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Axiomatic Regularity on Metric Spaces

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Introduction

The problem of the regularity of solutions to partial differential equations with prescribed boundary values and of regular variational problems constitutes one of the most interesting chapters in analysis, which has its origins mostly starting from the year 1900, when Hilbert formulated his famous 23 problems in an address delivered before the International Congress of Mathematicians at Paris. The essential parts of the 20th problem on existence of solutions and its related 19th problem about the regularity itself read as follows.

19th problem: "Are the solutions of regular problems in the calculus of variations always necessarily analytic?"

20th problem: "Has not every regular variational problem a solution, provided certain assumptions regarding the given boundary conditions are satisfied, and provided also if need be that the notion of a solution shall be suitably extended?"

It is known that in the Euclidean space the problem of minimizing a variational integral in a set of functions with prescribed boundary values is closely related to solving the corresponding Dirichlet problem for its Euler–Lagrange equation. In particular, for the Dirichlet p-energy integral

$$\int_{\Omega\subset\mathbb{R}^n} |\nabla u(x)|^p \, dx,$$

the corresponding Euler–Lagrange equation, for $1 < p < \infty$, is

$$\operatorname{div}(|\nabla u(x)|^{p-2} \nabla u(x)) = 0.$$

Starting with the remarkable result of Bernstein in 1904 that any C^3 solution of an elliptic nonlinear analytic equation in two variables is necessarily analytic, and through the works of many authors, in particular in the works of Leray and Schauder in 1934, it was proved that every sufficiently smooth, say $C^{0,\alpha}$ (Hölder continuous), stationary point of a regular variational problem with analytic integrand is analytic. On the other hand, by direct methods of the calculus of variations one can prove in general the existence of solutions that have derivatives only in a generalized sense and satisfy the equation only in a correspondingly weak form.

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Thus arose the problem of proving that such "generalized solutions" are "regular"—namely, that they possess enough smoothness to satisfy the differential equation in a classical sense. In this respect, Hilbert's 20th problem of existence of classical solutions becomes precisely the problem of regularity of generalized solutions.

This *problem of regularity,* by which we now mean the problem to show that solutions, or extremals, that belong to a Sobolev space are in fact Hölder continuous, resisted many attempts, but finally in 1957 De Giorgi [10] and Nash [39], independently of each other, provided a proof of it. Later, in 1960, Moser [31], by entirely different methods, gave another proof of their result. Moser's argument was later extended by J. Serrin, N. S. Trudinger, and others. Although this approach (known as Moser's iteration technique), which is based on a differential equation, has proved to be very useful for investigating various problems in the Euclidean spaces, it is not readily generalized to the case when one wants to deal with regularity questions on a general metric space (see, however, [8]), since the concept of a partial derivative is (generally) meaningless on a metric space, and thus there is no differential (Euler–Lagrange) equation. However, since it is possible to define a substitute for the modulus of the usual gradient to the case of general metric spaces, the approach of De Giorgi, which is essentially a variational one, can be used. This approach was developed and generalized to certain cases of nonlinear equations by O. Ladyzhenskaya, N. Ural'tseva, G. Stampacchia, and others. Later, in the 1980s, Giaquinta [12] (see also [13]), and then, in the 1990s, Malý and Ziemer [29] tried to give the method of De Giorgi a more transparent form.

The regularity of extremal functions for the Dirichlet energy on an abstract metric space has been studied in the 2001 paper of Kinnunen and Shanmugalingam [25]. One should stress that there are several definitions of the notion of Dirichlet energy (or equivalently of Sobolev space) on a general metric measure space. The first one is probably the notion introduced by Hajłasz in [16], but the most popular way of defining the energy of a function is via the concept of *upper gradients* first introduced by Heinonen and Koskela in [19]. Shanmugalingam has defined and studied a structured notion of Sobolev space based on upper gradients; she called this space the *Newtonian space* on the metric measure space. Other approaches are the Gol'dshtein–Troyanov axiomatic theory of Sobolev spaces on metric spaces [14; 15], where the pseudo-gradients are described by a set of rules they must satisfy, and the notion of Sobolev functions based on a Poincaré inequality first considered in [26] and extensively studied in [17]. The last two approaches are quite different from the Newtonian space in their methods and spirit.

In their article, Kinnunen and Shanmugalingam studied the regularity problem in the context of the Newtonian space. The goal of this paper is to show that De Giorgi's method can also be applied within the Poincaré inequality framework and even to more general situations.

Our strategy is to reduce the De Giorgi argument to a system of three axioms, which we will call Hypotheses H1–H3, that a function u defined on a measure metric space may satisfy. These hypotheses are very general; they can be formulated for any function u in L^p, and in particular we do not assume that a notion of

Sobolev space has been defined. The essence of the De Giorgi argument is first to show that a function satisfying the three hypotheses is Hölder continuous and then to show that a function minimizing an appropriate energy must satisfy the three hypotheses. The technique leaves us a lot of freedom to choose various kinds of energies. We show in the second part of the paper how this technique can be applied to prove the Hölder regularity of extremal functions in the class of functions satisfying a Poincaré inequality. This result is formulated in Theorem 2.5.

The paper is organized as follows. In the first section we formulate the hypotheses H1–H3 we are going to work with. Then we show that a function u satisfying Hypotheses H1 and H2 in the pair with some function g is locally bounded, and if, in addition, Hypothesis H3 is satisfied for the functions u and g, then the function u is locally Hölder continuous. In Section 2 we recall the approach to Sobolev spaces on a metric space via Poincaré inequalities from [17] and verify that functions from the Poincaré–Sobolev space, which have an additional property (De Giorgi condition), satisfy Hypotheses H1–H3 and, thus, are Hölder continuous. The last section addresses some questions on further potential applicability of our main result.

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In the initially submitted version of this paper we also applied our abstract De Giorgi method to establish the regularity of quasi-minimizers in the axiomatic framework of Gol'dshtein–Troyanov mentioned previously. This result was proved under assumptions of strong locality of the corresponding D-structure on a metric space and the validity of a Poincaré inequality. Recently, Nageswari Shanmugalingam informed the author that the axiomatic Sobolev space that satisfies these extra hypotheses is equivalent to a Newtonian space based on upper gradients, and thus the regularity of the p-energy quasi-minimizers in axiomatic Sobolev spaces is a consequence of the regularity results from [25]. We would like to thank Nages for her observation, which simplified and shortened the paper.

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1. De Giorgi Argument in an Abstract Setting

Throughout the paper (X, d) will be a metric space equipped with a Borel regular outer measure μ such that $0 < \mu(B) < \infty$ for any ball $B = B(R) = B(z, R) = \{x \in X : d(x, z) < R\}$ in X of positive radius. If $\sigma > 0$ and $B = B(z, R)$ is a ball, we denote by σB the ball $B(z, \sigma R)$.

For convenience we will suppose that the space X is locally compact and separable. For $1 \leq p < \infty$, $L^p_{\text{loc}}(X) = L^p_{\text{loc}}(X, d, \mu)$ is the space of measurable functions on X that are p-integrable on every relatively compact subset of X.

We will also assume that the measure μ is *doubling*—in other words, that there exists a constant $C_d \geq 1$ such that for all balls $B \subset X$ we have

$$\mu(2B) \le C_d \mu(B).$$

C_d is called the *doubling constant.*

At the beginning of this section we want to emphasize that in the sequel the notation $g_{(u)}$ for a function from $L^p(X)$ means no a priori dependence of this function on the given function $u \in L^p_{\text{loc}}(X)$.

Let Ω be an open subset of X and let u be a function in $L^p(\Omega)$. In this section we prove that if the functions u and $-u$ satisfy Hypotheses H1 and H2 in the pairs with some functions $g_{(u)}, g_{(-u)} \in L^p(\Omega)$, respectively, and if, in addition, the pair $(u, g_{(u)})$ satisfies Hypothesis H3, then u (and, of course, $-u$) is Hölder continuous inside the set Ω.

Unless otherwise stated, C denotes a positive constant whose exact value is unimportant, can change even within a line, and depends only on fixed parameters, such as X, d, μ, p, and others.

1.1. List of Hypotheses

The hypotheses for two functions $u, g_{(u)} \in L^p(\Omega)$ that we shall need are the following.

HYPOTHESIS H1 (De Giorgi condition). There exist constants $C > 0$ and $k^* \in \mathbb{R}$ such that for all $k \ge k^*$, $z \in \Omega$, and $0 < \rho < R \le \operatorname{diam}(X)/3$ so that $B(z, R) \subset \Omega$, the following Caccioppoli-type inequality on the "upper-level" sets of the function u holds:

$$\int_{A(k,\rho)} g_{(u)}^p \, d\mu \le \frac{C}{(R-\rho)^p} \int_{A(k,R)} (u-k)^p \, d\mu, \tag{1}$$

where $A(k,r) = A_z(k,r) = \{x \in B(z,r) = B(r) : u(x) > k\}$ with $z \in \Omega$ being fixed.

Let η be a $\frac{C}{R-\rho}$-Lipschitz (cutoff) function for some $C > 0$ such that $0 \le \eta \le 1$, the support of η is contained in $B\left(\frac{R+\rho}{2}\right)$, and $\eta = 1$ on $B(\rho)$.

HYPOTHESIS H2. There exists a constant $C > 0$ such that for functions $v = \eta(u-k)_+$ and $g_{(v)} = g_{(u)}\chi_{A(k,(R+\rho)/2)} + \frac{C}{R-\rho}(u-k)_+$ and for some t and q, $t > p > q$, we have

$$\left(\fint_{B((R+\rho)/2)} v^t \, d\mu \right)^{1/t} \le CR \left(\fint_{B((R+\rho)/2)} g_{(v)}^q \, d\mu \right)^{1/q}, \tag{2}$$

where k, ρ, and R are as in Hypothesis H1. Here, as usual, $(u-k)_+ = \max\{u-k, 0\}$, and $\chi_{A(k,(R+\rho)/2)}$ is the characteristic function of the set $A\left(k, \frac{R+\rho}{2}\right)$. The bar in the sign of integral means that the corresponding integral is divided by the measure of the set over which the integral is taken.

HYPOTHESIS H3. There exist constants $C > 0$ and $\sigma \ge 1$ such that for all $h, k \in \mathbb{R}$, $h > k \ge k^*$, for the functions

$$w = u_k^h := \min\{u, h\} - \min\{u, k\} = \begin{cases} h - k & \text{if } u \ge h, \\ u - k & \text{if } k < u < h, \\ 0 & \text{if } u \le k, \end{cases}$$

and $g_{(w)} = g_{(u)}\chi_{\{k<u\leq h\}}$ we have

$$\left(\int_{B(R)} w^q \, d\mu\right)^{1/q} \leq CR\left(\int_{B(\sigma R)} g_{(w)}^q \, d\mu\right)^{1/q}, \tag{3}$$

where q is as in Hypothesis H2.

Note that Hypotheses H2 and H3 are the characteristics of the whole Sobolev space of functions for which our abstract results apply (cf. the section on Poincaré–Sobolev spaces), whereas Hypothesis H1 is the property of some particular functions, the functions whose regularity we want to establish. Hypotheses H2 and H3 are Sobolev-type inequalities that are typically true for pairs $(u, g_{(u)})$ in a sufficiently nice metric measure space; they essentially assert that the associated Poincaré inequality remains stable under cutoffs and truncations.

1.2. Boundedness and Hölder Continuity

In this section we show that a function $u \in L^p_{\text{loc}}(\Omega)$ satisfying Hypotheses H1 and H2 with some function $g_{(u)} \in L^p(\Omega)$ is locally bounded in Ω. If, in addition, Hypothesis H3 is valid for the pair $(u, g_{(u)})$, then u is locally Hölder continuous in Ω.

THEOREM 1.1. *Suppose that a pair of functions $(u, g_{(u)})$ satisfies Hypotheses H1 and H2. If $k' \geq k^*$, then there exist constants $C > 0$ and $\theta > 1$ such that*

$$\operatorname*{ess\,sup}_{B(R/2)} u \leq k' + C\left(\fint_{B(R)} (u-k')_+^p \, d\mu\right)^{1/p}\left(\frac{\mu(A(k',R))}{\mu(B(R/2))}\right)^{\theta/p}$$

for all $z \in \Omega$ and $0 < R \leq \operatorname{diam}(X)/3$.

COROLLARY 1.2. *If the functions u and $-u$ satisfy Hypotheses H1 and H2 with some functions $g_{(u)}$ and $g_{(-u)}$, respectively, $u, g_{(u)}, g_{(-u)} \in L^p(\Omega)$, then*

$$\operatorname*{ess\,sup}_{B(R/2)} |u| \leq k + C\left(\fint_{B(R)} |u|^p \, d\mu\right)^{1/p} \tag{4}$$

for all $z \in \Omega$, $k \geq k^*$, $0 < R \leq \operatorname{diam}(X)/3$, and some $C > 0$.

THEOREM 1.3. *Assume that u and $-u \in L^p(\Omega)$ satisfy in the pairs with some $g_{(u)}, g_{(-u)} \in L^p(\Omega)$ Hypotheses H1 and H2. If Hypothesis H3 is also satisfied for the pair $(u, g_{(u)})$, then u is locally Hölder continuous.*

The proofs of Theorems 1.1 and 1.3 closely follow the classical De Giorgi argument [10] mentioned in the Introduction (see also [13; 29] for details in Euclidean spaces and [25] for corresponding proofs in case of upper gradients on metric spaces). For detailed proofs in our abstract setting we refer to [45, Thms. 4.1 and 4.3].

2. Regularity in the Class of Poincaré–Sobolev Functions

In this section we briefly recall the approach to Sobolev spaces on a metric space using Poincaré inequalities (see [17] for the definitions we give) and prove the Hölder continuity of certain extremal functions in these spaces.

2.1. Poincaré–Sobolev Functions

DEFINITION 2.1 (Poincaré inequality). Let $u \in L^1_{\mathrm{loc}}(X)$ and $g\colon X \to [0,\infty]$ be Borel measurable functions. We say that the pair (u,g) satisfies a (s,q)-*Poincaré inequality* in $\Omega \subset X$, $s,q \geq 1$, if there exist two constants $\sigma \geq 1$ and $C_P > 0$ such that the inequality

$$\left(\fint_B |u-u_B|^s \, d\mu\right)^{1/s} \leq C_P r \left(\fint_{\sigma B} g^q \, d\mu\right)^{1/q} \tag{5}$$

holds on every ball B with $\sigma B \subset \Omega$, where r is the radius of B.

Recall that

$$u_B = \fint_B u \, d\mu = \frac{1}{\mu(B)} \int_B u \, d\mu.$$

By the Hölder inequality, a weak (s,q)-Poincaré inequality implies weak (s',q')-Poincaré inequalities with the same σ for all $s' \leq s$ and $q' \geq q$. On the other hand, by [17, Thm. 5.1], a weak $(1,q)$-Poincaré inequality implies a weak (s,q)-Poincaré inequality for some $s > q$ and possibly a new σ.

DEFINITION 2.2 (Poincaré–Sobolev functions). A function $u \in L^1_{\mathrm{loc}}(X)$ for which there exists $0 \leq g \in L^q(X)$ such that the pair (u,g) satisfies a $(1,q)$-Poincaré inequality in X is called a *Poincaré–Sobolev function*. We denote by $PW^{1,q}(X)$ the set of all Poincaré–Sobolev functions.

The Poincaré inequality (5) is the only relationship between the functions u and g. Working in this setting, Hajłasz and Koskela developed in [17] quite a rich theory of these Sobolev-type functions on metric spaces.

Given a function v and $\infty > t_2 > t_1 > 0$, we set

$$v_{t_1}^{t_2} = \min\{\max\{0, v - t_1\}, t_2 - t_1\}.$$

In the sequel we will need also the following definitions.

DEFINITION 2.3 (Truncation property). Let the pair (u,g) satisfy a $(1,q)$-Poincaré inequality in Ω. Assume that for every $b \in \mathbb{R}$, $0 < t_1 < t_2 \leq \infty$, and $\varepsilon \in \{-1,1\}$, the pair $(v_{t_1}^{t_2}, g\chi_{\{t_1 < v \leq t_2\}})$, where $v = \varepsilon(u-b)$, satisfies the $(1,q)$-Poincaré inequality in Ω (with fixed constants C_P, σ). Then we say that the pair (u,g) has the *truncation property.*

The truncation property for Poincaré–Sobolev functions is the notion that reflects some localization properties of the Sobolev space under consideration. Note that in the Euclidean space $\mathbb{R}^n$ this condition means that the gradient of a function, which is constant on some set, equals zero a.e. on that set.

The quasi-minimizers of the p-Dirichlet energy both in the Newtonian spaces (see [25]) and in the axiomatic Sobolev spaces (see [45]) satisfy the De Giorgi condition (Hypothesis H1). For the class of Poincaré–Sobolev functions, the possible notion of energy is not consistent; in particular, it is not clear how it would be possible to prove the existence of corresponding minimizers, since in this case the corresponding Sobolev space is not a Banach space (it is, in fact, only a quasi-Banach space). But the De Giorgi condition is still legitimate for the Poincaré–Sobolev functions. Thus, as it seems that there exists an intimate connection between extremal functions and the functions satisfying the De Giorgi condition, in the case of Poincaré–Sobolev functions, the functions whose regularity we are going to establish will be those that satisfy the following property.

DEFINITION 2.4 (p-De Giorgi condition). We say that a Poincaré–Sobolev function u (satisfying a $(1, q)$-Poincaré inequality with some function g) enjoys the *p-De Giorgi condition* on the set Ω if for all $k \in \mathbb{R}$, $z \in X$, and $0 < \rho < R \leq \operatorname{diam}(X)/3$, the inequality

$$\int_{A(k,\rho)} g^p \, d\mu \leq \frac{C}{(R-\rho)^p} \int_{A(k,R)} (u-k)^p \, d\mu \tag{6}$$

holds, provided $\mu(\Omega \setminus A(k, R)) = 0$, where $A(k,r) = B(z,r) \cap \{x : u(x) > k\}$, $p \in \mathbb{R}$, and $p > q$.

2.2. Regularity of Extremal Functions

In this section we impose the following condition on the measure μ. For every $z \in X$ and $0 < R \leq \operatorname{diam}(X)/3$ we assume that there exists γ, $0 < \gamma < 1$, such that

$$\frac{\mu(B(z, R/2))}{B(z, R)} \leq \gamma.$$

Note that in the case of Newtonian spaces this condition follows from a Poincaré inequality on the space X; in the axiomatic Sobolev spaces it follows from a Poincaré inequality and the strong locality of D-structure.

We will also assume that any pair (u, g), $u \in L^1_{\mathrm{loc}}(X)$, $g \in L^q(X)$, satisfying a $(1, q)$-Poincaré inequality in X has the truncation property. We have the following theorem.

THEOREM 2.5. *Let $u \in PW^{1,q}(X)$ (satisfying a $(1, q)$-Poincaré inequality with some function $g \in L^q(X)$). Suppose that the pairs (u, g) and $(-u, g)$ enjoy the p-De Giorgi condition on the set Ω. Then both (u, g) and $(-u, g)$ satisfy Hypotheses H1 and H2. In addition, one of these pairs satisfies Hypothesis H3 and, thus, the function u is locally Hölder continuous in Ω.*

Proof. Hypothesis H1 is the definition of the p-De Giorgi condition.

Hypothesis H2: Let η be the Lipschitz function as in Hypothesis H2 and $v = \eta(u-k)_+$ (k, ρ, and R are fixed). Since the pair (u, g) satisfies a $(1, q)$-Poincaré inequality, by the truncation property, for every $h \in \mathbb{R}$, $k < h < \infty$, the functions

$$u_k^h = \min\{\max\{0, u-k\}, h-k\} = \begin{cases} h-k & \text{if } u \geq h, \\ u-k & \text{if } k < u < h, \\ 0 & \text{if } u \leq k, \end{cases}$$

and the $g\chi_{\{k<v\leq h\}}$ satisfy this $(1, q)$-Poincaré inequality as well. Hence they satisfy a (t, q)-Poincaré inequality for some t, $t > q$, and thus a (q, q)-Poincaré inequality (see the remark after Definition 2.1).

Let $\{h_i\}_{i\in\mathbb{N}}$ be a sequence of real numbers such that $h_i > k$, $i \in \mathbb{N}$, and $h_i \to \infty$ as $i \to \infty$. Denote $u_i := u_k^{h_i}$. Then, the sequence of functions $\{u_i\}_{i\in\mathbb{N}}$ converges in L^q_{loc} topology to the function $(u-k)_+$. Indeed, for any $i \in \mathbb{N}$,

$$0 \le u_i \le (u-k)_+,$$

and the fact follows from the dominated convergence theorem.

Similarly, the functions $g_i := g\chi_{\{k<v\le h_i\}}$ converge in L^q_{loc} topology to the function $g\chi_{\{u>k\}}$. Since for every $i \in \mathbb{N}$ the pair (u_i, g_i) satisfies a (q,q)-Poincaré inequality, it follows that the pair $((u-k)_+, g\chi_{\{u>k\}})$ also satisfies it.

Denote $\varphi := (u-k)_+$. For all $x, y \in \Omega$ and some ball $B \subset \Omega$ we have

$$\begin{aligned}|\eta(x)\varphi(x) - (\eta\varphi)_B| &\le |\eta(x)\varphi(x) - \eta(x)\varphi_B| + |\eta(x)\varphi_B - (\eta\varphi)_B| \\ &\le \sup|\eta||\varphi(x) - \varphi_B| + |\eta(x)\varphi_B - (\eta\varphi)_B| \\ &\le |\varphi(x) - \varphi_B| + |\eta(x)\varphi_B - (\eta\varphi)_B| =: \Psi(x).\end{aligned}$$

Integrating the last expression $\Psi(x)$ to the power q and using classical inequalities and the definition of Lipschitz functions we get

$$\begin{aligned}&\fint_B \Psi(x)^q \, d\mu(x) \\ &\quad= \fint_B \{|\varphi(x) - \varphi_B| + |\eta(x)\varphi_B - (\eta\varphi)_B|\}^q \, d\mu(x) \\ &\quad= \fint_B \left\{|\varphi(x) - \varphi_B| + \left|\eta(x) \fint_B \varphi(y)\, d\mu(y) - \fint_B \eta(y)\varphi(y)\, d\mu(y)\right|\right\}^q d\mu(x) \\ &\quad= \fint_B \left\{|\varphi(x) - \varphi_B| + \left|\fint_B (\eta(x)\varphi(y)) - \eta(y)\varphi(y))\, d\mu(y)\right|\right\}^q d\mu(x) \\ &\quad\le \fint_B \left\{|\varphi(x) - \varphi_B| + \fint_B |\varphi(y)||\eta(x) - \eta(y)|\, d\mu(y)\right\}^q d\mu(x) \\ &\quad\le \fint_B \left\{|\varphi(x) - \varphi_B| + \mathrm{Lip}(\eta)\,\mathrm{diam}(B) \fint_B |\varphi(y)|\, d\mu(y)\right\}^q d\mu(x) \\ &\quad\le 2^{q-1} \fint_B \left\{|\varphi(x) - \varphi_B|^q + (\mathrm{Lip}(\eta)\,\mathrm{diam}(B))^q \left(\fint_B |\varphi(y)|\, d\mu(y)\right)^q\right\} d\mu(x) \\ &\quad= 2^{q-1} \fint_B |\varphi(x) - \varphi_B|^q \, d\mu(x) + 2^{q-1}(\mathrm{Lip}(\eta)\,\mathrm{diam}(B))^q \left(\fint_B |\varphi(y)|\, d\mu(y)\right)^q.\end{aligned}$$

Now, the pair $(\varphi = (u-k)_+, g\chi_{\{u>k\}})$ satisfies the following (q,q)-Poincaré inequality:

$$\fint_B |\varphi - \varphi_B|^q \, d\mu \le \left(C_P \frac{\mathrm{diam}(B)}{2}\right)^q \fint_{\tau B} (g\chi_{\{u>k\}})^q \, d\mu,$$

where $\tau \ge 1$. Hence

$$\fint_B \Psi(x)^q \, d\mu(x) \le C(\mathrm{diam}(B))^q \fint_{\tau B} \{(g\chi_{\{u>k\}})^q + (\mathrm{Lip}(\eta)|\varphi|)^q\} \, d\mu,$$

where the constant C depends only on τ, q, C_P, and on the doubling constant C_d. We thus have proved that

$$\begin{aligned}
&\left(\fint_B |\eta(x)(u(x)-k)_+ - (\eta(u-k)_+)_B|^q \, d\mu(x)\right)^{1/q} \\
&\qquad \le \left(\fint_B \Psi(x)^q \, d\mu(x)\right)^{1/q} \\
&\qquad \le C \operatorname{diam}(B)\left(\fint_{\tau B} (g\chi_{\{u>k\}} + \operatorname{Lip}(\eta)(u-k)_+)^q \, d\mu\right)^{1/q}.
\end{aligned}$$

In particular, recalling that $v = \eta(u-k)_+$, for the ball $B(R+\rho)$ we have

$$\begin{aligned}
&\left(\fint_{B(R+\rho)} |v - v_{B(R+\rho)}|^q \, d\mu\right)^{1/q} \\
&\qquad\qquad \le C(R+\rho)\left(\fint_{B(\tau(R+\rho))}\left(g\chi_{\{u>k\}} + \frac{C}{R-\rho}(u-k)_+\right)^q d\mu\right)^{1/q}
\end{aligned}$$

for some $C > 0$.

Obviously, $v = v\chi_{\{v>0\}}$. Repeating the argument in the very beginning of the proof of Hypothesis H2, it is easy to show that the truncation property implies that a (q,q)-Poincaré inequality holds for the pair of v and $\big(g\chi_{\{u>k\}} + \frac{C}{R-\rho}(u-k)_+\big)\chi_{\{v>0\}}$. A (t,q)-Poincaré inequality also holds for these functions.

Therefore, we have

$$\begin{aligned}
&\left(\fint_{B(R+\rho)} v^t \, d\mu\right)^{1/t} \\
&\quad \le \left(\fint_{B(R+\rho)} |v - v_{B(R+\rho)}|^t \, d\mu\right)^{1/t} + |v_{B(R+\rho)}| \\
&\quad \le C(R+\rho)\left(\fint_{B(\lambda(R+\rho))}\left(g\chi_{\{u>k\}} + \frac{C}{(R-\rho)}(u-k)_+\right)^q \chi_{\{v>0\}} \, d\mu\right)^{1/q} \\
&\qquad + |v_{B(R+\rho)}| \\
&\quad \le C(R+\rho)\left(\fint_{B(\lambda(R+\rho))}\left(g\chi_{A(k,(R+\rho)/2)} + \frac{C}{(R-\rho)}(u-k)_+\right)^q \chi_{\{v>0\}} \, d\mu\right)^{1/q} \\
&\qquad + |v_{B(R+\rho)}| \\
&\quad \le CR\left(\fint_{B((R+\rho)/2)} g_{(v)}^q \, d\mu\right)^{1/q} + |v_{B(R+\rho)}| \qquad (7)
\end{aligned}$$

for some $\lambda > 0$. In the last inequality we denoted

$$g_{(v)} = g\chi_{A(k,(R+\rho)/2)} + \frac{C}{(R-\rho)}(u-k)_+$$

and used the doubling property of μ and the fact that $\{v > 0\} \subset B\big(\frac{R+\rho}{2}\big)$.

By the Hölder inequality we obtain

$$|v_{B(R+\rho)}| = \frac{1}{\mu(B(R+\rho))} \int_{B(R+\rho)} v\, d\mu = \frac{1}{\mu(B(R+\rho)} \int_{B(R+\rho)} v\chi_{\{v>0\}}\, d\mu$$
$$\leq \left(\fint_{B(R+\rho)} v^t\, d\mu \right)^{1/t} \left(\frac{\mu(\{x \in B(R+\rho) : v(x) > 0\})}{\mu(B(R+\rho))} \right)^{1-1/t}.$$

Then, the condition for the measure μ stated at the beginning of this section implies that

$$\frac{\mu(\{v > 0\})}{\mu(B(R+\rho))} \leq \frac{\mu\big(B\big(\frac{R+\rho}{2}\big)\big)}{\mu(B(R+\rho))} \leq \gamma$$

for some γ, $0 < \gamma < 1$. Hence from the previous inequality and the inequality (7) we obtain

$$(1 - \gamma^{1-1/t}) \left(\fint_{B(R+\rho)} v^t\, d\mu \right)^{1/t} \leq CR \left(\fint_{B((R+\rho)/2)} g_{(v)}^q\, d\mu \right)^{1/q}.$$

From the doubling property of μ finally we have

$$\left(\fint_{B((R+\rho)/2)} v^t\, d\mu \right)^{1/t} \leq CR \left(\fint_{B((R+\rho)/2)} g_{(v)}^q\, d\mu \right)^{1/q}$$

for some $C > 0$. Hypothesis H2 is thus verified.

Hypothesis H3 follows from the truncation property, the doubling condition, and the fact that a $(1,q)$-Poincaré inequality on the doubling metric measure space implies a (t,q)-Poincaré inequality with some $t > q$ and, thus, a (q,q)-Poincaré inequality. Indeed, in Definition 2.3 of the truncation property take $\varepsilon = 1$, $b = 0$, $t_1 = k$, and $t_2 = h$ and note that, in this case,

$$v_{t_1}^{t_2} = u_k^h = \min\{\max\{0, u-k\}, h-k\} = \min\{u,h\} - \min\{u,k\} = w.$$

Then we repeat the proof of Hypothesis H3 in [45, Prop. 6.3] with the functions u^* replaced by u and g_{u^*} replaced by g. □

3. Final Remarks

Let us conclude this paper with some remarks on the potential applicability of our methods.

3.1. The Theory of Dirichlet Forms

Another possible way of doing analysis on a metric space is to use the theory of *Dirichlet forms,* which is well adapted to construct some equivalents of the Laplacian and the Dirichlet energy for $p = 2$. In [5], Biroli and Mosco prove a Harnack inequality, à la De Giorgi–Nash–Moser, for minimizers of canonical variational problems in the framework of the abstract theory of Dirichlet forms on general metric spaces. This result implies the Hölder continuity of extremal functions. In their work, Biroli and Mosco assume a Poincaré inequality for the Dirichlet form and a doubling property for the measure. Another important assumption is the existence of a density for the local energies of the Dirichlet form with respect to the underlying volume measure on the space. The corresponding results for non-linear case of $p > 1$ are obtained in [7], using the approach of the measure-valued homogeneous p-Lagrangians introduced in [28].

3.2. Analysis on Fractals

There exist several approaches in the literature to do analysis on fractal spaces. In particular, Umberto Mosco and his co-authors have studied certain analytic notions on fractal media with the help of Dirichlet forms and the theory of associated metric variational fractals (see e.g. [32; 33; 34; 35; 36; 37; 38] and also [1; 2; 3; 4; 21] for a related probabilistic approach). In this respect one has a number of possible strategies to investigate the continuity of extremal functions on a fractal.

1. Often, the continuity holds for free. This is the case for spaces of homogeneous dimension less than 2, such as the Sierpinski gasket and other self-similar fractals. On such spaces the functions of finite Dirichlet energy are always Hölder continuous. This easily follows from the Morrey-type inequalities established in [6] (see also [9] for the general nonlinear case).
2. One may hope to apply the Biroli–Mosco result [5], but unfortunately, the density for the local energies of the Dirichlet form does not exist in general, even for some standard fractals (see e.g. [27]).
3. One may also try to apply the technique of this paper—that is, associate to any extremal function u for the Dirichlet energy a function g such that the pair (u, g) satisfies Hypotheses H1–H3. This function g need neither be a pseudo-gradient of a D structure nor the density for the energy associated to the Dirichlet form, and any construction of such a function would imply the regularity of u. (Another delicate point is that in the theory of Dirichlet forms, the distance is constructed after the definition of the energy and not before, and for fractals it is often not a metric but only a quasi-metric; but this issue should not be a fatal problem).

One should also mention the works of Kigami, Strichartz and their co-authors. They have studied a notion of Laplacian on the Sierpinski gasket and other post-critically finite fractals through iterative procedures and renormalizations from a somewhat different perspective; see [22; 23; 24; 30; 42; 43]. Their approach also leads to a notion of p-energy and p-harmonic function [20; 44]. It is, however, not clear how to apply a De Giorgi argument in this context. The main obstacles are the same as before: nonexistence of densities associated to the Dirichlet energy and "automatic" continuity of the functions of finite energy.

References

[1] M. T. Barlow, *Some remarks on the elliptic Harnack inequality,* Bull. London Math. Soc. 37 (2005), 200–208.

[2] M. T. Barlow and R. F. Bass, *Brownian motion and harmonic analysis on Sierpinski carpets,* Canad. J. Math. 51 (1999), 673–744.

[3] ———, *Divergence form operators on fractal-like domains,* J. Funct. Anal. 175 (2000), 214–247.
[4] M. T. Barlow, R. F. Bass, and T. Kumagai, *Stability of parabolic Harnack inequalities on metric measure spaces,* J. Math. Soc. Japan 58 (2006), 485–519.
[5] M. Biroli and U. Mosco, *A Saint-Venant type principle for Dirichlet forms on discontinuous media,* Ann. Mat. Pura Appl. (4) 169 (1995), 125–181.
[6] ———, *Sobolev and isoperimetric inequalities for Dirichlet forms on homogeneous spaces,* Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 6 (1995), 37–44.
[7] M. Biroli and P. Vernole, *Harnack inequality for nonlinear Dirichlet forms,* Quad. 584-P Dip. Mat. Politecnico di Milano, 2004.
[8] A. Björn and N. Marola, *Moser iteration for (quasi)minimizers on metric spaces,* Manuscripta Math. 121 (2006), 339–366.
[9] R. Capitanelli, *Functional inequalities for measure-valued Lagrangians on homogeneous spaces,* Adv. Math. Sci. Appl. 13 (2003), 301–313.
[10] E. De Giorgi, *Sulla differenziabilità e l'analicità delle estremali degli integrali multipli regolati,* Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. 3 (1957), 25–43.
[11] K. Gafaïti, *Algèbres de Royden et homéomorphismes à p-dilatation borneé entre espaces métriques mesurés,* thèse, EPF-Lausanne (2001).
[12] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems,* Ann. of Math. Stud., 105, Princeton Univ. Press, Princeton, NJ, 1983.
[13] ———, *Introduction to regularity theory for nonlinear elliptic systems,* Lectures Math. ETH Zurich, Birkhäuser, Basel, 1993.
[14] V. M. Gol'dshtein and M. Troyanov, *Axiomatic theory of Sobolev spaces,* Exposition. Math. 19 (2001), 289–336.
[15] ———, *Capacities on metric spaces,* Integral Equations Operator Theory 44 (2002), 212–242.
[16] P. Hajłasz, *Sobolev spaces on an arbitrary metric space,* Potential Anal. 5 (1996), 403–415.
[17] P. Hajłasz and P. Koskela, *Sobolev met Poincaré,* Mem. Amer. Math. Soc. 145 (2000).
[18] J. Heinonen, *Lectures on analysis on metric spaces,* Springer-Verlag, New York, 2001.
[19] J. Heinonen and P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry,* Acta Math. 181 (1998), 1–61.
[20] P. E. Herman , R. Peirone, and R. Strichartz, *p-Energy and p-harmonic functions on Sierpinski gasket type fractals,* Potential Anal. 20 (2004), 125–148.
[21] A. Jonsson, *Dirichlet forms and Brownian motion penetrating fractals,* Potential Anal. 13 (2000), 69–80.
[22] J. Kigami, *A harmonic calculus on the Sierpinski spaces,* Japan J. Appl. Math. 6 (1989), 259–290.
[23] ———, *Analysis on fractals,* Cambridge Tracts in Math., 143, Cambridge Univ. Press, Cambridge, 2001.
[24] J. Kigami, R. S. Strichartz, and K. C. Walker, *Constructing a Laplacian on the diamond fractal,* Experiment. Math. 10 (2001), 437–448.
[25] J. Kinnunen and N. Shanmugalingam, *Regularity of quasi-minimizer on metric spaces,* Manuscripta Math. 105 (2001), 401–423.
[26] P. Koskela and P. MacManus, *Quasiconformal mappings and Sobolev classes,* Studia Math. 131 (1998), 1–17.

[27] S. Kusuoka, *Diffusion process on nested fractals,* Lecture Notes in Math., 1567, Springer-Verlag, Berlin, 1993.
[28] J. Malý and U. Mosco, *Remarks on measure-valued Lagrangians on homogeneous spaces,* Ricerche Mat. 48 (1999), 217–231.
[29] J. Malý and W. P. Ziemer, *Fine regularity of solutions of elliptic partial differential equations,* Math. Surveys Monogr., 51, Amer. Math. Soc., Providence, RI, 1997.
[30] R. Meyers, R. S. Strichartz, and A. Teplyaev, *Dirichlet forms on the Sierpinski gasket,* Pacific J. Math. 217 (2004), 149–174.
[31] J. Moser, *A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations,* Comm. Pure Appl. Math. 13 (1960), 457–468.
[32] U. Mosco, *Variational metrics on self-similar fractals,* C. R. Acad. Sci. Paris Sér. I Math. 321 (1995), 715–720.
[33] ———, *Variational fractals,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 25 (1997), 683–712.
[34] ———, *Dirichlet forms and self-similarity,* New directions in Dirichlet forms, AMS/IP Stud. Adv. Math., 8, pp. 117–155, Amer. Math. Soc., Providence RI, 1998.
[35] ———, *Lagrangian metrics and fractal dynamics,* Fractal geometry and stochastics, II (Greifswald/Koserow, 1998), Progr. Probab., 46, pp. 269–283, Birkhäuser, Basel, 2000.
[36] ———, *Harnack inequalities on recurrent metric fractals,* Tr. Mat. Inst. Steklova 236 (2002), 503–508.
[37] ———, *Energy functionals on certain fractal structures,* J. Convex Anal. 9 (2002), 581–600.
[38] ———, *Harnack inequalities on scale irregular Sierpinski gaskets,* Nonlinear problems in mathematical physics and related topics, II, Int. Math. Ser. (N.Y.), 2, pp. 305–328, Kluwer/Plenum, New York, 2002.
[39] J. Nash, *Continuity of solutions of parabolic and elliptic equations,* Amer. J. Math. 80 (1958), 931–954.
[40] N. Shanmugalingam, *Newtonian spaces: An extension of Sobolev spaces to metric measure spaces,* Rev. Mat. Iberoamericana 16 (2000), 243–279.
[41] ———, *Harmonic functions on metric spaces,* Illinois J. Math. 45 (2001), 1021–1050.
[42] R. S. Strichartz, *Analysis on fractals,* Notices Amer. Math. Soc. 46 (1999), 1199–1208.
[43] ———, *Function spaces on fractals,* J. Funct. Anal. 198 (2003), 43–83.
[44] R. S. Strichartz and C. Wong, *The p-Laplacian on the Sierpinski gasket,* Nonlinearity 17 (2004), 595–616.
[45] S. Timoshin, *Regularity in metric spaces,* Ph.D. thesis, Lausanne, 2006, ⟨http://biblion.epfl.ch/EPFL/theses/2006/3571/EPFL_TH3571.pdf⟩.

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