Michigan Math. J. 55 (2007)

Local Lipschitz Numbers and Sobolev Spaces

Thomas Zürcher

1. Introduction

Given a metric space (X,d) and a continuous function $f\colon X \to \mathbb{R}$, following the terminology in [BaC] we consider for each point $x \in X$ the *upper* and *lower scaled oscillation functions* of f as

$$L_f(x) := \limsup_{r\to 0} \frac{\sup_{d(x,y)\le r}|f(y)-f(x)|}{r},$$

$$l_f(x) := \liminf_{r\to 0} \frac{\sup_{d(x,y)\le r}|f(y)-f(x)|}{r}.$$

These quantities are also known as *pointwise infinitesimal Lipschitz numbers* (see [H2]); for a Lipschitz function they are always finite, but for general functions they can be infinite at many points. These functions, as shown by Keith [Ke2], play an important role in the study of generalizations of the theorem of Rademacher to metric measure spaces. The theorem of Rademacher states that a Lipschitz function between Euclidean spaces is differentiable almost everywhere (see e.g. [EG, p. 81] or [H1, Thm. 6.15]). As shown by Stepanov, a function $f\colon \mathbb{R}^n \to \mathbb{R}$ with $L_f(x) < \infty$ for $x \in \mathbb{R}^n$ is also differentiable almost everywhere (see [M] for a simple proof). A generalization of this theorem to metric measure spaces was recently obtained in [BaRZ]. One may ask if it is possible to replace L_f by l_f; in [BaC] it was shown that this is not the case. However, by putting additional restrictions on l_f, Balogh and Csörnyei proved the following two regularity theorems (for the definitions see Section 2).

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain and $f\colon \Omega \to \mathbb{R}$ a continuous function. Assume that $l_f(x) < \infty$ for $x \in \Omega \setminus E$, where the exceptional set E has σ-finite $(n-1)$-dimensional Hausdorff measure. Assume that $l_f \in L^p_{\mathrm{loc}}(\Omega)$ for some $1 \le p \le \infty$. Then f is in the Sobolev space $W^{1,p}_{\mathrm{loc}}(\Omega)$. If, in addition, $p > n$, then*

$$l_f(x) = L_f(x) = \|\nabla f(x)\| \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \Omega.$$

Theorem 1.2. *Let (X,d,μ) be a Q-regular metric measure space and $f\colon X \to \mathbb{R}$ a continuous function. Assume that there exists a set $E \subset X$ and an exponent $1 \le p < Q$ such that $l_f \in L^p_{\mathrm{loc}}(X)$ and $l_f(x) < \infty$ for $x \in X \setminus E$, where $\mathcal{H}^{Q-p}(E) = 0$. Then it follows that f is in the Newtonian space $N^{1,p}_{\mathrm{loc}}(X)$.*

Received June 12, 2006. Revision received May 16, 2007.
The author was supported by the Swiss National Science Foundation.

The Newtonian space $N^{1,p}(X)$ is a natural generalization (due to [S]) of the Sobolev space to the metric setting.

Note that in Theorem 1.2 the exceptional set E must be of dimension less than or equal to $Q - p$. In contrast, in the Euclidean setting the corresponding dimension is $Q - 1$. Here we shall improve the size of the exceptional set in the metric setting by imposing additional conditions on the space (X, d, μ). Our main result is as follows.

THEOREM 1.3. *Let (X, d, μ) be a Q-regular and proper metric measure space, where $Q > 1$. Suppose that $1 \le q \le p$ and that (X, d, μ) supports a $(1, q)$-Poincaré inequality. Assume that $f \colon X \to \mathbb{R}$ is continuous and that $l_f(x) < \infty$ for $x \in X \setminus E$, where E has σ-finite $(Q - q)$-dimensional Hausdorff measure. If l_f is in $L^p_{\mathrm{loc}}(X)$, then f is in the Newtonian space $N^{1,p}_{\mathrm{loc}}(X)$.*

There are many examples of spaces supporting a Poincaré inequality. Among others, Carnot groups equipped with the Lebesgue measure and the Carnot–Carathéodory metric as well as complete Riemannian manifolds with nonnegative Ricci curvature admit a $(1,1)$-Poincaré inequality; see [HaK; H2]. Thus, for $q = 1$ we obtain a generalization of Theorem 1.1.

To obtain Theorem 1.3, we use the method developed in [BaKR]. The technique therein is used to obtain regularity results for another class of mappings (i.e., quasiconformal mappings). In fact these ideas work for Sobolev spaces as well. However, we needed to sharpen some of the statements (with new proofs) from [BaKR] and [BaC] in order to obtain our result. Before detailing the proof of Theorem 1.3 at the beginning of Section 3, we list needed definitions and results of a general nature in the next section.

ACKNOWLEDGMENTS. My thanks go to my supervisor Zoltán Balogh for his many suggestions. I would also like to express my gratitude to Kurt Falk, who suggested improvements, to Pekka Pankka for his remarks, to Pekka Koskela for answering one of my questions, to Simon Bünzli for his tool that made writing easier, and to the referees for carefully reading the article and for their suggestions.

2. Preliminary Definitions and Facts

To start with, we fix some notation; the definitions can be found in [H1]. The standard *Sobolev space* for $\Omega \subset \mathbb{R}^n$ is denoted by $W^{1,p}(\Omega)$ and the *Hausdorff measure* by $\mathcal{H}^k$. For the n-dimensional *Lebesgue measure* we will use the symbol $\mathcal{L}^n$. We say that a metric space is *proper* if closed balls are compact.

A set is *of σ-finite measure* if it can be written as a countable union of closed subsets with finite measure. By a *locally finite measure* we mean a measure with the property that every point has a neighborhood of finite measure.

By a *path* we denote a continuous map $\gamma \colon I \to X$, where I is some real interval; abusing notation we use "path" to refer to its image as well. We will use Γ_{rect} to denote all nonconstant paths in the space with finite length and compact domain ("rect" = rectifiable). The symbol $\gamma_{a,b}$ will denote a path with endpoints a and b.

If we write λB for a ball B and a scaling factor $\lambda > 0$, we mean the ball with the same center as B but with its radius scaled by a factor λ.

Before defining Newtonian spaces (as a generalization of the Sobolev spaces to the metric setting), we need to recall the concept of the modulus of a path family and the concept of a weak upper gradient.

Definition 2.1 (Modulus). Let Γ be a collection of paths in X. The *p-modulus* of the family Γ, denoted $\operatorname{mod}_p(\Gamma)$, is defined to be the number

$$\inf\left\{\|\rho\|_{L^p}^p : \rho\colon X \to \mathbb{R} \text{ Borel}, \int_\gamma \rho\, ds \geq 1\right\}.$$

Such functions ρ used to define the p-modulus of Γ are said to be *admissible* for the family Γ.

The modulus is an outer measure.

Definition 2.2 (Weak upper gradient). Assume that u is a real-valued function on X and that ρ is a nonnegative Borel function on X. Suppose

$$|u(x) - u(y)| \leq \int_\gamma \rho\, ds \tag{2.3}$$

holds for all points $x, y \in X$ and for all paths $\gamma \in \Gamma_{\text{rect}}$ connecting them. Then ρ is said to be an *upper gradient* of u. If there exists a family $\Gamma \subset \Gamma_{\text{rect}}$ such that $\operatorname{mod}_p(\Gamma) = 0$ and (2.3) holds for all points $x, y \in X$ and all paths $\gamma \in \Gamma_{\text{rect}} \setminus \Gamma$ connecting them, then ρ is said to be a *p-weak upper gradient* of u.

The definition of Newtonian spaces is due to Shanmugalingam [S]. A treatise on Sobolev spaces in metric measure spaces can be found in [Ha].

Definition 2.4 (Newtonian space). The *Newtonian space* corresponding to the index p ($1 \leq p < \infty$), denoted $N^{1,p}(X)$, is defined to be the space of equivalence classes of all real-valued p-integrable functions u on X that have a p-integrable weak upper gradient. The equivalence classes are constructed with respect to the semi-norm

$$\|u\|_{N^{1,p}} := \|u\|_{L^p} + \inf_\rho \|\rho\|_{L^p},$$

where the infimum is taken over all p-integrable weak upper gradients of u.

Let us now fix our setting. The triple (X, d, μ) always denotes a *metric measure space*—that is, a metric space equipped with a measure. We wish to consider metric measure spaces that have some regularity. The following definition links the measure and the metric.

Definition 2.5 (Q-regularity). Let (X, d, μ) be given with μ a locally finite Radon measure in X. We say that (X, d, μ) is *Ahlfors regular of dimension Q* (or *Q-regular*) if

$$ar^Q \leq \mu(B(x,r)) \leq Ar^Q \quad \text{for any ball } B(x,r) \subset X \text{ with } r < \operatorname{diam} X$$

for suitable constants $a, A > 0$.

The Q-regularity of a measure allows us to compare the measures of balls of comparable size. The following fact is based on this; see [H1, Exer. 2.10] or [Bo, Lemma 4.2].

LEMMA 2.6 (Bojarski lemma). *Let (X, d, μ) be a proper and Q-regular metric measure space. Further, let $B_0 = B(x_0, r_0)$ and fix $1 \le p < \infty$. Then there exist positive constants δ and C such that, given any collection $B_1, B_2, \ldots,$ of balls in B_0 with radii of at most δ and any nonnegative numbers a_i, we have the estimate*

$$\int_{B_0}\Big(\sum_i a_i \chi_{6B_i}\Big)^p \, d\mu \le C \int_{B_0}\Big(\sum_i a_i \chi_{B_i/6}\Big)^p \, d\mu.$$

Given the definition of Newtonian spaces, we can prevent $N^{1,p}(X)$ from becoming trivial by requiring the existence of many paths in X. A condition that imposes this is the Poincaré inequality (see [Ke1] for more details on this relationship). There are several different definitions of the Poincaré inequality. We will use the following one (see [HaK]).

DEFINITION 2.7 (Poincaré inequality). The metric measure space (X, d, μ) is said to support a $(1, q)$-*Poincaré inequality* if there exist constants $C, \lambda > 0$ such that, for all open balls B in X and all pairs of functions f and ρ defined on B, if f is continuous and if ρ is an upper gradient of f on B and f is integrable on B, then

$$\fint_B |f - f_B| \, d\mu \le C \operatorname{diam}(B)\Big(\fint_{\lambda B} \rho^q \, d\mu\Big)^{1/q},$$

where, for a measurable function u on X,

$$u_B := \frac{1}{\mu(B)} \int_B u \, d\mu =: \fint_B u \, d\mu.$$

If a Q-regular metric measure space admits a Poincaré inequality, then it allows for first-order differential calculus similar to that in Euclidean space; see [H2].

As pointed out to the author by P. Koskela, a space satisfying a $(1, q)$-Poincaré inequality for upper gradients fulfills also a $(1, q)$-Poincaré inequality for q-weak upper gradients. See [KMac, Lemma 2.4] for a proof.

LEMMA 2.8. *Assume that (X, d, μ) is a metric measure space supporting a $(1, q)$-Poincaré inequality for some $q \ge 1$ and constants C and λ. Suppose that ρ is a q-weak upper gradient of the function $u \colon X \to \mathbb{R}$. Then the pair (u, ρ) also satisfies a $(1, q)$-Poincaré inequality with the same constants.*

3. Proof of Theorem 1.3

Because our statement is local, we may assume without loss of generality that X *is compact.*

Sketch of the Proof. A necessary condition for a function f to be in $N^{1,p}(X)$ is that it be absolutely continuous on a p-a.e. rectifiable path. (This means that f must be absolutely continuous on all rectifiable paths except a path family of p-modulus 0.) Assume that E is the set where l_f is infinite. Having in mind the Cantor function (see e.g. [GeO, p. 95]), we see that a path γ with $\mathcal{H}^1(f(\gamma \cap E)) > 0$ may be a path on which f is not absolutely continuous. Thus our first goal is to show that

$$\operatorname{mod}_p(\{\gamma \in \Gamma_{\text{rect}} : \mathcal{H}^1(f(\gamma \cap E)) > 0\}) = 0.$$

However, we are only able to prove that

$$\operatorname{mod}_q(\{\gamma \in \Gamma_{\text{rect}} : \mathcal{H}^1(f(\gamma \cap E)) > 0\}) = 0. \tag{3.1}$$

It is the dimension of the set E that plays an important role here. After proving (3.1), we show that f is absolutely continuous on paths γ with $\mathcal{H}^1(f(\gamma \cap E)) = 0$. This enables us then to prove that l_f is a q-weak upper gradient. If $p = q$ then we are done. Otherwise, we use the Poincaré inequality to show that f is in the Hajłasz space $M^{1,p}(X)$. We define $M^{1,p}(X)$ later but note here that a continuous function in $M^{1,p}(X)$ is also in $N^{1,p}(X)$, which allows us to conclude the proof.

Let us start now with the proof of Theorem 1.3. We begin with a proposition, an improvement of [BaKR, Lemma 3.5] that validates (3.1).

Proposition 3.2. *Let $f: X \to Y$ be a continuous mapping between metric spaces, and assume that X is proper and supports a Q-regular measure μ. Let $E \subset X$ have σ-finite $\mathcal{H}^{Q-q}$-measure for some $1 \le q \le Q$. Then*

$$\operatorname{mod}_q(\{\gamma \in \Gamma_{\text{rect}} : \mathcal{H}^1(f(\gamma \cap E)) > 0\}) = 0.$$

Proof. In light of the subadditivity of mod_q, it suffices to consider the case where $\mathcal{H}^{Q-q}(E) < \infty$. Fix $\varepsilon > 0$ and let

$$\Gamma_\varepsilon := \{\gamma \in \Gamma_{\text{rect}} : \mathcal{H}^1(f(\gamma \cap E)) > \varepsilon\}.$$

We want to show that this set has q-modulus 0; the claim will then follow, again by the subadditivity of the modulus. We will construct admissible functions by means of collections of balls that cover the exceptional set E. Here we use our assumption that E has finite $\mathcal{H}^{Q-q}$-measure.

Since for every rectifiable path we can find a ball that contains this path, it suffices (by the subadditivity of the modulus) to assume that $E \subset \frac{1}{2}B_0$ for a ball B_0 with radius bounded by the diameter of X. Since X is proper, the closed ball $\bar{B}_0$ is compact, and since f is continuous, its restriction to $\bar{B}_0$ is uniformly continuous. Hence for $k \in \mathbb{N}$ there exists a $\delta_k > 0$ such that, for all $x, x' \in \bar{B}_0$,

$$d_X(x, x') < \delta_k \implies d_Y(f(x), f(x')) < \frac{\varepsilon}{2^{k+3}}. \tag{3.3}$$

We can assume that $(\delta_k)_k$ is a sequence of positive numbers decreasing to zero. Fix $\tilde{\varepsilon} > 0$. Using the definition of the Hausdorff measure and applying the $5r$-covering theorem (see [H1, Thm. 1.2]), we find a sequence of balls $(B_i^k)_i$ such that

- $2B_i^k \subset B_0$,
- $B_i^k \cap B_j^k = \emptyset$ for $i \neq j$,
- $\operatorname{diam}(B_i^k) < \delta_k/5$,
- $E \subset \bigcup_i 5B_i^k$, and
- $\sum_k (\operatorname{diam}(B_i^k))^{Q-q} < \mathcal{H}^{Q-q}(E) + \tilde{\varepsilon}$.

Let $\gamma \in \Gamma_\varepsilon$, and note that $\mathcal{H}^1(f(\gamma \cap E)) > \varepsilon$. This implies that, for all sufficiently large integers k, there exist points $y_1, \dots, y_{2^{k-2}} \in f(\gamma \cap E)$ such that

$$d_Y(y_i, y_j) > \frac{\varepsilon}{2^{k+2}} \quad \text{for } i \neq j. \tag{3.4}$$

By inequalities (3.3) and (3.4), we see that at least 2^{k-2} sets $f(5B_i^k)$ will be needed to cover $f(\gamma \cap E)$. Hence there are at least 2^{k-2} balls $5B_i^k$ that hit γ. We define the following sequences $(\rho_k)_k$ and $(\widehat{\rho_k})_k$ of Borel functions:

$$\rho_k(x) = \frac{4}{2^k} \sum_i \frac{1}{\operatorname{diam}(B_i^k)} \chi_{12B_i^k}(x),$$

$$\widehat{\rho_k}(x) = \sum_{j=k}^{\infty} \rho_j(x).$$

Next we want to show that the functions $\widehat{\rho_k}$ are admissible for the modulus. For any $\gamma \in \Gamma_\varepsilon$ and any $j \in \mathbb{N}$, we find $k \geq j$ such that γ is not entirely contained in any ball $12B_i^k$ and there exist 2^{k-2} points y_m such that (3.4) holds. Then

$$\begin{aligned}\int_\gamma \widehat{\rho_j}\, ds \geq \int_\gamma \rho_k\, ds &= \frac{4}{2^k} \sum_i \int_\gamma \frac{\chi_{12B_i^k}}{\operatorname{diam}(B_i^k)}\, ds \\ &\geq \frac{4}{2^k} \sum_{5B_i^k \cap \gamma \neq \emptyset} \frac{\operatorname{diam}(B_i^k)}{\operatorname{diam}(B_i^k)} \geq \frac{4}{2^k} \cdot 2^{k-2} \geq 1,\end{aligned}$$

where we have used the estimated number of balls intersecting γ and the fact that γ is not entirely contained in any ball $12B_i^k$.

We now use Lemma 2.6 to estimate the modulus from above. The constant C may vary from line to line, but it depends only on the regularity constants and on q. We obtain for k with $\delta_k < \delta$, where δ is the constant from Lemma 2.6,

$$\begin{aligned}\int_X \rho_k^q\, d\mu = \frac{4^q}{2^{kq}} \int_X \Bigl(\sum_i \frac{\chi_{12B_i^k}}{\operatorname{diam}(B_i^k)} \Bigr)^q d\mu &\leq \frac{C}{2^{kq}} \sum_i \int_X \Bigl(\frac{\chi_{B_i^k}}{\operatorname{diam}(B_i^k)} \Bigr)^q d\mu \\ &\leq \frac{C}{2^{kq}} \sum_i \frac{\mu(B_i^k)}{\operatorname{diam}(B_i^k)^q}.\end{aligned}$$

By the Q-regularity of μ, we finally conclude that

$$\int_X \rho_k^q\, d\mu \leq \frac{C}{2^{kq}} \sum_i (\operatorname{diam}(B_i^k))^{Q-q} \leq \frac{C}{2^{kq}} (\mathcal{H}^{Q-q}(E) + \tilde{\varepsilon}). \tag{3.5}$$

Consequently

$$\left(\int_X \widehat{\rho_j}^q \, d\mu\right)^{1/q} = \left(\int_X \left(\sum_{k=j}^{\infty} \rho_k\right)^q d\mu\right)^{1/q} \le \sum_{k=j}^{\infty}\left(\int_X \rho_k^q \, d\mu\right)^{1/q}$$
$$\le \sum_{k=j}^{\infty}\left(\frac{C}{2^{kq}}(\mathcal{H}^{Q-q}(E)+\tilde{\varepsilon})\right)^{1/q} \le C(\mathcal{H}^{Q-q}(E)+\tilde{\varepsilon})^{1/q}\frac{1}{2^{j-1}}.$$

Raising to the power q and letting $j \to \infty$ finishes the proof. □

The next result will be needed to show that l_f is a q-weak upper gradient. It is a stronger version of [BaC, Lemma 1.1].

LEMMA 3.6. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $l_f(x) < \infty$ for $x \in [a, b] \setminus E$, where $\mathcal{L}^1(f(E)) = 0$. Assume also that $l_f \in L^1([a, b])$. Then f is an absolutely continuous function and*

$$|f'(x)| = l_f(x) \quad \textit{for } \mathcal{L}^1\textit{-a.e. } x \in [a, b].$$

Proof. We first prove that f is absolutely continuous. For $x \in [a, b] \setminus E$ we define the function $\tilde{l}_f(x) := \max\{l_f(x), 1\}$. It is clear that $\tilde{l}_f$ is also in $L^1([a, b])$. The absolute continuity follows essentially from (3.7).

Claim. For $c, d \in [a, b]$ with $c < d$, we have

$$|f(d) - f(c)| \le 8 \cdot \int_c^d \tilde{l}_f \, dx. \tag{3.7}$$

Proof of the claim. We fix $c, d \in [a, b]$ with $c < d$. First we define sets where $\tilde{l}_f$ is controlled from above and below. For $k \in \mathbb{N} \cup \{0\}$, we set

$$A_k := \{x \in [c, d] \setminus E : 2^k \le \tilde{l}_f(x) < 2^{k+1}\}.$$

Observe that the A_k are disjoint Borel sets (see [Ke2]) and that

$$[c, d] \setminus E = \bigcup_k A_k.$$

The proof of the claim is based on a covering argument. First we approximate the Borel sets A_k by open sets U_k as follows. Fix a small $\varepsilon > 0$ and choose, for each k, an open set U_k such that $A_k \subset U_k$ and

$$\mathcal{L}^1(U_k) \le \mathcal{L}^1(A_k) + \frac{\varepsilon}{2^{2k}}.$$

For $x \in A_k$ with $\tilde{l}_f(x) = l_f(x)$ we obtain

$$\liminf_{r\to 0} \frac{\mathcal{L}^1(f(B(x, r)))}{r} \le 2 \liminf_{r\to 0} \frac{\sup_{|x-y|\le r} |f(y) - f(x)|}{r} = 2l_f(x) < 2^{k+2}.$$

If $\tilde{l}_f(x) = 1$ then, as before, we conclude that

$$\liminf_{r\to 0} \frac{\mathcal{L}^1(f(B(x, r)))}{r} \le 2l_f(x) \le 2 < 2^{k+2}.$$

Thus we can find for each $x \in A_k$ a radius $r_x > 0$ such that the following two properties are fulfilled:

(a) $B(x, r_x) \subset U_k$;
(b) $\mathcal{L}^1(f(B(x, r_x))) \le 2^{k+2} \cdot r_x$.

We denote by $\mathcal{B}$ the collection of balls with the properties (a) and (b) just obtained. It is clear that

$$[c, d] \setminus E = \bigcup_k A_k \subset \bigcup_{B \in \mathcal{B}} B.$$

We can choose a countable subcollection $\{B_i\}_i$ of $\mathcal{B}$, $B_i = B(x_i, r_i)$, that covers the set $[c, d] \setminus E$ with overlap bounded by 2; that is,

$$\sum_i \chi_{B_i}(x) \le 2 \quad \text{for all } x \in [c, d] \setminus E. \tag{3.8}$$

Using the continuity of f we conclude that

$$\begin{aligned} |f(d) - f(c)| \le \mathcal{L}^1(f([c, d] \setminus E)) &\le \sum_k \mathcal{L}^1(f(A_k)) \le \sum_k \sum_{\{i : x_i \in A_k\}} \mathcal{L}^1(f(B_i)) \\ &\le \sum_k 2^{k+2} \sum_{\{i : x_i \in A_k\}} r_i. \end{aligned}$$

By (3.8) we finally obtain

$$\begin{aligned} |f(d) - f(c)| &\le 2 \sum_k 2^{k+2} \Big(\mathcal{L}^1(A_k) + \frac{\varepsilon}{2^{2k}} \Big) \le 8 \Big(\sum_k \int_{A_k} \tilde{l}_f \, dx + 2\varepsilon \Big) \\ &\le 8 \int_c^d \tilde{l}_f \, dx + 16\varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ gives inequality (3.7) and thus the claim.

Now the claim—together with the fact that $\tilde{l}_f$ is in $L^1([a, b])$—gives the absolute continuity. By the general differentiability properties of absolutely continuous functions, it follows that f is differentiable for $\mathcal{L}^1$-a.e. $x \in [a, b]$. A bit of calculation shows that, in points of differentiability of f, the equality

$$l_f(x) = |f'(x)|$$

holds. $\square$

The next lemma proves Theorem 1.3 in the case $p = q$.

LEMMA 3.9. *Let (X, d, μ) be a proper and Q-regular metric measure space, $E \subset X$. If E has σ-finite $\mathcal{H}^{Q-q}$-measure (where $1 \le q \le Q$) and if $f : X \to \mathbb{R}$ is continuous with $l_f(x) < \infty$ for $x \in X \setminus E$, then l_f is a q-weak upper gradient of f.*

Proof. Let

$$\Gamma_E := \{\gamma \in \Gamma_{\text{rect}} : \mathcal{H}^1(f(\gamma \cap E)) > 0\}.$$

By Proposition 3.2 we have that

$$\operatorname{mod}_q(\Gamma_E) = 0.$$

Consequently, for every rectifiable path $\gamma \in \Gamma_{\text{rect}}$ with $\mathcal{H}^1(f(\gamma \cap E)) = 0$, we want to show that

$$|f(p_2) - f(p_1)| \le \int_\gamma l_f \, ds,$$

where p_1 and p_2 denote the endpoints of γ. Let us fix a path $\gamma \in \Gamma_{\text{rect}}$ such that $\mathcal{H}^1(f(\gamma \cap E)) = 0$. Without loss of generality, γ is parameterized by arc length. Denote its domain by $[a, b]$, and let

$$h := f \circ \gamma \colon [a, b] \to \mathbb{R}.$$

By the following claim we may apply Lemma 3.6 to h.

Claim. $l_h(t) \le l_f(\gamma(t))$ for all $t \in [a, b]$.

Proof of the claim. Because γ is 1-Lipschitz, it follows that $|t - t'| \le r$ implies $d(\gamma(t), \gamma(t')) \le r$ for all $t, t' \in [a, b]$ and all $r \ge 0$. Then

$$\begin{aligned} l_h(t) &= \liminf_{r \to 0} \frac{1}{r} \sup_{|t-t'| \le r} |h(t') - h(t)| = \liminf_{r \to 0} \frac{1}{r} \sup_{|t-t'| \le r} |f(\gamma(t)) - f(\gamma(t'))| \\ &\le \liminf_{r \to 0} \frac{1}{r} \sup_{d(\gamma(t), y) \le r} |f(\gamma(t)) - f(y)| = l_f(\gamma(t)). \end{aligned}$$

This proves the claim.

If l_h is not in $L^1[a, b]$, then

$$|f(p_2) - f(p_1)| \le \infty = \int_\gamma l_f \, ds.$$

Otherwise we obtain by Lemma 3.6 that h is absolutely continuous and so, by the foregoing claim,

$$\begin{aligned} |f(p_2) - f(p_1)| = |h(b) - h(a)| &\le \int_a^b |h'| \, ds = \int_a^b |l_h| \, ds \\ &\le \int_\gamma |l_f| \, ds = \int_\gamma l_f \, ds. \end{aligned}$$ □

We introduce now the Hajłasz spaces (see e.g. [Ha]).

DEFINITION 3.10 (Hajłasz space). Let (X, d, μ) be a metric measure space. For $1 \le p < \infty$, the *Hajłasz space* $M^{1,p}(X)$ is the collection of L^p-equivalence classes of functions u such that there exists a p-integrable nonnegative function g, called a *Hajłasz gradient* of u, satisfying the inequality

$$|u(x) - u(y)| \le d(x, y)(g(x) + g(y))$$

for μ-almost all x, y in X. The corresponding norm for functions u in $M^{1,p}(X)$ is given by

$$\|u\|_{M^{1,p}} := \|u\|_{L^p} + \inf_g \|g\|_{L^p},$$

where the infimum is taken over all Hajłasz gradients g of u. With this norm, $M^{1,p}(X)$ is a Banach space.

Lemma 4.7 in [S] relates $M^{1,p}(X)$ and $N^{1,p}(X)$. For further connections between Hajłasz spaces and Newtonian spaces the reader may consult [KeZh].

LEMMA 3.11. *The set of all equivalence classes of continuous functions f in $M^{1,p}(X)$ embeds into $N^{1,p}(X)$.*

We will construct a Hajłasz upper gradient of f by applying the maximal operator to l_f^q.

DEFINITION 3.12 (Maximal operator). Assume that (X, d, μ) is a metric measure space and that $u\colon X \to \mathbb{R}$ is a locally integrable real-valued function in X. Define

$$Mu(x) := \sup_{0<r} \fint_{B(x,r)} |u| \, d\mu \quad \text{and}$$
$$M_R u(x) := \sup_{0<r<R} \fint_{B(x,r)} |u| \, d\mu$$

to be the *maximal operator* and the *restricted maximal operator,* respectively.

If the space supports a Poincaré inequality then we have the following estimate, which is Theorem 3.2 in [HaK].

PROPOSITION 3.13. *Let (X, d, μ) be a Q-regular metric measure space. Assume that the pair (u, ρ) satisfies a $(1, q)$-Poincaré inequality, $u \in L^1_{\mathrm{loc}}(X)$, and $q > 0$. Then, for a.e. $x, y \in X$,*

$$|u(x) - u(y)| \le Cd(x, y)((M_{2\lambda d(x,y)}\rho^q(x))^{1/q} + (M_{2\lambda d(x,y)}\rho^q(y))^{1/q}).$$

We are now ready to prove our main theorem.

Proof of Theorem 1.3. Since the case $q = p$ is covered by Lemma 3.9, we may assume that $q < p$. By Lemma 3.9 we know that l_f is a q-weak upper gradient of f. We shall first apply Proposition 3.13 to show that $f \in M^{1,p}(X)$. By Lemma 2.8 we know that the pair (f, l_f) satisfies a $(1, q)$-Poincaré inequality. Since $l_f \in L^p(X)$, it follows that $l_f^q \in L^{p/q}(X)$. Using the theorem of Hardy and Littlewood on L^p-boundedness of maximal operators (see e.g. [H1, Thm. 2.2] or [AT, Thm. 5.2.10]), we obtain that $Ml_f^q \in L^{p/q}(X)$. Applying Proposition 3.13 then yields

$$|f(x) - f(y)| \le Cd(x, y)((Ml_f^q(x))^{1/q} + (Ml_f^q(y))^{1/q})$$

for a.e. $x, y \in X$. Since $(Ml_f^q)^{1/q} \in L^p(X)$, we see that $f \in M^{1,p}(X)$. By Lemma 3.11, we conclude that $f \in N^{1,p}(X)$. □

Using Theorem 1.3 and Corollary 4.3 in [BaRZ], we obtain the following result.

Corollary 3.14. *Let (X, d, μ) be a Q-regular and proper space, where $Q > 1$. Suppose that (X, d, μ) supports a $(1, q)$-Poincaré inequality for $1 \le q \le Q$. Assume further that $f : X \to \mathbb{R}$ is continuous and that $l_f(x) < \infty$ for $x \in X \setminus E$, where E has σ-finite $(Q - q)$-dimensional Hausdorff measure. If, moreover, l_f is in $L^p_{\mathrm{loc}}(X)$ for a p with $Q < p$, then f is locally Hölder continuous and is differentiable a.e. in the sense of Cheeger.* (For the notion of Cheeger differentiability see [BaRZ; Ch].)

Remark 3.15. According to [BBS, Prop. 1.2], in every metric measure space (X, d, μ) wherein continuous functions are dense in $N^{1,p}(X)$, every $u \in N^{1,p}(X)$ has a representative that is quasicontinuous. Quasicontinuity is a weaker notion of continuity; see [BBS, Def. 2.7]. Therefore, our assumption that f needs to be continuous is not as strong as it seems. However, it is not clear if we can replace the continuity assumption by p-quasicontinuity in Theorem 1.3. The problem is that we do not know if the Poincaré inequality still holds on the subspace where f is continuous.

4. Examples

In this section we discuss examples and applications of Theorem 1.3.

The bound of the dimension of the exceptional set in Theorem 1.3 is important, as the following example demonstrates.

Example 4.1. According to [BaC, Thm. 1.4], there exists a nowhere differentiable and continuous function $f : [0,1] \to \mathbb{R}$ such that $l_f(x) = 0$ for $\mathcal{L}^1$-a.e. $x \in [0,1]$. We define a continuous function $g : [0,1]^n \to \mathbb{R}$ as follows:

$$g(x_1, x_2, \dots, x_n) := f(x_1).$$

We claim that g is not in $N^{1,n}([0,1]^n)$ even though $l_g \in L^n([0,1]^n)$. Let us denote

$$E_f := \{x \in [0,1] : l_f(x) = \infty\},$$
$$E_g := \{x \in [0,1]^n : l_g(x) = \infty\}.$$

Observe that

$$E_g = E_f \times [0,1]^{n-1}$$

and that $l_g(x) = 0$ for $\mathcal{L}^n$-a.e. $x \in [0,1]^n$. Since g is not absolutely continuous on n-almost every line parallel to the coordinate axis x_1, it follows that g is not in $N^{1,n}([0,1]^n)$.

Next we consider a class of spaces that are constructed by gluing together two spaces (see [HK, p. 43]). Assume that (X, d_X, μ_X) and (Y, d_Y, μ_Y) are two proper Q-regular metric measure spaces. Suppose further that A is a closed subset of X that has an isometric copy inside Y; in other words, suppose there exists an isometric embedding $i : A \to Y$. We fix this embedding and consider A as subset of both X and Y. The space

$$X \cup_A Y$$

is the disjoint union of X and Y with points in the two copies of A identified. It is possible to extend the metrics from X and Y by defining a metric d on $X \cup_A Y$:

$$d(x, y) = \inf_{a\in A}\{d_X(x, a) + d_Y(a, y)\} \quad \text{for } x \in X \text{ and } y \in Y.$$

If x, y are contained in the same space, then we take the metric of this space. The measures μ_X and μ_Y add up to a Q-regular measure μ on $X \cup_A Y$. For such spaces we obtain the following corollary of Theorem 1.3.

COROLLARY 4.2. *Let (X_1, d_1, μ_1) and (X_2, d_2, μ_2) be two Q-regular and proper metric measure spaces, where $Q > 1$. Suppose that $1 \le q \le p$ and $1 < p < Q$ and that both spaces support a $(1, q)$-Poincaré inequality. Assume that A is a closed set with*

$$\mathcal{H}^{Q-p}(A) < \infty.$$

Then we construct $X := X_1 \cup_A X_2$ as described previously with metric d and measure μ. Let us further assume that $f : X \to \mathbb{R}$ is continuous and that $l_f(x) < \infty$ for $x \in X \setminus E$, where E has σ-finite $(Q - q)$-dimensional Hausdorff measure. If l_f is in $L^p_{\text{loc}}(X)$, then f is in $N^{1,p}_{\text{loc}}(X)$.

Proof. We assume without loss of generality that X is compact. The idea of the proof is to split a given path in X into its components in X_1 and X_2 and then to apply Theorem 1.3 on the individual components.

We define $f_1 : X_1 \to \mathbb{R}$ and $f_2 : X_2 \to \mathbb{R}$ as restrictions of f to X_1 and X_2, respectively. By applying Theorem 1.3 on f_1 and f_2, we obtain p-weak upper gradients ρ_1 and ρ_2 of f_1 and f_2 that are in $L^p(X_1)$ and $L^p(X_2)$ respectively. Set

$$\tilde{\Gamma}(X) := \{\gamma \in \Gamma_{\text{rect}}(X) : \text{the cardinality of } \gamma \cap A \text{ is infinite}\}.$$

By [BaKR, Lemma 3.4], we see that

$$\text{mod}_p(\tilde{\Gamma}(X)) = 0.$$

Take $\gamma \in \Gamma_{\text{rect}}(X) \setminus \tilde{\Gamma}(X)$, $\gamma : [0, l] \to X$. Without loss of generality, we can assume that γ is parameterized by arc length. We set

$$\begin{aligned}
\{a_1, a_2, \dots, a_m\} &= \gamma \cap A,\\
\delta &:= \min\{d(a_i, a_j),\ i \ne j\},\\
T &:= \{t \in [0, l] : \gamma(t) \in A\}.
\end{aligned}$$

We assume that $m \ge 2$ and leave the case $m = 1$ as an exercise for the reader. If T were infinite then, for every natural N, we could extract a subset with N elements. But for each N we would then have $(N - 1)\delta$ as lower bound for the length of γ, contradicting the rectifiability. In conclusion, we can write γ as a finite union of subpaths each lying entirely in one of the spaces X_1 or X_2. For every $\gamma \in \Gamma_{\text{rect}}(X)$ we fix such a representation as a finite union: $\gamma = \bigcup_{i\in I} \gamma_i$, where $\gamma_i \subset X_1$ or $\gamma_i \subset X_2$. We define

$$\begin{aligned}
\Gamma := \{\gamma \in \Gamma_{\text{rect}}(X) \setminus \tilde{\Gamma}(X) :&\\
&\text{there is a } \gamma_i \text{ on which } f \text{ is not absolutely continuous}\}.
\end{aligned}$$

We also define $\rho := \rho_1 \chi_{X_1} + \rho_2 \chi_{X_2} \in L^p(X)$. Our goal is to show that

$$\operatorname{mod}_p(\Gamma) = 0 \tag{4.3}$$

and

$$|f(b) - f(a)| \leq \int_\gamma \rho \, ds \tag{4.4}$$

for all $\gamma \in \Gamma_{\mathrm{rect}}(X) \setminus \Gamma$ connecting a and b. From (4.3) and (4.4) it will then follow that $f \in N^{1,p}_{\mathrm{loc}}(X)$.

To show (4.3), we define

$$\Gamma_1 := \{\gamma \in \Gamma_{\mathrm{rect}}(X_1) : f_1 \text{ is not absolutely continuous on } \gamma\},$$

$$\Gamma_2 := \{\gamma \in \Gamma_{\mathrm{rect}}(X_2) : f_2 \text{ is not absolutely continuous on } \gamma\}.$$

By Theorem 1.3 and Proposition 3.1 in [S] (note that the measures μ_i are comparable to μ and so we can take the moduli with respect to the space (X, d, μ)), we obtain

$$\operatorname{mod}_p(\Gamma_1) = \operatorname{mod}_p(\Gamma_2) = 0;$$

since every path in Γ has a subpath in $\Gamma_1 \cup \Gamma_2$, we obtain equation (4.3):

$$\operatorname{mod}_p(\Gamma) \leq \operatorname{mod}_p(\Gamma_1 \cup \Gamma_2) \leq \operatorname{mod}_p(\Gamma_1) + \operatorname{mod}_p(\Gamma_2) = 0.$$

Let us now take $\gamma = \bigcup_{i \in I} \gamma_i \in \Gamma_{\mathrm{rect}}(X) \setminus \Gamma$. We denote by a_i and b_i the starting point and endpoint of γ_i, respectively. Observe that if $b_i \neq b$ then there is a a_j with $b_i = a_j$, and if $a_i \neq a$ then there is a b_j such that $a_i = b_j$. Consequently,

$$|f(b) - f(a)| \leq \sum_{i \in I} |f(b_i) - f(a_i)| \leq \sum_{i \in I} \int_{\gamma_i} \rho \, ds \leq \int_\gamma \rho \, ds.$$

This concludes the proof. □

REMARK 4.5. Note that in Corollary 4.2 there is no Poincaré inequality required for the space X. In [HK, pp. 43–45], the authors glue together spaces with a $(1, q)$-Poincaré inequality such that the resulting space does admit a (weaker) Poincaré inequality.

REMARK 4.6. Assume that in Theorem 1.2 we additionally require X to be proper. Then, by Lemma 3.9, we can replace the condition that $\mathcal{H}^{Q-p}(E) = 0$ with the requirement that E have σ-finite $\mathcal{H}^{Q-p}$-measure; then the statement of the theorem still holds.

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Department of Mathematics
University of Bern
Sidlerstrasse 5
3012 Bern
Switzerland

thomas.zuercher@math-stat.unibe.ch