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On Abelian Coverings of Surfaces

E. B. Vinberg

In this paper we consider only orientable compact topological surfaces without boundary, which for brevity are simply called *surfaces.* All autohomeomorphisms of surfaces are presumed to be orientation preserving.

We are interested in a classification of finite abelian coverings of surfaces up to the following equivalence relation: two coverings $\pi_1 \colon T_1 \to S_1$ and $\pi_2 \colon T_2 \to S_2$ are *equivalent* if there are homeomorphisms $\varphi \colon S_1 \to S_2$ and $\psi \colon T_1 \to T_2$ such that the diagram

$$\begin{tikzcd} T_1 \arrow[r, "\psi"] \arrow[d, "\pi_1"'] & T_2 \arrow[d, "\pi_2"] \\ S_1 \arrow[r, "\varphi"'] & S_2 \end{tikzcd}$$

commutes.

If $\pi : T \to S$ is a Galois covering with Galois group G (acting on T), then $S \simeq T/G$. Conversely, if G is a finite group of autohomeomorphisms of a surface T acting on T freely (i.e., with trivial stabilizers), then the factorization map $\pi : T \to T/G = S$ is a Galois covering with Galois group G.

Thus, instead of considering finite abelian coverings of surfaces one can consider pairs (T, G), where T is a surface and G is a finite abelian group of autohomeomorphisms of G acting on T freely. The foregoing equivalence relation for coverings corresponds to the following notion of isomorphism of pairs: two pairs (T_1, G_1) and (T_2, G_2) are *isomorphic* if there exist a homeomorphism $\psi : T_1 \to T_2$ and an isomorphism $f : G_1 \to G_2$ such that the diagram

$$\begin{tikzcd} T_1 \arrow[r, "\psi"] \arrow[d, "g"'] & T_2 \arrow[d, "f(g)"] \\ T_1 \arrow[r, "\psi"'] & T_2 \end{tikzcd}$$

commutes for any $g \in G_1$.

Given a finite abelian group G_0, one can consider the problem of classification of free G_0-actions on surfaces. This is not the same as classifying pairs (T, G) with $G \simeq G_0$. To each such pair there corresponds a set of isomorphism classes

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of free G_0-actions on surfaces of genus $h = \operatorname{gen} T$ that is in a bijection with $\operatorname{Aut} G/\operatorname{Aut}_T G$, where $\operatorname{Aut}_T G$ denotes the group of automorphisms of G realized by homeomorphisms of T.

Edmonds [E] (see also [Zim]) defined a canonical injection from the set of isomorphism classes of free G_0-actions on a surface T into $H_2(G_0, \mathbb{Z}) = \bigwedge^2 G_0$ and proved that it becomes a bijection as soon as $g \doteqdot \operatorname{gen} T/G_0 \geq \operatorname{rk} G_0$.

Some authors [E; CN1] define *weak equivalence* of G_0-actions by allowing the twist of an action by an automorphism of G_0. Classification of free G_0-actions on surfaces up to weak equivalence is exactly the same as classification of pairs (T, G) with $G \simeq G_0$.

The latter problem was solved by Nielsen [Nie] for G_0 cyclic and by Costa and Natanzon [CN1] for $G_0 = (C_p)^m$ with p prime. Recently it was independently solved by Costa and Natanzon [CN2] and by George Michael [G] for $G_0 = (C_{p^k})^m$.

In this paper we suggest a new approach to the problem, which permits us to easily recover the known results and to obtain new ones. We give a complete solution in the following three cases:

(1) $G \simeq (C_{p^k})^m$ (Theorem 5.3);
(2) $G \simeq (C_p)^{m_1} \times (C_{p^2})^{m_2}$ (Theorem 6.1);
(3) $g = 2$ (Section 9, especially Theorem 9.1).

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NOTATION.

C_n: the (multiplicative) cyclic group of order n
$\mathbb{Q}_p$: the field of p-adic numbers
$\mathbb{Z}_p$: the ring of integer p-adic numbers
$\mathbb{Z}_p^*$: the group of invertible (= not divisible by p) elements of $\mathbb{Z}_p$

1. Reduction to an Algebraic Problem

For a pair (T, G) of the type described previously, set $S = T/G$, $h = \operatorname{gen} T$, and $g = \operatorname{gen} S$. Then

$$h = |G|(g-1) + 1.$$

The fundamental group $\pi_1(T)$ is embedded into $\pi_1(S)$ as a normal subgroup, so

$$\pi_1(S)/\pi_1(T) \simeq G.$$

The pair $(S, \pi_1(T))$ (where $\pi_1(T)$ is considered as a normal subgroup of $\pi_1(S)$) determines the pair (T, G) up to isomorphism according to the uniformization diagram

$$\begin{tikzcd} & X \arrow[dl, "q"'] \arrow[dd, "p"] \\ T \arrow[dr, "\pi"'] & \\ & S \end{tikzcd},$$

where X is the simply connected covering of S (homeomorphic to $\mathbb{R}^2$ unless T is the sphere and G is trivial) and where p, q, π are the factorization maps defined (respectively) by the actions of $\pi_1(S), \pi_1(T), G$.

Moreover, since (by the Dehn–Nielsen theorem) any isomorphism of the fundamental groups of two surfaces is induced by a homeomorphism of the surfaces (see e.g. [ZVC, Thm. 5.6.2]), the pair (T, G) is determined up to isomorphism by the group–subgroup pair $(\pi_1(S), \pi_1(T))$.

Since $G \simeq \pi_1(S)/\pi_1(T)$ is abelian, it follows that $\pi_1(T) \supset (\pi_1(S), \pi_1(S))$. Set

$$L = H_1(S, \mathbb{Z}) = \pi_1(S)/(\pi_1(S), \pi_1(S)),$$
$$M = \pi_1(T)/(\pi_1(S), \pi_1(S)) \subset L.$$

The group L is free abelian of rank $2g$, and M is a subgroup of finite index such that $L/M \simeq G$. Let ω be the intersection form on L. It is known that ω is a unimodular integral 2-form. By a theorem of Poincaré ([P]; see also [MKS, Chap. 3, Thm. N13]), any symplectic automorphism of L is induced by an automorphism of $\pi_1(S)$. Hence the pair $(\pi_1(S), \pi_1(T))$ is determined up to isomorphism by the triple (L, ω, M).

Our classification problem therefore reduces to the classification problem for triples (L, ω, M), where L is a free abelian group of a given (even) rank n, ω is a unimodular integral 2-form on L, and M is a subgroup of finite index such that L/M is isomorphic to a given finite abelian group G.

2. Symplectic Modules

Let A be a principal ideal domain and let L be a free A-module of finite rank. The following description of 2-forms on L (with values in A) can be found, for example, in [B, Chap. IX, Sec. 5, Thm. 1].

PROPOSITION 2.1. *For any 2-form ω on L, there exists a basis of L in which the matrix of ω has the form*

$$\operatorname{diag}\left(\begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & a_s \\ -a_s & 0 \end{pmatrix}, 0, \dots, 0\right), \tag{2.1}$$

where $a_1, \dots, a_s \neq 0$ and $a_i \mid a_{i+1}$ for $i = 1, \dots, s-1$. The ideals $(a_1), \dots, (a_s)$ are uniquely determined.

A 2-form ω on L is called *unimodular* if its discriminant (in any basis of L) is an invertible element of A. In this case there exists a basis of L in which the matrix of ω has the form

$$\operatorname{diag}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right); \tag{2.2}$$

such a basis is called *symplectic.*

A free A-module of finite rank supplied with a unimodular 2-form is called a *unimodular symplectic A-module*. A unimodular symplectic $\mathbb{Z}$-module is called a *unimodular symplectic lattice*. The existence of a symplectic basis implies that all unimodular symplectic A-modules of the same rank are isomorphic.

For a prime number p, let $\mathbb{Z}_p$ denote the ring of p-adic integers. If L is a free abelian group (= a free $\mathbb{Z}$-module) of finite rank, then $L_p = \mathbb{Z}_p \otimes L$ is a free $\mathbb{Z}_p$-module (of the same rank) called the *p-adic completion* of L. Any 2-form ω on L uniquely extends to a 2-form ω_p on L_p.

If M is a subgroup of finite index in L with $L/M = G$, then $M_p = \mathbb{Z}_p \otimes M$ is a subgroup of finite index (and a $\mathbb{Z}_p$-submodule) in L_p with $L_p/M_p \simeq \mathrm{Tor}_p\, G$, the p-primary component of G. In particular, $M_p = L_p$ for all but finitely many p. Conversely, the strong approximation theorem for the unimodular group [Kne] implies the following.

PROPOSITION 2.2. *Let a submodule M_p of finite index in L_p be given for each p in such a way that $M_p = L_p$ for all but finitely many p. Then there exists a unique subgroup $M \subset L$ such that $\mathbb{Z}_p \otimes M = M_p$ for any p.*

Let (L, ω) be a unimodular symplectic lattice and let $M, M' \subset L$ be two subgroups of finite index.

PROPOSITION 2.3. *If for any prime p there exists a symplectic automorphism φ_p of L_p such that $\varphi_p(M_p) = M'_p$, then there exists a symplectic automorphism φ of L such that $\varphi(M) = M'$.*

Proof. Let S be the (finite) set of prime numbers p for which $M_p \neq L_p$. For any $p \in S$ there exists a natural number $\nu(p)$ such that $M_p \supset p^{\nu(p)} L_p$. By the strong approximation theorem for the symplectic group [Kne], there exists a symplectic automorphism φ of L such that

$$\varphi \equiv \varphi_p \pmod{p^{\nu(p)}} \quad \forall p \in S.$$

Then $\varphi(M)_p = M'_p$ for all p and hence $\varphi(M) = M'$. □

Thus, a triple (L, ω, M) is determined up to isomorphism by its p-adic completions (L_p, ω_p, M_p), and the latter may be arbitrary provided $M_p = L_p$ for all but finitely many p. Our classification problem thus reduces to the classification of submodules of finite index in unimodular symplectic $\mathbb{Z}_p$-modules.

3. Changing the Point of View

Let L be a free $\mathbb{Z}_p$-module of rank n and let M be a submodule of finite index in L. It is known (see e.g. [B, Chap. VII, Sec. 4, Thm. 1]) that there exist a basis $\{e_1, \dots, e_n\}$ of L and nonnegative integers $k_1 \le \cdots \le k_n$ such that $\{p^{k_1} e_1, \dots, p^{k_n} e_n\}$ is a basis of M. Such a basis of L is called *compatible* with M, and $p^{k_1}, \dots, p^{k_n}$ are called the *invariant factors* of the submodule $M \subset L$. If $0 = k_1 = \cdots = k_s < k_{s+1}$, then $p^{k_{s+1}}, \dots, p^{k_n}$ are the invariant factors of the group $G = L/M$; that is,

$$G \simeq C_{p^{k_{s+1}}} \times \cdots \times C_{p^{k_n}},$$

where C_q denotes the (multiplicative) cyclic group of order q. It follows that M is determined by the group G up to automorphism of L.

This permits us to look at the problem from another point of view. Namely, we can fix M and classify unimodular 2-forms ω on L up to the action of the group $\mathrm{Aut}(L, M)$ of automorphisms of L preserving M.

The group $\mathrm{Aut}(L, M)$ is described as follows. As before, let $\{e_1, \dots, e_n\}$ be a basis of L that is compatible with M. An automorphism φ of L is given by a matrix $A = (a_{ij})_{i,j=1}^n$ with entries in $\mathbb{Z}_p$ such that

$$\varphi(e_j) = \sum_{i=1}^{n} a_{ij} e_i.$$

Clearly, $\varphi(M) = M$ if and only if

$$p^{k_i - k_j} | a_{ij} \quad \text{for } i > j. \tag{3.1}$$

In particular, the following "elementary" automorphisms belong to $\mathrm{Aut}(L, M)$:

(1) $\varphi(e_j) = e_j + a e_i$, where $i < j$ or $i > j$ and $p^{k_i - k_j} | a$; and $\varphi(e_k) = e_k$ for $k \neq j$;

(2) $\varphi(e_j) = c e_j$, where $c \in \mathbb{Z}_p^*$; and $\varphi(e_k) = e_k$ for $k \neq j$.

One can show that the group $\mathrm{Aut}(L, M)$ is generated by the elementary automorphisms, but we do not need this fact.

Under these elementary automorphisms, the matrix $\Omega = (\omega_{ij})_{i,j=1}^n$ of the form ω is transformed as follows:

(1) the ith column multiplied by a is subtracted from the jth column and, simultaneously, the ith row multiplied by a is subtracted from the jth row;

(2) the jth column and the jth row are divided by c.

Instead of changing the form ω with the group $\mathrm{Aut}(L, M)$ leaving the basis $\{e_1, \dots, e_n\}$ invariant, one can change the basis $\{e_1, \dots, e_n\}$ with the same group leaving the form ω invariant. We will use one or the other approach as may be convenient.

Let L_k be the submodule of L generated by the e_i with $k_i = k$. Then

$$L = \bigoplus_k L_k, \qquad M = \bigoplus_k p^k L_k, \tag{3.2}$$

and any automorphism of the form

$$\varphi = \bigoplus_k \varphi_k \quad (\varphi_k \in \mathrm{Aut}\, L_k)$$

belongs to $\mathrm{Aut}(L, M)$.

It is likely that our classification problem can be reasonably settled only in some particular cases—when there are only few different invariant factors. We consider three such cases in subsequent sections.

4. Preliminaries

In the sequel, L is a free $\mathbb{Z}_p$-module of rank n, ω is a unimodular 2-form on L, and M is a submodule of finite index in L with invariant factors $p^{k_1}, \dots, p^{k_n}$. The number n will be called the *rank* of the triple (L, ω, M).

Let us call such a triple (L, ω, M) *decomposable* if the module L decomposes into a direct sum of orthogonal (with respect to ω) nontrivial submodules L' and L'' such that $M = M' + M''$, where $M' \subset L'$ and $M'' \subset L''$. In this case, the forms $\omega' = \omega|_{M'}$ and $\omega'' = \omega|_{M''}$ are automatically unimodular, and the triple (L, ω, M) is called the *direct sum* of the triples (L', ω', M') and (L'', ω'', M''). Clearly, any triple can be decomposed into a direct sum of indecomposable triples.

For any submodule $N \subset L$, set $N^{\perp} = \{y \in L \colon \omega(x, y) = 0\ \forall x \in N\}$. A submodule N is called *unimodular* if the form $\omega|_N$ is unimodular. In this case $L = N \oplus N^{\perp}$. Moreover, if $N \subset M$ then $M = N \oplus (M \cap N^{\perp})$, so the triple (L, ω, M) decomposes into a direct sum of the triple $(N, \omega|_N, N)$ and its orthogonal complement $(N^{\perp}, \omega|_{N^{\perp}}, M \cap N^{\perp})$.

Consider the symplectic vector space $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} L$ over the field $\mathbb{Q}_p$ of p-adic numbers, and set

$$M^* = \{y \in V : \omega(x, y) \in \mathbb{Z}_p\ \forall x \in M\}.$$

This is a $\mathbb{Z}_p$-submodule of V, containing L, that is called the *dual submodule* of M. It is generated by the basis $\{f_1, \ldots, f_n\}$ of V dual (with respect to ω) to the basis $\{p^{k_1}e_1, \ldots, p^{k_n}e_n\}$ of M. In particular, M is a submodule of finite index in M^*.

PROPOSITION 4.1. *The invariant factors of M in M^* are the invariant factors of the* (*skew-symmetric*) *matrix $\mathcal{D}\Omega\mathcal{D}$, where*

$$\mathcal{D} = \operatorname{diag}(p^{k_1}, \ldots, p^{k_n}).$$

Proof. We express the basis $\{p^{k_1}e_1, \ldots, p^{k_n}e_n\}$ of M in terms of the basis $\{f_1, \ldots, f_n\}$ of M^*:

$$p^{k_j}e_j = \sum_{i=1}^{n} a_{ij} f_i.$$

The invariant factors of M in M^* are the invariant factors of the matrix $A = (a_{ij})_{i,j=1}^n$. But

$$a_{ij} = \omega(p^{k_i}e_i, p^{k_j}e_j) = p^{k_i+k_j}\omega_{ij},$$

whence $A = \mathcal{D}\Omega\mathcal{D}$. $\square$

It follows from Proposition 2.1 that the invariant factors of any skew-symmetric matrix have even multiplicities. Consequently, the invariant factors of M in M^* have even multiplicities. Taken with halves of their multiplicities, they will be called the *symplectic invariant factors* of the triple (L, ω, M).

Obviously, the set of invariant factors (resp., symplectic invariant factors) of a direct sum of triples is the union of the sets of those of the summands. A triple (L, ω, M) of rank 2 is determined up to isomorphism by the invariant factors p^{k_1} and p^{k_2}. Indeed, take any basis $\{e_1, e_2\}$ of L compatible with M. Dividing e_1 by ω_{12} (which is an invertible element of $\mathbb{Z}_p$), we may assume that $\omega_{12} = 1$. This determines the triple up to isomorphism.

The triple of rank 2 with invariant factors p^{k_1} and p^{k_2} will be denoted by $T_2(k_1, k_2)$. For such a triple, we have

$$\mathcal{D}\Omega\mathcal{D} = \begin{pmatrix} 0 & p^{k_1+k_2} \\ -p^{k_1+k_2} & 0 \end{pmatrix},$$

so the symplectic invariant factor is $p^{k_1+k_2}$.

If all invariant factors of M are equal to p^k, then any basis of L is compatible with M. It follows that such a triple decomposes into a direct sum of triples of type $T_2(k,k)$.

We also need the following result.

PROPOSITION 4.2. *Let A be a matrix with entries in $\mathbb{Z}_p$ and let $[A]_p$ be its reduction modulo p. Suppose that the rows of $[A]_p$ are linearly independent. Then the matrix A can be put in the form*

$$\left(\begin{array}{ccc|c} 1 & & 0 & \\ & \ddots & & 0 \\ 0 & & 1 & \end{array}\right) \tag{4.1}$$

by elementary transformations of its rows and columns over the ring $\mathbb{Z}_p$.

Proof. The matrix $[A]_p$ can be put in the form (4.1) by elementary transformations of its rows and columns over the field $\mathbb{Z}/p\mathbb{Z}$. Lifting these transformations to some elementary transformations of A over $\mathbb{Z}_p$, one may assume that $[A]_p$ already has the form (4.1). Then the standard Gauss algorithm applied to the rows of A (without their permutation) allows us to put A in the form

$$\left(\begin{array}{ccc|c} 1 & & 0 & \\ & \ddots & & B \\ 0 & & 1 & \end{array}\right).$$

Finally, by subtracting suitable linear combinations of the first columns from the columns of B, we put A in the form (4.1). □

5. Case $G = C_{p^k} \times \cdots \times C_{p^k}$

In this case

$$L = L_0 \oplus L_k \quad \text{and} \quad M = L_0 \oplus p^k L_k.$$

We set $\operatorname{rk} L_0 = n_0$ and $\operatorname{rk} L_k = n_k$ (so $n = n_0 + n_k$).

Let $T_4(k;\ell)$ denote the triple of rank 4 of this type, for which $n_0 = n_k = 2$ and the matrix of ω is

$$\Omega = \left(\begin{array}{cc|cc} 0 & p^\ell & 1 & 0 \\ -p^\ell & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right). \tag{5.1}$$

Observe that, if $\ell = 0$, then the submodule L_0 is unimodular and so the triple $T_4(k;0)$ is decomposable. More precisely,

$$T_4(k;0) \simeq T_2(0,0) \oplus T_2(k,k). \tag{5.2}$$

If $\ell \geq k$ then we can annul the left upper corner of Ω, subtracting the third column multiplied by p^ℓ from the second column and doing the same with the rows. This yields

$$T_4(k;\ell) \simeq T_2(0,k) \oplus T_2(0,k) \quad \text{for } \ell \geq k. \tag{5.3}$$

PROPOSITION 5.1. *For $0 < \ell < k$, the symplectic invariant factors of $T_4(k;\ell)$ are*

$$p^\ell \text{ and } p^{2k-\ell}.$$

Proof. By Proposition 4.1, the symplectic invariant factors of $T_4(k;\ell)$ are the invariant factors of the matrix

$$\mathcal{D}\Omega\mathcal{D} = \left(\begin{array}{cc|cc} 0 & p^\ell & p^k & 0 \\ -p^\ell & 0 & 0 & p^k \\ \hline p^k & 0 & 0 & 0 \\ 0 & -p^k & 0 & 0 \end{array}\right),$$

taken with halves of their multiplicities. Adding the first row multiplied by $p^{k-\ell}$ to the fourth row and then subtracting the second row multiplied by $p^{k-\ell}$ from the third row, we obtain the matrix

$$\left(\begin{array}{cc|cc} 0 & p^\ell & p^k & 0 \\ -p^\ell & 0 & 0 & p^k \\ \hline 0 & 0 & 0 & -p^{2k-\ell} \\ 0 & 0 & p^{2k-\ell} & 0 \end{array}\right),$$

whose invariant factors are obviously p^ℓ, p^ℓ, $p^{2k-\ell}$, and $p^{2k-\ell}$. $\square$

COROLLARY 5.2. *The triple $T_4(k;\ell)$ is indecomposable for $0 < \ell < k$.*

THEOREM 5.3. *Any triple (L,ω,M) with $G = L/M \simeq C_{p^k} \times \cdots \times C_{p^k}$ decomposes into a direct sum of triples of types*

$$T_2(0,0),\ T_2(0,k),\ T_2(k,k),\ T_4(k;\ell) \quad (0 < \ell < k). \tag{5.4}$$

The summands of such decompositions are uniquely determined up to isomorphism.

Proof. Let

$$\Omega = \begin{array}{c} \left(\begin{array}{cc} \Omega_0 & \Omega_1 \\ -\Omega_1^\top & \Omega_2 \end{array}\right)\begin{array}{l} \}\, n_0 \\ \}\, n_k \end{array} \\ \begin{array}{cc} \underbrace{\hphantom{-\Omega_1^\top}}_{n_0} & \underbrace{\hphantom{\Omega_2}}_{n_k} \end{array} \end{array}$$

be the matrix of ω in a basis of L compatible with M, where the superscript $\top$ denotes the transposition of a matrix.

If the (i,j)th entry ω_{ij} of Ω_0 is not divisible by p, then the submodule N generated by e_i and e_j is unimodular and hence the triple (L,ω,M) decomposes into a direct sum of the triple $(N, \omega|_N, N)$ (of type $T_2(0,0)$) and its orthogonal complement. Thus, we may assume that all the entries of Ω_0 are divisible by p.

Since $\det[\Omega]_p \neq 0$ but $[\Omega_0]_p = 0$, the rows of $[\Omega_1]_p$ are linearly independent. Elementary automorphisms of L_0 and L_k result in elementary transformations of rows and columns of Ω_1, respectively. By Proposition 4.2, the matrix Ω_1 can be put in the form (4.1) with such transformations. Thus, we may assume that

$$\Omega = \left(\begin{array}{c|c|c} \Omega_0 & \begin{matrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{matrix} & 0 \\ \hline \begin{matrix}-1 & & 0\\ & \ddots & \\ 0 & & -1\end{matrix} & \multicolumn{2}{c}{\Omega_2} \\ \cline{1-1} 0 & \multicolumn{2}{c}{} \end{array}\right).$$

Furthermore, subtracting suitable linear combinations of the first n_0 basis elements from the last n_k ones, we can put the matrix Ω in the form

$$\left(\begin{array}{c|c|c} \Omega_0 & \begin{matrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{matrix} & 0 \\ \hline \begin{matrix}-1 & & 0\\ & \ddots & \\ 0 & & -1\end{matrix} & 0 & 0 \\ \hline 0 & 0 & \Omega_3 \end{array}\right).$$

In particular, if $n_0 < n_k$ then the triple (L, ω, M) decomposes into a direct sum of two triples, the second of which has all invariant factors equal to p^k and hence decomposes into a direct sum of triples of type $T_2(k,k)$. Thus, we may assume that $n_0 = n_k$ and

$$\Omega = \left(\begin{array}{c|c} \Omega_0 & \begin{matrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{matrix} \\ \hline \begin{matrix}-1 & & 0\\ & \ddots & \\ 0 & & -1\end{matrix} & 0 \end{array}\right). \tag{5.5}$$

Under this assumption, the modules L_0 and L_k are in duality with respect to ω. Taking any basis in L_0 and the dual basis in L_k, we retain the form (5.5) of Ω. Therefore, making use of Proposition 2.1, we may assume that

$$\Omega_0 = \operatorname{diag}\left(\begin{pmatrix} 0 & p^{\ell_1} \\ -p^{\ell_1} & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & p^{\ell_s} \\ -p^{\ell_s} & 0 \end{pmatrix}, 0, \dots, 0\right),$$

where $0 < \ell_1 \leq \cdots \leq \ell_s$. But then the triple (L, ω, M) decomposes into a direct sum of triples of types $T_4(k;\ell)$ $(\ell > 0)$ and $T_2(0,k)$. Taking into account (5.3), we obtain the first assertion of the theorem.

To prove the second assertion, let us look at the following table of symplectic invariant factors of the triples of types (5.4).

Type	Symplectic invariant factors
$T_2(0,0)$	1
$T_2(0,k)$	p^k
$T_2(k,k)$	p^{2k}
$T_4(k;\ell)$	$p^\ell, p^{2k-\ell}$

We see that they have nothing in common, so the symplectic invariant factors of the triple (L, ω, M) permit us to determine the number of summands of each type. □

Note that summands of type $T_4(k;\ell)$ appear only for $k > 1$.

COROLLARY 5.4. *For $g \geq m$, the number of equivalence classes of Galois coverings of a surface of genus g with Galois group $G \simeq (C_{p^k})^m$ is equal to $\frac{(m'+1)(m'+2)\cdots(m'+k)}{k!}$, where $m' = \left[\frac{m}{2}\right]$.*

Proof. Let T be the triple corresponding to a covering of the considered type, and let x, y, z_0, z_ℓ $(0 < \ell < k)$ denote the numbers of summands of the types (5.4) in the decomposition of T. Comparing then the invariant factors gives

$$2x + y + 2\sum_{\ell=1}^{k-1} z_\ell = 2g - m, \tag{5.6}$$

$$y + 2\sum_{\ell=0}^{k-1} z_\ell = m \tag{5.7}$$

or, equivalently,

$$x = g - m + z_0, \tag{5.8}$$

$$y' + \sum_{\ell=0}^{k-1} z_\ell = m', \tag{5.9}$$

where $y' = \left[\frac{y}{2}\right]$.

Conversely, let $(y', z_0, z_1, \dots, z_{k-1})$ be any solution to (5.9). Set $y = 2y'$ or $y = 2y' + 1$ so that $y \equiv m \pmod 2$, and set $x = g - m + z_0$. (*Note:* $x \geq 0$ owing to our assumption that $g \geq m$.) Then $x, y, z_0, z_1, \dots, z_{k-1}$ satisfy (5.6) and (5.7), so the sum of the triples (5.4) taken with these multiplicities has the invariant factors as needed.

It remains to observe that the number of (nonnegative integer) solutions to (5.9) is equal to $\frac{(m'+1)(m'+2)\cdots(m'+k)}{k!}$. □

6. Case $G = C_p \times \cdots \times C_p \times C_{p^2} \times \cdots \times C_{p^2}$

In this case,

$$L = L_0 \oplus L_1 \oplus L_2 \quad \text{and} \quad M = L_0 \oplus pL_1 \oplus p^2 L_2.$$

We set $\operatorname{rk} L_0 = n_0$, $\operatorname{rk} L_1 = n_1$, and $\operatorname{rk} L_2 = n_2$ (so $n = n_0 + n_1 + n_2$).

THEOREM 6.1. *Any triple (L, ω, M) with $G = L/M \simeq C_p \times \cdots \times C_p \times C_{p^2} \times \cdots \times C_{p^2}$ decomposes into a direct sum of triples of types*

$$T_2(k_1, k_2)\ (0 \le k_1 \le k_2 \le 2),\ T_4(2; 1). \tag{6.1}$$

The summands of such decompositions are uniquely determined up to isomorphism.

Proof. Let

$$\Omega = \begin{array}{c} \left(\begin{array}{ccc} \Omega_0 & \Omega_1 & \Omega_2 \\ -\Omega_1 & \Omega_3 & \Omega_4 \\ \underbrace{-\Omega_2^\top}_{n_0} & \underbrace{-\Omega_4^\top}_{n_1} & \underbrace{\Omega_5}_{n_2} \end{array} \right) \end{array} \begin{array}{l} \}\, n_0 \\ \}\, n_1 \\ \}\, n_2 \end{array}$$

be the matrix of Ω in a basis of L compatible with M. As in the proof of Theorem 5.3, we may assume that all the entries of Ω_0 are divisible by p.

Suppose now that the (i, j)th entry ω_{ij} of Ω_1 $(0 < i \le n_0 < j \le n_0 + n_1)$ is not divisible by p. Let N be the submodule generated by e_i and e_j. Subtracting suitable linear combinations of e_i and e_j from the other basis elements of L, we can annul all the entries of the ith and jth rows and columns of Ω except ω_{ij} and ω_{ji}. (Our assumption that all the entries of Ω_0 are divisible by p means that the element e_j will be subtracted from the first n_0 basis elements with coefficients divisible by p.) Therefore, the triple (L, ω, M) decomposes into a direct sum of the triple $(N, \omega|_N, M \cap N)$ (of type $T_2(0,1)$) and its orthogonal complement. Thus, we may assume that all the entries of Ω_1 are also divisible by p.

Since $\det[\Omega]_p \ne 0$ but $[\Omega_0]_p = [\Omega_1]_p = 0$, the rows of $[\Omega_2]_p$ are linearly independent. Reasoning as in the proof of Theorem 5.3, we can put the matrix Ω_2 in the form (4.1). Furthermore, subtracting suitable linear combinations of the first n_0 basis elements of L_2 from the basis elements of L_1, we can annul the matrix Ω_1 (by our assumption that all its entries are divisible by p). Finally, subtracting suitable linear combinations of the basis elements of L_0 from the other basis elements of L, we can put the matrix Ω in the form

$$\left(\begin{array}{c|c|c|c}
\Omega_0 & 0 & \begin{matrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{matrix} & 0 \\
\hline
0 & \Omega_3 & 0 & \Omega_6 \\
\hline
\begin{matrix}-1 & & 0\\ & \ddots & \\ 0 & & -1\end{matrix} & 0 & 0 & 0 \\
\hline
0 & -\Omega_6^{\top} & 0 & \Omega_7
\end{array}\right).$$

It follows that the triple (L, ω, M) decomposes into a direct sum of two triples (L', ω', M') and (L'', ω'', M'') whose invariant factors are $1, \dots, 1, p^2, \dots, p^2$ and $p, \dots, p, p^2, \dots, p^2$, respectively. By Theorem 5.3, the triple (L', ω', M') decomposes into a direct sum of triples of types

$$T_2(0,0),\ T_2(0,2),\ T_2(2,2),\ T_4(2;1).$$

In the triple (L'', ω'', M'') the submodule M'' can be divided by p, which gives a triple with invariant factors $1, \dots, 1, p, \dots, p$. By Theorem 5.3, the latter decomposes into a direct sum of triples of types $T_2(0,0)$, $T_2(0,1)$, $T_2(1,1)$. Hence the triple (L'', ω'', M'') decomposes into a direct sum of triples of types

$$T_2(1,1),\ T_2(1,2),\ T_2(2,2).$$

This completes the proof of the first assertion of the theorem.

To prove the second assertion, consider the following table of the invariant factors and the symplectic invariant factors of the triples (6.1).

Type	Invariant factors	Symplectic invariant factors
$T_2(0,0)$	$1, 1$	1
$T_2(0,1)$	$1, p$	p
$T_2(0,2)$	$1, p^2$	p^2
$T_2(1,1)$	p, p	p^2
$T_2(1,2)$	p, p^2	p^3
$T_2(2,2)$	p^2, p^2	p^4
$T_4(2;1)$	$1, 1, p^2, p^2$	p, p^3

Calculating the invariant factors and the symplectic invariant factors of the triple (L, ω, M) in terms of its decomposition into a direct sum of triples of types (6.1), we obtain eight linear equations for the seven multiplicities of summands of this decomposition. It is easy to check that these equations permit us to determine the multiplicities. $\square$

7. Exchange Numbers

In general, the invariant factors and the symplectic invariant factors are not sufficient for distinguishing nonisomorphic triples. In this section we introduce some new invariants.

Fix a decomposition of the set $\nu = \{1, \dots, n\}$ into s blocks $\nu_1, \dots, \nu_s$ of consecutive integers of sizes $n_1, \dots, n_s$ $(n_1 + \dots + n_s = n)$. Denote by $S(n_1, \dots, n_s)$ the subgroup of the symmetric group S_n formed by the permutations that leave invariant each block ν_p $(p = 1, \dots, s)$. Clearly, $S(n_1, \dots, n_s) \simeq S_{n_1} \times \dots \times S_{n_s}$.

For an involution $\sigma \in S_n$ without fixed points, define the *exchange numbers*

$$\mathrm{ex}_{pq}(\sigma) = \#\{i \in \nu_p : \sigma(i) \in \nu_q\}.$$

Two involutions without fixed points are conjugate by means of $S(n_1, \dots, n_s)$ if and only if their exchange numbers coincide.

According to the foregoing decomposition of ν, every $n \times n$ matrix A decomposes into s^2 blocks A_{pq} $(p, q = 1, \dots, s)$, where A_{pq} is an $n_p \times n_q$ matrix.

Let V be a vector space with basis $\{e_1, \dots, e_n\}$ over a field F. Set $V_p = \langle e_i : i \in \nu_p \rangle$. Define a parabolic subgroup $P(n_1, \dots, n_s)$ of the group $\mathrm{GL}(V) = \mathrm{GL}_n(F)$ as the group of nondegenerate blockwise upper triangular matrices. Note that it contains $\mathrm{GL}(V_1) \times \dots \times \mathrm{GL}(V_s)$.

The following proposition must be known, but I was unable to find a reference for it. Let E_{ij} denote the matrix unit whose (i, j)th entry is 1 while all other entries are 0.

PROPOSITION 7.1. *By means of the group $P(n_1, \dots, n_s)$, the matrix of any nondegenerate skew-symmetric bilinear form ω on V can be put in the form*

$$A(\sigma) = \sum_{\sigma(i) > i} (E_{i,\sigma(i)} - E_{\sigma(i),i}), \tag{7.1}$$

where σ is an involution without fixed points defined up to conjugation by permutations of $S(n_1, \dots, n_s)$.

Proof. Let us first prove by induction on n that the matrix of ω can be put in the form (7.1) using only the usual upper triangular matrices.

Let A be the matrix of ω in the basis $\{e_1, \dots, e_n\}$. Because $\det A \neq 0$, there exist k such that

$$a_{11} = a_{12} = \dots = a_{1,k-1} = 0, \quad a_{1k} \neq 0.$$

One may suppose that $a_{1k} = 1$. Subtracting suitable linear combinations of e_1 and e_k from the last $n - k$ basis vectors and subtracting suitable multiples of e_1 from $e_2, \dots, e_{k-1}$, one can transform the matrix A into a direct sum of the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and some matrix A_1 of order $n - 2$ (whose index sets intertwine if $k > 2$). Then one can apply the induction hypothesis to the matrix A_1.

To prove the second statement of the proposition, consider the rank $r_{pq}(A)$ of the left upper corner of A obtained when intersecting the first p block rows with the first

q block columns. Clearly, this rank is invariant under the action of $P(n_1, \dots, n_s)$. On the other hand, $r_{pq}(A(\sigma))$ is the number of ± 1 in the corresponding left upper corner of $A(\sigma)$, whence

$$r_{pq}(A(\sigma)) = \sum_{k=1}^{p} \sum_{\ell=1}^{q} \mathrm{ex}_{k\ell}(\sigma).$$

It is easy to see that knowing the latter sums for all $p, q = 1, \dots, s$ allows us to determine the exchange numbers $\mathrm{ex}_{pq}(\sigma)$ and thereby to determine the involution σ up to conjugation by means of $S(n_1, \dots, n_s)$.

It remains to note that if the involutions σ and τ are conjugate by means of $S(n_1, \dots, n_s)$, then the matrices $A(\sigma)$ and $A(\tau)$ are transformed into one another by means of the group $\mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_s) \subset P(n_1, \dots, n_s)$. $\square$

8. The Pre-Canonical Form of the Matrix of ω

Let now (L, ω, M) be a $\mathbb{Z}_p$-triple with invariant factors $p^{k_1}, \dots, p^{k_n}$ as in Sections 4–6, and let $([L]_p, [\omega]_p, [M]_p)$ denote its reduction modulo p. Define blocks $\nu_1, \dots, \nu_s \subset \{1, \dots, n\}$ as follows: numbers i and j belong to the same block if $k_i = k_j$. Then the reduction of the group $\mathrm{Aut}(L, M)$ modulo p is just the group $P(n_1, \dots, n_s)$ (over the field $F = \mathbb{Z}/p\mathbb{Z}$). Let σ be any involution associated to the form $[\omega]_p$ according to Proposition 7.1.

Lifting the element of $P(n_1, \dots, n_s)$ that transforms $[\omega]_p$ into the form (7.1) to a suitable element of $\mathrm{Aut}(L, M)$, we obtain

$$\omega_{i,\sigma(i)} = 1 \quad \text{for } \sigma(i) > i, \tag{8.1}$$

$$\omega_{ij} \equiv 0 \pmod p \quad \text{for } j \neq \sigma(i). \tag{8.2}$$

After that, subtracting suitable multiples of e_1 from $e_2, \dots, e_n$, then suitable multiples of e_2 from $e_3, \dots, e_n$, and so forth, yields

$$\omega_{ij} = 0 \quad \text{for } j > \sigma(i) \text{ or } i > \sigma(j). \tag{8.3}$$

A matrix of the form ω satisfying (8.1)–(8.3) will be called *pre-canonical*. We remark that this notion depends on the choice of the involution σ in the class of $S(n_1, \dots, n_s)$-conjugacy associated to $[\omega]_p$.

9. Case $n = 4$

A $\mathbb{Z}_p$-triple (L, ω, M) of rank 4 either is indecomposable or decomposes into a direct sum of two triples of rank 2. In the latter case, comparing the symplectic invariant factors shows that the summands are uniquely determined up to isomorphism. Thus, it remains to classify indecomposable triples of rank 4.

We fix the notation $p^{k_1}, p^{k_2}, p^{k_3}, p^{k_4}$ for the invariant factors of the triple. Now consider all possiblilities for the involution $\sigma \in S_4$ associated to the triple according to Section 7.

Case 1: $\sigma = (12)(34)$. In this case, the pre-canonical form of the matrix of ω is

$$\Omega = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

so the triple is decomposable.

Case 2: $\sigma = (13)(24)$. If $k_2 = k_3$ then one can also choose (12)(34) for σ. To avoid repetitions, we will assume that

$$k_1 \leq k_2 < k_3 \leq k_4. \tag{9.1}$$

Here the pre-canonical form of the matrix of ω is

$$\Omega = \begin{pmatrix} 0 & a & 1 & 0 \\ -a & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad p|a.$$

If $a = 0$ then the triple is decomposable. If $a \neq 0$ then, after renormalizing e_1 and e_3, one may assume that

$$\Omega = \begin{pmatrix} 0 & p^\ell & 1 & 0 \\ -p^\ell & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \ell > 0. \tag{9.2}$$

If $\ell \geq k_3 - k_2$ then one can annul the entry p^ℓ by subtracting $p^\ell e_3$ from e_2; hence we may assume that

$$0 < \ell < k_3 - k_2. \tag{9.3}$$

Let $T_4(k_1, k_2, k_3, k_4; \ell)$ denote the triple so defined. We can easily see that its symplectic invariant factors are

$$p^{k_1+k_2+\ell} \quad \text{and} \quad p^{k_3+k_4-\ell}. \tag{9.4}$$

Since $k_1 + k_2 + \ell$ cannot be equal to any sum of two of the numbers k_1, k_2, k_3, k_4, it follows that the triple is indecomposable.

Note that the triple $T_4(k, \ell)$ defined in Section 5 is simply $T_4(0, 0, k, k; \ell)$.

Case 3: $\sigma = (14)(23)$. If $k_1 = k_2$ or $k_3 = k_4$, then one can choose (13)(24) for σ. So we will assume that

$$k_1 < k_2 \leq k_3 < k_4. \tag{9.5}$$

The pre-canonical form of the matrix of ω is now

$$\Omega = \begin{pmatrix} 0 & a & b & 1 \\ -a & 0 & 1 & 0 \\ -b & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad p|a, \ p|b.$$

If $a = b = 0$ then the triple is decomposable. If $a = 0$ and $b \neq 0$ then, after renormalizing e_2 and e_3, one may assume that

$$\Omega = \begin{pmatrix} 0 & 0 & p^\ell & 1 \\ 0 & 0 & 1 & 0 \\ -p^\ell & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \ell > 0. \tag{9.6}$$

If $\ell \geq k_2 - k_1$ or $\ell \geq k_4 - k_3$, then one can annul the entry p^ℓ by subtracting $p^\ell e_2$ from e_1 or $p^\ell e_4$ from e_3. We may therefore assume that

$$0 < \ell < \min\{k_2 - k_1, k_4 - k_3\}. \tag{9.7}$$

Let $T_4'(k_1, k_2, k_3, k_4; \ell)$ denote the triple defined by (9.6) with restrictions (9.7) (which imply that each of the differences $k_2 - k_1$ and $k_4 - k_3$ is greater than 1). Its symplectic invariant factors are

$$p^{k_1+k_3+\ell} \quad \text{and} \quad p^{k_2+k_4-\ell}. \tag{9.8}$$

Since $k_1 + k_3 + \ell$ cannot be equal to any sum of two of the numbers k_1, k_2, k_3, k_4, it follows that the triple $T_4'(k_1, k_2, k_3, k_4; \ell)$ is indecomposable.

If $a \neq 0$ and $b = 0$ then, after renormalizing e_2 and e_3, we may assume that

$$\Omega = \begin{pmatrix} 0 & p^\ell & 0 & 1 \\ -p^\ell & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \ell > 0. \tag{9.9}$$

If $\ell \geq k_3 - k_1$ or $\ell \geq k_4 - k_2$, then one can annul the entry p^ℓ by adding $p^\ell e_3$ to e_1 or subtracting $p^\ell e_4$ from e_2. Hence we may assume that

$$0 < \ell < \min\{k_3 - k_1, k_4 - k_2\}. \tag{9.10}$$

Let $T_4''(k_1, k_2, k_3, k_4; \ell)$ denote the triple defined by (9.9) with restrictions (9.5) and (9.10). Its symplectic invariant factors are

$$p^{k_1+k_2+\ell} \quad \text{and} \quad p^{k_3+k_4-\ell}. \tag{9.11}$$

If this triple is decomposable, then comparing the associated involutions shows that it can be isomorphic only to the direct sum of $T_2(k_1, k_4)$ and $T_2(k_2, k_3)$. The symplectic invariant factors of this sum are, up to permutation,

$$p^{k_1+k_4} \quad \text{and} \quad p^{k_2+k_3}.$$

By (9.10), however, $k_1 + k_2 + \ell$ cannot be equal to $k_1 + k_4$ or $k_2 + k_3$. Hence, the triple $T_4''(k_1, k_2, k_3, k_4; \ell)$ is indecomposable.

Finally, if $a, b \neq 0$ then, after renormalizing e_1, e_2, e_3, e_4, we may assume that

$$\Omega = \begin{pmatrix} 0 & p^{\ell_1} & \varepsilon p^{\ell_2} & 1 \\ -p^{\ell_1} & 0 & 1 & 0 \\ -\varepsilon p^{\ell_2} & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \ell_1, \ell_2 > 0, \tag{9.12}$$

where ε is an invertible element of $\mathbb{Z}_p$ defined up to multiplication by a square. (There are two possibilities for ε if $p \neq 2$ and four possibilities if $p = 2$.)

If $\ell_1 - \ell_2 \geq k_3 - k_2$, then one can annul the entry p^{ℓ_1} by subtracting $\varepsilon^{-1} p^{\ell_1 - \ell_2} e_3$ from e_2. If $\ell_2 \geq k_2 - k_1$ or $\ell_2 \geq k_4 - k_3$ or $\ell_2 \geq \ell_1$, then one can annul the entry εp^{ℓ_2}. Thus, we may assume that

$$0 < \ell_2 < \min\{k_2 - k_1, k_4 - k_3\}, \tag{9.13}$$

$$0 < \ell_1 - \ell_2 < k_3 - k_2. \tag{9.14}$$

(This implies that each of the differences $k_2 - k_1$, $k_3 - k_2$, and $k_4 - k_3$ is greater than 1.)

Let $T_4(k_1, k_2, k_3, k_4; \ell_1, \ell_2; \varepsilon)$ denote the triple defined by (9.12) with restrictions (9.13) and (9.14). Its symplectic invariant factors are

$$p^{k_1 + k_2 + \ell_1} \quad \text{and} \quad p^{k_3 + k_4 - \ell_1}. \tag{9.15}$$

The same reasoning as in the case of $T_4''(k_1, k_2, k_3, k_4; \ell)$ shows that the triple $T_4(k_1, k_2, k_3, k_4; \ell_1, \ell_2; \varepsilon)$ is indecomposable.

Thus, we have proved the first statement of the following theorem.

THEOREM 9.1. *The triples* $T_4(k_1, k_2, k_3, k_4; \ell)$, $T_4'(k_1, k_2, k_3, k_4; \ell)$, $T_4''(k_1, k_2, k_3, k_4; \ell)$, *and* $T_4(k_1, k_2, k_3, k_4; \ell_1, \ell_2; \varepsilon)$ *defined previously are all indecomposable* $\mathbb{Z}_p$*-triples of rank* 4. *They are mutually nonisomorphic with the following exceptions.*

1. $T_4'(k_1, k_2, k_3, k_4; \ell) \simeq T_4''(k_1, k_2, k_3, k_4; \ell)$ *if* $k_2 = k_3$.
2. *For* $p = 2$, $T_4(k_1, k_2, k_3, k_4; \ell_1, \ell_2; \varepsilon) \simeq T_4(k_1, k_2, k_3, k_4; \ell_1, \ell_2; \varepsilon')$ *if* (a) *one of the numbers* ℓ_2, $\ell_1 - \ell_2$, $k_2 - k_1 - \ell_2$, $k_4 - k_3 - \ell_2$, *and* $k_3 - k_2 - \ell_1 + \ell_2$ *is equal to* 2 *and* $\varepsilon \equiv \varepsilon' \pmod 4$ *or* (b) *one of those numbers is equal to* 1.

It remains to prove that the triples $T_4'(k_1, k_2, k_3, k_4; \ell)$, $T_4''(k_1, k_2, k_3, k_4; \ell)$, and $T_4(k_1, k_2, k_3, k_4; \ell_1, \ell_2; \varepsilon)$ are mutually nonisomorphic (with the exceptions indicated in the theorem). This will be done in the next section.

10. Stable Matrices

Each nonzero element $a \in \mathbb{Z}_p$ is uniquely represented as $a = \varepsilon p^\nu$, where $\varepsilon \in \mathbb{Z}_p^*$. The number ν is called the *exponent* of a, denoted $\nu(a)$.

Let $k_1, \dots, k_n$ be nonnegative integers such that

$$k_1 < \cdots < k_n. \tag{10.1}$$

Set $[k]_+ = \max\{k, 0\}$ for $k \in \mathbb{Z}$. An $n \times n$ matrix $\Omega = (\omega_{ij})$ with entries in $\mathbb{Z}_p$ will be called *stable* with respect to the set $(k_1, \dots, k_n)$ if, for any different nonzero entries ω_{ij} and ω_{st},

$$\nu(\omega_{ij}) < \nu(\omega_{st}) + [k_s - k_i]_+ + [k_t - k_j]_+. \tag{10.2}$$

Obviously, a stable matrix remains stable when some of its entries are replaced with zeros.

Let now (L, ω, M) be a $\mathbb{Z}_p$-triple for which (10.1) holds, and let $\{e_1, \dots, e_n\}$ and $\{e'_1, \dots, e'_n\}$ be two bases of L that are compatible with M. Then

$$e'_j = \sum_{i=1}^{n} a_{ij} e_i,$$

where (3.1) holds. Note also that $a_{ii} \in \mathbb{Z}_p^*$ $(i = 1, \dots, n)$ because the matrix $A = (a_{ij})$ is invertible over $\mathbb{Z}_p$.

PROPOSITION 10.1. *Assume that the matrix $\Omega = (\omega_{ij})$ of the form ω in the basis $\{e_1, \dots, e_n\}$ is stable. Then the matrix $\Omega' = (\omega'_{ij})$ of ω in the basis $\{e'_1, \dots, e'_n\}$ satisfies the congruences*

$$\omega'_{ij} \equiv a_{ii} a_{jj} \omega_{ij} \pmod{p^{\nu(\omega_{ij})+1}} \tag{10.3}$$

for any $\omega_{ij} \neq 0$. In particular, if $\omega_{ij} \neq 0$ then $\omega'_{ij} \neq 0$ and $\nu(\omega'_{ij}) = \nu(\omega_{ij})$.

Proof. This can be shown by a straightforward calculation. □

The inequalities (9.13) and (9.14) mean exactly that the matrix (9.12) is stable. Furthermore, the matrices (9.6) and (9.9) are stable. Applying the last statement of Proposition 10.1 then yields the second statement of Theorem 9.1, except for the possible isomorphisms

$$T_4(k_1, k_2, k_3, k_4; \ell_1, \ell_2; \varepsilon) \simeq T_4(k_1, k_2, k_3, k_4; \ell_1, \ell_2; \varepsilon'). \tag{10.4}$$

If $p \neq 2$ then there are two possibilities for ε, depending on whether or not $[\varepsilon]_p$ is a square in $\mathbb{Z}/p\mathbb{Z}$. In the notation of Proposition 10.1, suppose that Ω and Ω' have the form (9.12) (after replacing ε by ε' in the case of Ω'). Then (10.3) gives

$$\begin{aligned} \varepsilon' &\equiv a_{11} a_{33} \varepsilon \pmod{p}, \\ a_{11} a_{22} &\equiv 1 \pmod{p}, \quad \text{and} \\ a_{22} a_{33} &\equiv 1 \pmod{p}, \end{aligned}$$

from which it follows that

$$\varepsilon' \equiv a_{11}^2 \varepsilon \pmod{p}.$$

The case $p = 2$ is handled similarly by using a refinement of Proposition 10.1.

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Department of Mechanics and Mathematics
Moscow State University
Moscow 119992, GSP-2
Russia

vinberg@zebra.ru