = \mathcal{O}_X$ (by [15, Sec. 4]) and so $H^0(X, \omega_{X/Y}) = R$, the converse inclusion is obvious:

$$\begin{aligned}
\operatorname{adj}_X(I^n) &= \bigcap_{v} \{r \in K \mid v(r) \geq n \cdot v(I) - v(J_{R_v/R})\} \cap R \\
&\subseteq \bigcap_{v \in \mathcal{RV}(I)} \{r \in R \mid v(r) \geq n \cdot v(I) - v(J_{R_v/R})\},
\end{aligned}$$

and we conclude that

$$\operatorname{adj}_X(I^n) = \bigcap_{v \in \mathcal{RV}(I)} \{r \in R \mid v(r) \geq n \cdot v(I) - v(J_{R_v/R})\}.$$

Thus Theorem 5.2(1) is equivalent to

$$\operatorname{adj}_X(I^n) = \operatorname{adj}_Z(I^n) \quad \text{for all } \ n \in \mathbb{N}$$

for all $f\colon Z \to Y$ as before.

First assume part (2); this direction is implicit in [12] (cf. [12, 1.3.2(b)]). Let $f\colon Z \to Y$ be as before. By assumption, we have trivially

$$H^0(X, I^n\pi_*\omega_{Z/Y}) = H^0(X, I^n\omega_{X/Y}),$$

which implies by the calculations preceeding $(*)$ that $\operatorname{adj}_X(I^n) = \operatorname{adj}_Z(I^n)$ for all positive integers n. Thus part (1) follows.

Conversely, suppose that part (1) holds (i.e., that $\operatorname{adj}(I^n) = \operatorname{adj}_X(I^n)$ for all positive integers n). Then by $(*)$ it must follow that the canonical inclusions

$$H^0(X, I^n\pi_*\omega_{Z/Y}) = \operatorname{adj}_Z(I^n) \hookrightarrow \operatorname{adj}_X(I^n) = H^0(X, I^n\omega_{X/Y})$$

are isomorphisms for all positive integers n. If X' denotes the blow-up of I on Y, then $I\mathcal{O}_{X'}$ is a very ample invertible sheaf on X'. The extension X/X' is finite and so $I\mathcal{O}_X$ is an ample invertible sheaf on X; hence the previous isomorphisms imply that the canonical inclusion

$$\pi_*\omega_{Z/Y} \hookrightarrow \omega_{X/Y}$$

is an isomorphism—in other words, that (2) holds. □

Examples are known for both parts of the theorem, as we will show.

Recall that two ideals $I, J \subseteq R$ are called *projectively equivalent* if there exist positive integers i, j with $\overline{I^i} = \overline{J^j}$ (cf. [2]).

COROLLARY 5.3. *Let R be a regular domain, let $x_1, \dots, x_d$ be a generalized regular system of parameters, and let I be an ideal projectively equivalent to an ideal generated by monomials $\underline{x}^{\underline{a}_1}, \dots, \underline{x}^{\underline{a}_s}$ in $x_1, \dots, x_d$. Then* $\operatorname{adj}(I)$ *is a monomial ideal in $x_1, \dots, x_d$ determined by the Rees valuations of I, and the normalized blow-up of I satisfies Theorem 5.2(2).*

Proof. We need only observe

$$\operatorname{Proj}(\overline{R[It]}) = \operatorname{Proj}(\overline{R[I^i t]}) = \operatorname{Proj}(\overline{R[\overline{I^i} t]});$$

then the corollary follows from Theorem 4.1. □

By the work of Lipman and Teissier we also know part (2) in some cases.

COROLLARY 5.4. *Let $(R, \mathfrak{m})$ be a two-dimensional regular domain. Then, for any nonzero ideal I,*

$$\operatorname{adj}(I) = \bigcap_{v \in \mathcal{RV}(I)} \{r \in R \mid v(r) \geq v(I) - v(J_{R_v/R})\}.$$

Proof. In the two-dimensional case, the normalized blow-up of I has only pseudo-rational singularities by [15, p. 103] and [11, 1.4]. Thus, part (2) of the theorem is satisfied. □

REMARK. In the case of two-dimensional regular rings, an elementary direct proof of Corollary 5.4 can be given as well. We may assume that $(R, \mathfrak{m})$ is local with infinite residue field and that I is $\mathfrak{m}$-primary. Then it follows from [6] and [7] (see also [12]) that, for a generic $x \in \mathfrak{m} \setminus \mathfrak{m}^2$, the ideals I and $\operatorname{adj}(I)$ are contracted from $S := R\left[\frac{\mathfrak{m}}{x}\right]$ and that $\operatorname{adj}(I)S = \frac{1}{x}\operatorname{adj}(IS)$. From this the corollary follows by an easy induction on the multiplicity $\operatorname{mult}(I)$ of I.

With this line of argument we can also give an easy proof of subadditivity of adjoint ideals in the two-dimensional case. Again we may assume that $(R, \mathfrak{m})$ is local with infinite residue field and that I and J are $\mathfrak{m}$-primary. For a generic $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ we have that I, J, IJ, $\operatorname{adj}(I)$, $\operatorname{adj}(J)$, $\operatorname{adj}(IJ)$, and $\operatorname{adj}(I)\operatorname{adj}(J)$ are all contracted from $S = R\left[\frac{\mathfrak{m}}{x}\right]$. Denoting by I' (resp. J') the strict transforms of I (resp. J), we may now conclude by induction on $\operatorname{mult}(I) + \operatorname{mult}(J)$ that

$$\begin{aligned}
\operatorname{adj}(IJ) &= \operatorname{adj}(IJ)S \cap R \\
&= \frac{1}{x}\operatorname{adj}(IJS) \cap R \\
&= x^{\operatorname{ord}(I)+\operatorname{ord}(J)-1}\operatorname{adj}(I'J') \cap R \\
&\subseteq x^{\operatorname{ord}(I)-1}\operatorname{adj}(I') \cdot x^{\operatorname{ord}(J)-1}\operatorname{adj}(J') \cap R \\
&= \operatorname{adj}(I)\operatorname{adj}(J)S \cap R \\
&= \operatorname{adj}(I)\operatorname{adj}(J).
\end{aligned}$$

Alternatively, the subadditivity result may be deduced from [13] and [12]. We note that Tagaki and Watanabe [22] proved subadditivity of adjoint ideals more generally for two-dimensional log-terminal singularities. The argument given here does not extend to their situation.

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