

# Dynamics of Symmetric Polynomial Endomorphisms of $\mathbf{C}^2$

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## 1. Introduction

We study dynamics of symmetric polynomial endomorphisms of  $\mathbf{C}^2$  and attempt to define an analogue of the Mandelbrot set in complex two-dimensional dynamics.

The Mandelbrot set  $M$  of the quadratic family  $p_c(z) = z^2 + c$  is defined by

$$M = \{c \in \mathbf{C} : \text{the orbit } \{p_c^n(0)\} \text{ is bounded}\}.$$

But here we consider quadratic polynomials of the form

$$q_c(z) = z^2 + cz, \tag{1.1}$$

since it is convenient to extend them to polynomial endomorphisms of  $\mathbf{C}^2$ . The connectedness locus of  $q_c(z)$  is known as the double Mandelbrot set (see [M]).

Ueda [Ue] classified 4 types of holomorphic maps of degree 2 in  $\mathbf{P}^2$ . The types are classified by an equivalence relation  $\sim$  defined by  $f \sim g$  if and only if there exist linear maps  $L_1$  and  $L_2$  satisfying

$$L_2^{-1} \circ f \circ L_1 = g.$$

Four types are written as

- (1)  $(x : y : z) \mapsto (x^2 : y^2 : z^2)$
- (2)  $(x : y : z) \mapsto (x^2 + yz : y^2 : z^2)$
- (3)  $(x : y : z) \mapsto (x^2 + cyz : y^2 + cxz : z^2)$
- (4)  $(x : y : z) \mapsto (x^2 + cxy + y^2 : z^2 + xy : yz)$ .

The dynamics of maps of type (1) is trivial. Maps of type (2) are polynomial skew products and their dynamics is studied in [J]. In this paper we begin a study of maps of type (3). When  $z \neq 0$ , they are maps on  $\mathbf{C}^2$  given by

$$f_c(x, y) = (x^2 + cy, y^2 + cx).$$

The maps  $f_c(x, y)$  restricted to the line  $(x = y)$  are the maps  $q_c(z)$ .

Let  $\Gamma_c$  be the critical set of  $f_c(x, y)$  in  $\mathbf{C}^2$ . We denote the set of points with bounded orbits of  $f_c(x, y)$  by  $K(f_c)$ . As in [FS2], we define the Mandelbrot set  $M$  of  $f_c$  by

$$M = \{c \in \mathbf{C} : K(f_c) \cap \Gamma_c \neq \emptyset\}.$$

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The critical set  $\Gamma_c$  of  $f_c(x, y)$  is an algebraic curve given by

$$4xy = c^2.$$

We parameterize this algebraic curve by a parameter  $t$ . That is, we represent  $\Gamma_c$  by

$$x = -\frac{c}{2}t, \quad y = -\frac{c}{2t}, \quad \text{where } t \in \mathbf{C} - \{0\}.$$

Our aim is to generate the Mandelbrot set of  $f_c(x, y)$  by a computer. One of the problems is how to check the condition

$$K(f_c) \cap \Gamma_c \neq \emptyset. \quad (1.2)$$

By symmetry (see Lemma 2.4), we know (see Proposition 3.2 and notes after it) that it is enough to check condition (1.2) for all parameters  $t$  satisfying

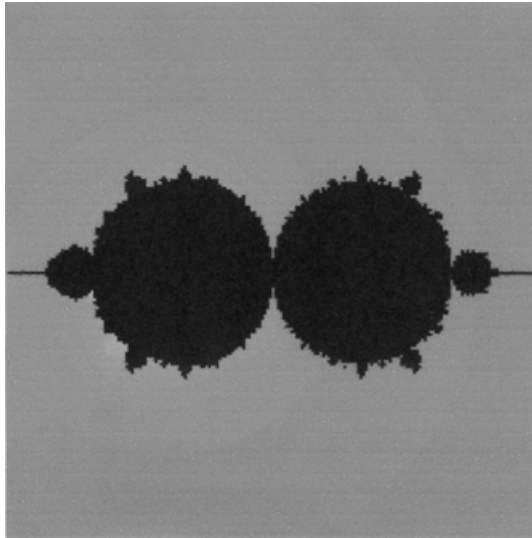
$$\frac{1}{6} \leq |t| \leq 1 \quad \text{and} \quad 0 \leq \arg t < \frac{\pi}{3}. \quad (1.3)$$

And it is also easily seen (see Lemma 3.1) that

$$\max\{|x|, |y|\} > |c| + 1 \implies \|f_c^n(x, y)\| \rightarrow \infty \quad (n \rightarrow \infty). \quad (1.4)$$

We therefore select about 150 values of the parameter  $t$ , including  $t = 1$ , that satisfy (1.3) and then check condition (1.4) for each point (supposedly for a finite number of iterates). Figure 1.1 consists of the values of  $c$  such that there exists at least one of our selected  $t$ -values satisfying that, for all  $n$  ( $1 \leq n \leq 50$ ),

$$|u_n|, |v_n| \leq |c| + 1, \quad \text{where } (u_n, v_n) := f_c^n\left(-\frac{c}{2}t, -\frac{c}{2t}\right).$$



**Figure 1.1** The Mandelbrot set of  $f_c(x, y)$

The figure consists of two clusters intersecting at  $c = 1$ . Both clusters are similar but the left part ( $\operatorname{Re} c \leq 1$ ) is a little bigger than the right part ( $\operatorname{Re} c \geq 1$ ). A surprising fact is that the right part looks like a half of the double Mandelbrot set. Compare Figure 1.1 with the double Mandelbrot set in [M]. One of our conjectures is that if  $\operatorname{Re} c \leq 1$  then the dynamics of  $f_c(x, y)$  is complex two-dimensional, whereas if  $\operatorname{Re} c \geq 1$  then the dynamics is essentially one-dimensional.

Uchimura [U3] studies the dynamics of  $f_c(x, y)$  restricted on  $\{x = \bar{y}\}$  when  $c$  is real. The function  $f_c$  admits an invariant plane  $(x = \bar{y})$  on which it acts as

$$g_c(x) = x^2 + c\bar{x}.$$

The map

$$g_c(z) = z^2 + c\bar{z}$$

has a connection with a physical model when  $c = -2$ ; it is Chebyshev map

$$g_{-2}(z) = z^2 - 2\bar{z}.$$

Lopes [L1; L2] considered the dynamics of  $g_{-2}$  as a special kind of Potts model and showed that triple point phase transition (three equilibrium states) exists. He conjectured that if  $c < -2$  then a Cantor set with expanding dynamics exists. For expanding systems, equilibrium states are unique.

Uchimura [U3] gave an affirmative answer to this conjecture of Lopes.

**THEOREM 1 [U3].**  *$K(g_c)$  is connected with the simply connected complement in  $\mathbf{P}^1$  if and only if  $-2 \leq c \leq 4$ .*

**THEOREM 2 [U3].** *If  $c < -2$ , then:*

- (1)  $K(g_c)$  is a Cantor set and expanding dynamics exists;
- (2) the two-dimensional Lebesgue measure of  $K(g_c)$  is 0;
- (3)  $g_c$  restricted to  $K(g_c)$  is topological conjugate to the shift on four symbols.

In this paper, we extend some of these results to polynomial endomorphisms of  $\mathbf{C}^2$ .

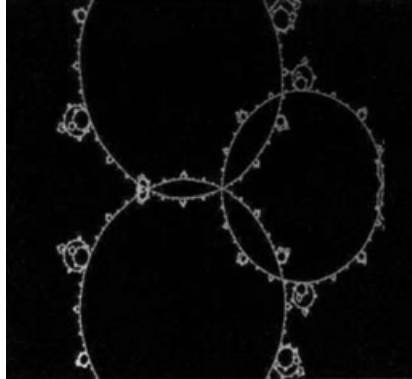
In Section 2 we study fixed points and periodic points of period 2 of  $f_c(x, y)$ . When  $\operatorname{Re} c \geq 1$ , the nonrepelling hyperbolic periodic points of period  $\leq 2$  are saddles, and their stable manifolds on  $(x = y)$  include the attractive basins of  $q_c(x)$ . When  $\operatorname{Re} c \leq 1$ , attractive periodic points exist and the domain's "fingers" appear in the  $c$ -plane.

In Section 3 we show that, if  $c < -2$  or  $c > 2 + 2\sqrt{2}$ , then  $K(f_c) \cap \Gamma_c = \emptyset$  and  $K(f_c)$  is a Cantor set that lies on the real plane  $(x = \bar{y})$ . Note that the double Mandelbrot set restricted on the real axis is the interval  $[-2, 4]$ . Then the dynamics of  $f_c(x, y)$  with  $c < -2$  correspond to the dynamics of  $q_c(z)$  with  $c < -2$ . We shall also show if  $-2 \leq c \leq 4$  then  $K(f_c) \cap \Gamma_c \neq \emptyset$ .

In Section 4, we consider J. Hubbard's conjectures concerning the dynamics of  $f_c(x, y)$ .

- (1) If  $c = 2$  then the saddle  $(-1, -1)$  is isolated.
- (2) Let  $U := \{c \in \mathbf{C} : \text{the set } K(f_c) \cap \Gamma_c \text{ consists of three points}\}$ . Then  $U$  is a nonempty set containing 2 in its interior.

We show that these conjectures can be answered in the affirmative. Assume that  $c$  is a complex number near 2. The saddle  $(-c + 1, -c + 1)$  is isolated in the non-wandering set. The set  $K(f_c) \cap \Gamma_c$  consists of three points and the forward orbits of those points approach the three saddles. Further, all periodic points other than those three saddles are repelling. This supports our observation that the right half of Figure 1.1 looks like a half of the double Mandelbrot set.



**Figure 1.2** The second Julia set of  $f_2$

S. Ushiki produced Figure 1.2, which consists of iterated preimages of a repelling fixed point  $(0, 0)$  under the map  $f_2(x, y)$ . Note that the exceptional set of  $f_2$  is empty (see [BCSh]). You can see three circles intersecting at  $(0, 0)$ . Uchimura [U4] shows that the second Julia set of  $f_2$  is connected.

Figure 1.2 looks like a Julia set in complex one-dimensional dynamics. Compare the second Julia sets in [DH]. Denker and Heinemann's paper [DH] shows that, in our setting, if  $c$  is near 0 then the second Julia set is homeomorphic to the torus.

## 2. Fixed Points and Periodic Points of Period 2

We consider the polynomial endomorphisms of  $\mathbf{C}^2$ ,

$$f_c(x, y) = (x^2 + cy, y^2 + cx).$$

We shall demonstrate some properties of fixed points and periodic points of period 2 of the map  $f_c$ , and we shall show the intimate relation between those periodic points of  $f_c$  and those of  $g_c$ .

The fixed points of  $f_c(x, y)$  may be listed as follows:

$$P_0 = (0, 0),$$

$$P_1 = (-c + 1, -c + 1),$$

$$P_2 = \left( \frac{1}{2}(1 + c - \sqrt{(c+1)(1-3c)}), \frac{1}{2}(1 + c + \sqrt{(c+1)(1-3c)}) \right),$$

$$P_3 = \left( \frac{1}{2}(1 + c + \sqrt{(c+1)(1-3c)}), \frac{1}{2}(1 + c - \sqrt{(c+1)(1-3c)}) \right).$$

**Table 2.1**

	Eigenvalues	Eigenvectors
$P_0$	$-c$	$(-1, 1)$
	$c$	$(1, 1)$
$P_1$	$-c + 2$	$(1, 1)$
	$-3c + 2$	$(-1, 1)$
$P_2$	$1 + c - \sqrt{1 - 2c - 2c^2}$	$(\sqrt{1 - 2c - 3c^2} + \sqrt{1 - 2c - 2c^2}, -c)$
	$1 + c + \sqrt{1 - 2c - 2c^2}$	$(\sqrt{1 - 2c - 3c^2} - \sqrt{1 - 2c - 2c^2}, -c)$
$P_3$	$1 + c - \sqrt{1 - 2c - 2c^2}$	$(-\sqrt{1 - 2c - 3c^2} + \sqrt{1 - 2c - 2c^2}, -c)$
	$1 + c + \sqrt{1 - 2c - 2c^2}$	$(-\sqrt{1 - 2c - 3c^2} - \sqrt{1 - 2c - 2c^2}, -c)$

Clearly,  $P_0$  and  $P_1$  lie on the diagonal ( $x = y$ ). This corresponds to 0 and  $-c + 1$  being fixed points of  $q_c$ . The eigenvalues and eigenvectors are listed in Table 2.1.

Next we show the periodic points of period exactly 2. There are twelve such points:

$$\begin{aligned}
 P_0 &= \left\{ -\frac{1}{2}(-c + 1) + \frac{1}{2}i\sqrt{3}(-c + 1), -\frac{1}{2}(-c + 1) - \frac{1}{2}i\sqrt{3}(-c + 1) \right\}, \\
 P_1 &= \left\{ -\frac{1}{2}(-c + 1) - \frac{1}{2}i\sqrt{3}(-c + 1), -\frac{1}{2}(-c + 1) + \frac{1}{2}i\sqrt{3}(-c + 1) \right\}, \\
 P_2 &= \left\{ \frac{1}{4}(-c - 1 - \sqrt{3(c + 1)(3c - 1)}) - \frac{1}{4}i(\sqrt{3}(-c - 1) + \sqrt{(c + 1)(3c - 1)}), \right. \\
 &\quad \left. \frac{1}{4}(-c - 1 - \sqrt{3(c + 1)(3c - 1)}) + \frac{1}{4}i(\sqrt{3}(-c - 1) + \sqrt{(c + 1)(3c - 1)}) \right\}, \\
 P_3 &= \left\{ \frac{1}{4}(-c - 1 + \sqrt{3(c + 1)(3c - 1)}) + \frac{1}{4}i(\sqrt{3}(-c - 1) - \sqrt{(c + 1)(3c - 1)}), \right. \\
 &\quad \left. \frac{1}{4}(-c - 1 + \sqrt{3(c + 1)(3c - 1)}) - \frac{1}{4}i(\sqrt{3}(-c - 1) - \sqrt{(c + 1)(3c - 1)}) \right\}, \\
 P_4 &= \left\{ \frac{1}{4}(-c - 1 + \sqrt{3(c + 1)(3c - 1)}) - \frac{1}{4}i(\sqrt{3}(-c - 1) - \sqrt{(c + 1)(3c - 1)}), \right. \\
 &\quad \left. \frac{1}{4}(-c - 1 + \sqrt{3(c + 1)(3c - 1)}) + \frac{1}{4}i(\sqrt{3}(-c - 1) - \sqrt{(c + 1)(3c - 1)}) \right\}, \\
 P_5 &= \left\{ \frac{1}{4}(-c - 1 - \sqrt{3(c + 1)(3c - 1)}) + \frac{1}{4}i(\sqrt{3}(-c - 1) + \sqrt{(c + 1)(3c - 1)}), \right. \\
 &\quad \left. \frac{1}{4}(-c - 1 - \sqrt{3(c + 1)(3c - 1)}) - \frac{1}{4}i(\sqrt{3}(-c - 1) + \sqrt{(c + 1)(3c - 1)}) \right\}, \\
 P_6 &= \left\{ \frac{1}{2}(-c - 1 - \sqrt{(c + 1)(c - 3)}), \frac{1}{2}(-c - 1 - \sqrt{(c + 1)(c - 3)}) \right\}, \\
 P_7 &= \left\{ -\frac{1}{4}(-c - 1 + \sqrt{(c + 1)(c - 3)}) + \frac{1}{4}i\sqrt{3}(-c - 1 + \sqrt{(c + 1)(c - 3)}), \right. \\
 &\quad \left. -\frac{1}{4}(-c - 1 + \sqrt{(c + 1)(c - 3)}) - \frac{1}{4}i\sqrt{3}(-c - 1 + \sqrt{(c + 1)(c - 3)}) \right\}, \\
 P_8 &= \left\{ -\frac{1}{4}(-c - 1 + \sqrt{(c + 1)(c - 3)}) - \frac{1}{4}i\sqrt{3}(-c - 1 + \sqrt{(c + 1)(c - 3)}), \right. \\
 &\quad \left. -\frac{1}{4}(-c - 1 + \sqrt{(c + 1)(c - 3)}) + \frac{1}{4}i\sqrt{3}(-c - 1 + \sqrt{(c + 1)(c - 3)}) \right\}, \\
 P_9 &= \left\{ \frac{1}{2}(-c - 1 + \sqrt{(c + 1)(c - 3)}), \frac{1}{2}(-c - 1 + \sqrt{(c + 1)(c - 3)}) \right\},
 \end{aligned}$$

**Table 2.2**

	Eigenvalues	Eigenvectors
$P_0$ & $P_1$	$(c - 2)^2$ $(3c - 2)^2$	$(\frac{1}{2}(-1 - \sqrt{3}i), 1)$ $(\frac{1}{2}(1 + \sqrt{3}i), 1)$
$P_2$ & $P_3$	$(1 + c - \sqrt{1 - 2c - 2c^2})^2$  $(1 + c + \sqrt{1 - 2c - 2c^2})^2$	$(2(-\sqrt{(c+1)(3c-1)} + c\sqrt{(c+1)(3c-1)} - i\sqrt{1 - 5c^2 - 6c^3 - 2c^4}), \sqrt{3}(c^2 + c) - i(c^2 + c))$  $(2(-\sqrt{(c+1)(3c-1)} + c\sqrt{(c+1)(3c-1)} + i\sqrt{1 - 5c^2 - 6c^3 - 2c^4}), \sqrt{3}(c^2 + c) - i(c^2 + c))$
$P_4$ & $P_5$	$(1 + c - \sqrt{1 - 2c - 2c^2})^2$  $(1 + c + \sqrt{1 - 2c - 2c^2})^2$	$(2\sqrt{(c+1)(3c-1)} - c\sqrt{(c+1)(3c-1)} - i\sqrt{1 - 5c^2 - 6c^3 - 2c^4}, \sqrt{3}(c^2 + c) - i(c^2 + c))$  $(2\sqrt{(c+1)(3c-1)} - c\sqrt{(c+1)(3c-1)} + i\sqrt{1 - 5c^2 - 6c^3 - 2c^4}, \sqrt{3}(c^2 + c) - i(c^2 + c))$
$P_6$ & $P_9$	$-c^2 + 2c + 4$ $3c^2 + 6c + 4$	$(1, 1)$ $(-1, 1)$
$P_7$ & $P_{11}$	$-c^2 + 2c + 4$ $3c^2 + 6c + 4$	$(\frac{1}{2}(-1 + \sqrt{3}i), 1)$ $(\frac{1}{2}(1 - \sqrt{3}i), 1)$
$P_8$ & $P_{10}$	$-c^2 + 2c + 4$ $3c^2 + 6c + 4$	$(\frac{1}{2}(-1 - \sqrt{3}i), 1)$ $(\frac{1}{2}(1 + \sqrt{3}i), 1)$

$$P_{10} = \left\{ -\frac{1}{4}(-c - 1 - \sqrt{(c+1)(c-3)}) + \frac{1}{4}i\sqrt{3}(-c - 1 - \sqrt{(c+1)(c-3)}), \right. \\ \left. -\frac{1}{4}(-c - 1 - \sqrt{(c+1)(c-3)}) - \frac{1}{4}i\sqrt{3}(-c - 1 - \sqrt{(c+1)(c-3)}) \right\},$$

$$P_{11} = \left\{ -\frac{1}{4}(-c - 1 - \sqrt{(c+1)(c-3)}) - \frac{1}{4}i\sqrt{3}(-c - 1 - \sqrt{(c+1)(c-3)}), \right. \\ \left. -\frac{1}{4}(-c - 1 - \sqrt{(c+1)(c-3)}) + \frac{1}{4}i\sqrt{3}(-c - 1 - \sqrt{(c+1)(c-3)}) \right\}.$$

The periodic points  $P_6$  and  $P_9$  lie on the plane  $(x = y)$ .

We obtain these periodic points of period 2 as follows. In [U2] we find twelve periodic points of

$$g_c(z) = z^2 + c\bar{z}.$$

Let  $u + iv \in \mathbf{C}$  be a periodic point of period 2 of  $g_c(z)$ . Then, we construct a new point  $P = (u + iv, u - iv)$  in  $\mathbf{C}^2$  and verify that  $f_c^2(P) = P$ . The eigenvalues and eigenvectors of the periodic points can be computed directly and are listed in Table 2.2.

We divide the parameter space of the map  $f_c$  into two parts,  $\{\operatorname{Re} c > 1\}$  and  $\{\operatorname{Re} c \leq 1\}$ . We show that the dynamics of  $f_c$  with  $\operatorname{Re} c > 1$  differs from the dynamics of  $f_c$  with  $\operatorname{Re} c \leq 1$ . First we consider the case when  $\operatorname{Re} c > 1$ .

LEMMA 2.1. *If  $\operatorname{Re} c > 1$ , then the following inequalities hold:*

$$\begin{aligned} |3c - 2| > 1, \quad |1 + c - \sqrt{1 - 2c - 2c^2}| > 1, \\ |1 + c + \sqrt{1 - 2c - 2c^2}| > 1, \quad |3c^2 + 6c + 4| > 1. \end{aligned}$$

The proof is obtained by direct calculation.

PROPOSITION 2.2. *Assume that  $\operatorname{Re} c > 1$ . Then the following statements hold.*

- (1) *All the fixed points of  $f_c$  are repelling except for  $P_1$ . The fixed point  $P_1$  is a saddle if and only if  $|c - 2| < 1$ .*
- (2) *All the periodic points of period exactly 2 are repelling except for  $P_0$  and  $P_1$ ,  $P_6$  and  $P_9$ ,  $P_7$  and  $P_{11}$ , and  $P_8$  and  $P_{10}$ . The periodic points  $P_0$  and  $P_1$  are saddle if and only if  $|c - 2| < 1$ . The periodic points  $P_6$  and  $P_9$ ,  $P_7$  and  $P_{11}$ , and  $P_8$  and  $P_{10}$  are saddles if and only if  $|-c^2 + 2c + 4| < 1$ .*
- (3) *When  $|c - 2| < 1$ , a local stable manifold of the saddle fixed point  $P_1$  lies on the line  $(x = y)$ . (The map  $f_c(x, y)$  restricted on the line  $(x = y)$  is the map  $q_c(x)$ .) The local stable manifold of  $P_1$  lies on the line  $(x = y)$  and is included in the attractive basin of the fixed point  $-c + 1$  of the one-dimensional map  $q_c(x)$ . A similar statement is true for the periodic points  $P_6$  and  $P_9$ .*

The proofs of (1) and (2) follow from the eigenvalues of periodic points and Lemma 2.1. The proof of (3) follows from the eigenvectors of periodic points.



Figure 2.1 The circular domains

The region  $\{|c - 2| < 1\}$  is a unit disk, and Figure 2.1 depicts the circular domains  $\{|-c^2 + 2c + 4| < 1\}$ . We consider the case when  $\operatorname{Re} c \leq 1$ . In this case, there exist attractive periodic points.

PROPOSITION 2.3. *Assume that  $\operatorname{Re} c \leq 1$ . Then the following statements hold.*

- (1) *The fixed point  $P_0$  is attractive if and only if  $|c| < 1$ . The fixed points  $P_2$  and  $P_3$  are attractive if and only if  $c$  lies in the domains*

$$\begin{aligned} D_{1,2} = \{c \in \mathbb{C} : |1 + c - \sqrt{1 - 2c - 2c^2}| < 1 \text{ and} \\ |1 + c + \sqrt{1 - 2c - 2c^2}| < 1\}. \end{aligned}$$

*There are no other attractive fixed points.*

(2) *The periodic points of period 2 ( $P_2$  and  $P_3$ ,  $P_4$  and  $P_5$ ) are attractive if and only if  $c$  lies in the domains  $D_{1,2}$ . The periodic points  $P_6$  and  $P_9$ ,  $P_7$  and  $P_{11}$ , and  $P_8$  and  $P_{10}$  are attractive if and only if  $c$  lies in the domains*

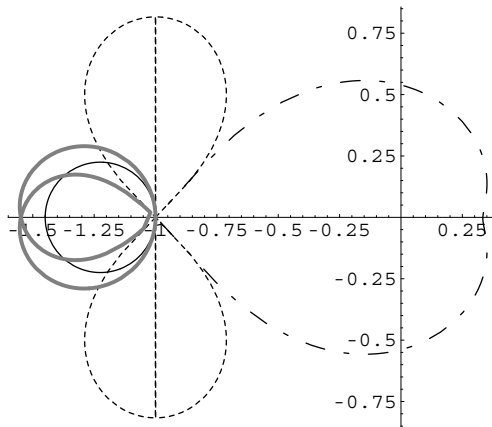
$$D_{2,1} = \{c \in \mathbf{C} : |-c^2 + 2c + 4| < 1 \text{ and } |3c^2 + 6c + 4| < 1\}.$$

*There are no other attractive periodic points of period exactly 2.*

Next we explain Figure 2.2, beginning with an explanation of the domains  $D_{1,2}$ . Set

$$\varphi(z) = 1 + z - \sqrt{1 - 2z - 2z^2},$$

and let  $\mathbf{D}$  be the unit disk. Then it can be easily proved that  $\varphi^{-1}(\mathbf{D}) \rightarrow \mathbf{D}$  is a 2-fold branch covering and that  $\varphi^{-1}(\mathbf{D})$  consists of two components. One is a domain bounded by a dot-dashed curve in Figure 2.2, and the other is a domain bounded by an outer thick gray curve. A branch point lies in the latter component. Hence it can be easily seen that  $D_{1,2}$  is equal to the domain bounded by an inner thick gray curve. If  $c$  is in  $D_{1,2}$  then periodic points of period 2,  $P_2$  and  $P_3$  as well as  $P_4$  and  $P_5$ , are attractive. This can be seen from the following lemma.



**Figure 2.2** Points  $c$  for which  $f_c(x, y)$  has an attracting periodic orbit

LEMMA 2.4. *Let  $(x_n, y_n) := f^n(x, y)$ .*

- (1) *If  $n$  is even, then  $f^n(\omega x, \omega^2 y) = (\omega x_n, \omega^2 y_n)$ .*
- (2) *If  $n$  is odd, then  $f^n(\omega x, \omega^2 y) = (\omega^2 x_n, \omega y_n)$ .*

*Here  $\omega$  is a cubic root of unity.*

The proof is obvious.

In Figure 2.2, we see that the domains

$$\{c \in \mathbf{C} : |3c^2 + 6c + 4| < 1\}$$



are the sets bounded by the dotted curves. The domain bounded by the black curve is one of the components of  $\{c \in \mathbf{C} : |-c^2 + 2c + 4| < 1\}$ . From these observations, you can find domains  $D_{2,1}$  in Figure 2.2.

REMARK 2.5. In Figure 2.2 we can see three fingers (Hubbard’s terminology) intersecting at  $c = -1$ . One finger is a domain bounded by an inner thick gray curve and other fingers are domains bounded by a black curve and dotted curves. The inner thick gray curves and the dotted curves have two common tangents. The slopes of the tangents are  $-1$  and  $1$ , and the outer thick gray curve is tangent to the vertical line through the point  $c = -1$ .

When  $c = -4/3$ ,  $c$  is in  $D_{1,2}$  and so there are two attractive fixed points and four attractive periodic points of period 2. They all lie in the plane  $(x = \bar{y})$ . The dynamics of  $f_{-4/3}(x, y)$  restricted on the plane  $(x = \bar{y})$  is studied in [U2].

### 3. Critical Sets and $K$ Sets

In this section we show that if  $c < -2$  then the set of points with bounded forward orbits for  $f_c(x, y)$  is a Cantor set. Here  $f_c(x, y) = (x^2 + cy, y^2 + cx)$ . Let

$$K(f_c) = \{(x, y) \in \mathbf{C}^2 : \{f_c^n(x, y) : n = 1, 2, \dots\} \text{ is bounded}\}.$$

We shall characterize the set of points with bounded orbits  $K(f_c)$ . Let  $\Gamma_c$  be the critical set of  $f_c(x, y)$  in  $\mathbf{C}^2$ . Our aim is to show that

$$c < -2 \vee c > 2 + 2\sqrt{2} \implies K(f_c) \cap \Gamma_c = \emptyset$$

and

$$-2 \leq c \leq 4 \implies K(f_c) \cap \Gamma_c \neq \emptyset.$$

Set  $(x_n, y_n) = f_c^n(x_0, y_0)$ . Then we have the following lemma.

LEMMA 3.1. *If  $\max\{|x_0|, |y_0|\} > |c| + 1$  and  $\lambda = \max\{|x_0|, |y_0|\} - |c|$ , then*

$$\max\{|x_n|, |y_n|\} > (|c| + 1)\lambda^n.$$

*Furthermore, when  $c \neq 0$ , there exist subsequences  $\{|x_{n_j}|\}$  and  $\{|y_{n_k}|\}$  such that  $|x_{n_j}| \rightarrow \infty$  and  $|y_{n_k}| \rightarrow \infty$ .*

*Proof.* We prove the inequality by induction on  $n$ . Assume without loss of generality that  $\max\{|x_0|, |y_0|\} = |x_0|$ . Then

$$|x_1| \geq |x_0|^2 - |c||y_0| \geq |x_0|^2 - |c||x_0| > (|c| + 1)\lambda.$$

Next we assume that  $|x_n| = \max\{|x_n|, |y_n|\} > (|c| + 1)\lambda^n$ . Then

$$|x_{n+1}| \geq |x_n|(|x_n| - |c|) \geq \lambda|x_n|.$$

Suppose that  $c \neq 0$  and that the set  $\{|x_n|\}$  is bounded. Consider the inequality

$$|x_{n+1}| \geq |c||y_n| - |x_n|^2.$$

Since

$$|y_n| \rightarrow \infty,$$

the right-hand side of the inequality tends to infinity—a contradiction.  $\square$

The critical set of the map  $f_c$  is an algebraic curve  $xy = \frac{c^2}{4}$ . We parameterize the curve by a complex variable  $t$ :

$$x = -\frac{c}{2}t, \quad y = -\frac{c}{2t}.$$

We denote the curve by  $\Gamma_c$ ; that is,

$$\Gamma_c = \left\{ \left( -\frac{c}{2}t, -\frac{c}{2t} \right) : t \in \mathbf{C} - \{0\} \right\}.$$

We say that  $f_c^n(\Gamma_c)$  tends to infinity, denoted by  $f_c^n(\Gamma_c) \rightarrow \infty$  ( $n \rightarrow \infty$ ), if

$$\left\| f_c^n \left( -\frac{c}{2}t, -\frac{c}{2t} \right) \right\| \rightarrow \infty$$

for any point  $(-\frac{c}{2}t, -\frac{c}{2t}) \in \Gamma_c$  with respect to the Euclidean norm.

Denote

$$f_c^n \left( -\frac{c}{2}t, -\frac{c}{2t} \right) = (u_n(t), v_n(t)).$$

Clearly,  $u_n(t)$  and  $v_n(t)$  are rational functions of  $t$  and satisfy

$$v_n(t) = u_n \left( \frac{1}{t} \right). \tag{3.1}$$

Set  $D_\delta = \{t \in \mathbf{C} : \delta \leq |t| \leq 1\}$  for  $\delta > 0$ .

**PROPOSITION 3.2.** *If  $|u_n(t)| \rightarrow \infty$  ( $n \rightarrow \infty$ ) for any  $t$  in  $D_\delta$  with  $\delta = |c|/(2|c| + 1)$ , then the set  $f_c^n(\Gamma)$  tends to infinity.*

*Proof.* We assume that  $|u_n(t)| \rightarrow \infty$  for  $t \in D_\delta$ . Then

$$\left\| f_c^n \left( -\frac{c}{2}t, -\frac{c}{2t} \right) \right\| \rightarrow \infty.$$

If  $|t| < \delta$ , then

$$\left| \frac{c}{2t} \right| > |c| + 1$$

and so, by Proposition 3.1, it follows that

$$\left\| f_c^n \left( -\frac{c}{2}t, -\frac{c}{2t} \right) \right\| \rightarrow \infty.$$

If  $|t| > 1$  then from (3.1) we obtain

$$v_n(t) = u_n \left( \frac{1}{t} \right) \rightarrow \infty.$$

Hence  $\|f_c^n(-\frac{c}{2}t, -\frac{c}{2t})\| \rightarrow \infty$ .  $\square$

Here we explain why it is enough to check condition (1.2) only for those  $t$  satisfying  $\frac{1}{6} \leq |t| \leq 1$ .

If  $|c| < 1$ , then the one-dimensional map  $q_c(x) = x^2 + cx$  has an attractive fixed point 0 that attracts a critical point  $-\frac{c}{2}$ . We therefore set  $t = 1$ . Then

$$f_c^n\left(-\frac{c}{2}, -\frac{c}{2}\right) \rightarrow (0, 0)$$

and hence  $K(f_c) \cap \Gamma_c \neq \emptyset$ . Thus we need only consider the case  $|c| \geq 1$ . Observe that

$$f_c\left(-\frac{c}{2}t, -\frac{c}{2}\frac{1}{t}\right) = \left(\frac{c^2}{4}\left(t^2 - \frac{2}{t}\right), \frac{c^2}{4}\left(\frac{1}{t^2} - 2t\right)\right).$$

If  $|t| < \frac{1}{6}$ , then

$$\left|\frac{c^2}{4}\left(\frac{1}{t^2} - 2t\right)\right| > |c| + 1.$$

Hence,

$$\left\|f_c^n\left(-\frac{c}{2}t, -\frac{c}{2}\frac{1}{t}\right)\right\| \rightarrow \infty.$$

Next we shall prove that if  $c < -2$  then  $u_n(t)$  has no zeros in the set

$$D^* = \{t \in \mathbf{C} : 0 < |t| \leq 1\}.$$

The rational function  $u_n(t)$  is holomorphic in  $D^*$  and has a pole at  $t = 0$ . To prove our claim, we use the argument principle. Let  $W(\gamma, p)$  be the winding number of a closed curve  $\gamma$  around a point  $p$ . We denote the unit circle  $\{e^{i\theta} : 0 \leq \theta < 2\pi\}$  by  $S^1$ . We study the winding numbers  $W(u_n(S^1), 0)$ . Let  $g_c(z) = z^2 + c\bar{z}$ , and set

$$-\frac{c}{2}S^1 = \left\{-\frac{c}{2}e^{i\theta} : 0 \leq \theta < 2\pi\right\}.$$

Although the relation from  $u_n(t)$  to  $u_{n+1}(t)$  is not a mapping, we can use the mapping  $g_c^n$  to study  $u_n(e^{i\theta})$ .

**PROPOSITION 3.3.** *If  $c$  is real, then*

$$W(u_n(S^1), 0) = W\left(g_c^n\left(-\frac{c}{2}S^1\right), 0\right).$$

*Proof.* We claim that

$$u_n(e^{i\theta}) = g_c^n\left(-\frac{c}{2}e^{i\theta}\right). \tag{3.2}$$

Indeed, by definition we have

$$f_c^n\left(-\frac{c}{2}e^{i\theta}, -\frac{c}{2}e^{-\theta i}\right) = (u_n(e^{i\theta}), v_n(e^{i\theta})).$$

Hence, from (3.1),

$$\begin{aligned} u_{n+1}(e^{\theta i}) &= (u_n(e^{\theta i}))^2 + cv_n(e^{\theta i}) \\ &= (u_n(e^{\theta i}))^2 + cu_n(e^{-\theta i}) \\ &= (u_n(e^{\theta i}))^2 + c\bar{u}_n(e^{\theta i}). \end{aligned}$$

Then (3.2) follows by induction. □

PROPOSITION 3.4. *If  $c < -2$ , then*

$$W\left(g_c^n\left(-\frac{c}{2}S^1\right), 0\right) = -2^{n-1}.$$

*Proof.* We use  $\Delta$  to denote the closed domain bounded by Steiner’s hypocycloid:

$$\{e^{2\theta i} - 2e^{-\theta i} : 0 \leq \theta < 2\pi\}.$$

Then

$$g_c\left(-\frac{c}{2}S^1\right) = \frac{c^2}{4}\partial\Delta \quad \text{and} \quad W\left(g_c\left(-\frac{c}{2}S^1\right), 0\right) = -1$$

(see [U3]).

Next we consider the case when  $n \geq 2$ . It is known that  $g_c((-\frac{c}{2})\Delta) = \frac{c^2}{4}\Delta$ . In [U3, Prop. 2.4] it is shown that the map  $g_c$  from  $\mathbf{C} - (-\frac{c}{2})\Delta$  to  $\mathbf{C} - \frac{c^2}{4}\Delta$  is a 2-fold covering mapping. Let  $\gamma$  be a simple path in  $\mathbf{C} - (-\frac{c}{2})\Delta$  whose initial point lies on the negative real axis, whose final point lies on the positive real axis, and whose intermediate points lie in the upper half-plane. Then we can prove that  $W(g_c(\gamma), 0) = -1$ . The same holds for a path in  $\mathbf{C} - (-\frac{c}{2})\Delta$  whose intermediate points lie in the lower half-plane. Then we have

$$W\left(g_c^2\left(-\frac{c}{2}S^1\right), 0\right) = -2.$$

By a similar method we can prove

$$W\left(g_c^n\left(-\frac{c}{2}S^1\right), 0\right) = -2^{n-1}. \quad \square$$

From Propositions 3.3 and 3.4 it follows that

$$W(u_n(S^1), 0) = -2^{n-1} \quad \text{when } c < -2.$$

We have also shown that  $u_n(t)$  does not have any zeros on the unit circle  $S^1$ .

Next we consider the multiplicity of the pole of  $u_n(t)$  at  $t = 0$ .

PROPOSITION 3.5. *The multiplicity of the pole of  $u_n(t)$  at  $t = 0$  is at most  $2^{n-1}$ .*

*Proof.* It suffices to show that

$$u_n(t) = a(2^n)t^{2^n} + \dots + a_0 + a(-1)t^{-1} + \dots + a(-2^{n-1})t^{-2^{n-1}},$$

where  $a(k)$  is a polynomial in the variable  $c$  with  $-2^{n-1} \leq k \leq 2^n$ . The equality can be proved by induction on  $n$ , because

$$u_{n+1}(t) = u_n(t)^2 + cv_n(t) = u_n(t)^2 + cu_n\left(\frac{1}{t}\right). \quad \square$$

We can apply the argument principle to the function  $u_n(t)$  on the unit circle  $S^1$ :

$$W(u_n(S^1), 0) = N - M,$$

where  $N$  is the number of zeros in the unit disk  $\mathbf{D}$  and  $M$  is the number of poles in  $\mathbf{D}$ .

Combining Propositions 3.3, 3.4, and 3.5 yields the following proposition.

PROPOSITION 3.6. *If  $c < -2$ , then  $u_n(t)$  has no zeros in the unit disk  $\mathbf{D}$ .*

Next we will show that if  $c < -2$  then  $f_c^n(\Gamma_c) \rightarrow \infty$ . It is enough to show that if  $c < -2$  then

$$|u_n(t)| \rightarrow \infty \quad \text{for any } t \in D_\delta$$

with a small positive number  $\delta$ . The functions  $u_n(t)$  in  $D_\delta$  are holomorphic and have no zeros. Therefore,  $|u_n(t)|$  has its minimum value on the boundary  $\partial D_\delta$ . On the boundary  $\{|t| = \delta\}$ ,  $|u_n(t)|$  is large. Hence  $|u_n(t)|$  takes its minimum value on the unit circle  $\{|t| = 1\}$ . Then proving our initial statement requires only proving that if  $c < -2$  then

$$u_n(e^{i\theta}) \rightarrow \infty \text{ as } n \rightarrow \infty, \quad \text{with } 0 \leq \theta < 2\pi.$$

Lemma 4.1 in [U3] tells us that if  $c < -2$  then

$$g_c^n\left(-\frac{c}{2}e^{i\theta}\right) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Hence, by (3.2), we conclude that if  $c < -2$  then

$$u_n(e^{i\theta}) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

We have thus proved that if  $c < -2$  then

$$f_c^n(\Gamma_c) \rightarrow \infty.$$

This is equivalent to the following result.

PROPOSITION 3.7. *If  $c < -2$ , then*

$$K(f_c) \cap \Gamma_c = \emptyset.$$

Theorem 3.8 of [FS2] reads as follows. Let  $\varphi$  be a polynomial map from  $\mathbf{C}^k$  to  $\mathbf{C}^k$  that extends as a holomorphic map into  $\mathbf{P}^k$ , and let  $\Gamma$  be the critical set for  $\varphi$ . Assume  $K(\varphi) \cap \Gamma = \emptyset$ . Then the following statements hold.

- (1) The map  $\varphi$  is strictly expanding on  $K(\varphi)$ .
- (2) Repelling periodic points are dense in  $K(\varphi)$ .
- (3)  $K(\varphi) = \text{support } \mu$ , where  $\mu$  is an invariant measure of maximal entropy.

Since the map  $f_c$  satisfies conditions (1)–(3), those assertions hold for  $f_c$ .

Based on these results, we show that if  $c < -2$  then  $K(f_c)$  is a Cantor set and lies on the real plane ( $x = \bar{y}$ ).

We begin with the study of the periodic points. From [FS1, Cor. 3.2] we know that the number of periodic points of period  $n$  of  $f_c$  is  $4^n$ .

LEMMA 3.8. *If  $c < -2$ , then there exist  $4^n$  periodic points of period  $n$  of the function  $g_c(z) = z^2 + c\bar{z}$ .*

*Proof.* By [U3, Prop. 2.2] we know that (a) the map  $g_c(z)$  restricted to  $(-\frac{c}{2})\Delta$ ,

$$g_c|_{(-\frac{c}{2})\Delta} : \left(-\frac{c}{2}\right)\Delta \rightarrow \frac{c^2}{4}\Delta,$$

is a 4-fold covering map such that  $(-\frac{c}{2})\Delta = \Delta_0 \cup \Delta_1 \cup \Delta_2 \cup \Delta_3$  and (b)  $g_c|_{\Delta_k}$  is a homeomorphism from  $\Delta_k$  onto  $\frac{c^2}{4}\Delta$  for any  $k = 0, 1, 2, 3$ . Then there exist inverse maps  $\varphi_k, k = 0, 1, 2, 3$ , from  $\frac{c^2}{4}\Delta$  to  $\Delta_k$  such that the composition  $g_c \circ \varphi_k$  is an identity map. Hence

$$\varphi_k\left(\frac{c^2}{4}\Delta\right) = \Delta_k \subset \left(-\frac{c}{2}\right)\Delta \subset \frac{c^2}{4}\Delta.$$

Clearly  $\frac{c^2}{4}\Delta$  is a closed topological disk. Applying the fixed point theorem to  $\varphi_k$  yields a fixed point  $p_k$  in  $\Delta_k$ , and then  $g_c(p_k) = p_k$ . Hence we have four fixed points of  $g_c$ . By a similar argument, we can prove this lemma when  $n > 1$ .  $\square$

From Lemma 3.8 we obtain the following proposition.

**PROPOSITION 3.9.** *If  $c < -2$ , then any periodic point of  $f_c(x, y)$  lies on the plane  $(x = \bar{y})$  and belongs to the set  $K(g_c)$ .*

Uchimura [U3, Thm. 5.1] shows that if  $c < -2$  then  $K(g_c)$  is a Cantor set. Hence Fornæss and Sibony’s statements (1) and (2), together with Proposition 3.9, allow us to conclude as follows.

**THEOREM 3.10.** *If  $c < -2$ , then:*

- (1)  $K(f_c) = \text{support } \mu \subset (x = \bar{y})$ ;
- (2)  $K(f_c) = K(g_c)$  is a Cantor set.

*Proof.* We have

$$\begin{aligned} K(f_c) &\subset \overline{\{\text{repelling periodic points of } f_c\}} \\ &\subset \overline{\{\text{periodic points of } f_c\}} \subset K(g_c) \subset K(f_c). \end{aligned} \quad \square$$

Next, we study the dynamics of  $f_c$  when  $c > 2 + 2\sqrt{2}$ .

**THEOREM 3.11.** *If  $c > 2 + 2\sqrt{2}$ , then:*

- (1)  $K(f_c) \cap \Gamma_c = \emptyset$ ;
- (2)  $K(f_c) = \text{support } \mu \subset (x = \bar{y})$ ;
- (3)  $K(f_c) = K(g_c)$  is a Cantor set.

The proof of this theorem is similar to that of Proposition 3.7 and Theorem 3.10. However, we need two additional lemmas to prove the result that is similar to Proposition 3.7.

**LEMMA 3.12** [U3, Prop. 4.1]. *If  $c > 2 + 2\sqrt{2}$  then, for any point  $z \in (g_c)^n(-\frac{c}{2}S^1)$  with  $n \geq 1$ , we have  $|z| > |c| + 1$ .*

**LEMMA 3.13.** *Assume that  $c > 0$ . Set*

$$A_j = \left\{ z \in \mathbf{C} : |z| \geq |c| + 1, \frac{j-1}{3}\pi \leq \arg z \leq \frac{j}{3}\pi \right\}, \quad j = 1, 2, \dots, 6.$$

Then

$$g_c(A_j) \subseteq \begin{cases} A_{2j-1} \cup A_{2j} & \text{if } 1 \leq j \leq 3, \\ A_{2(j-3)-1} \cup A_{2(j-3)} & \text{if } 4 \leq j \leq 6. \end{cases}$$

Furthermore, if  $r \geq |c| + 1$  then

$$\arg\left(g_c\left(r \exp\left(\frac{k\pi i}{3}\right)\right)\right) = \frac{2k\pi}{3}, \quad k = 0, 1, \dots, 5.$$

From these two lemmas it follows that, for  $n \geq 1$ ,

$$c > 2 + 2\sqrt{2} \implies W\left(g_c^n\left(-\frac{c}{2}S^1\right), 0\right) = -2^{n-1}.$$

Then, by an argument similar to that used in the proof of Proposition 3.7, we can prove Theorem 3.11.

Finally, we consider the relation between  $K(f_c)$  and  $\Gamma_c$  when  $-2 \leq c \leq 4$ .

PROPOSITION 3.14. *If  $-2 \leq c \leq 4$ , then*

$$K(f_c) \cap \Gamma_c \neq \emptyset.$$

*Proof.*

Case 1:  $-2 \leq c \leq \frac{2}{3}$ . From [U3, Prop. 3.2] we have

$$g_c^n\left(\frac{c^2}{4}\Delta\right) \subset D(0, |c| + 1) \quad \text{for any } n \in \mathbb{N},$$

where  $D(0, |c| + 1) = \{z \in \mathbb{C} : |z| < |c| + 1\}$ . Since

$$g_c\left(-\frac{c}{2}S^1\right) = \partial\frac{c^2}{4}\Delta \subset \frac{c^2}{4}\Delta, \tag{3.3}$$

it follows that

$$g_c^m(\Gamma_c \cap (x = \bar{y})) = g_c^m\left(-\frac{c}{2}S^1\right) \subset D(0, |c| + 1).$$

Then

$$K(g_c) \cap (\Gamma_c \cap (x = \bar{y})) \neq \emptyset$$

and so

$$K(f_c) \cap \Gamma_c \neq \emptyset.$$

Case 2:  $\frac{2}{3} \leq c \leq 4$ . Clearly the point  $(-\frac{c}{2}, -\frac{c}{2})$  belongs to the set  $\Gamma_c$  and lies on the line  $(x = y)$ , so we consider the function  $q_c(x) = x^2 + cx$ . Since  $q_c([-c, 0]) \subset [-c, 0]$ , the orbit  $\{q_c^n(-\frac{c}{2}) : n = 0, 1, \dots\}$  is bounded.  $\square$

We have shown that

$$c > 2 + 2\sqrt{2} \vee c < -2 \implies K(f_c) \cap \Gamma_c = \emptyset$$

and

$$-2 \leq c \leq 4 \implies K(f_c) \cap \Gamma_c \neq \emptyset.$$

When  $c < -2$ , the dynamics of  $f_c(x, y)$  is analogous to that of  $p_c(z)$ .

#### 4. Hubbard's Conjectures

In this section, we consider J. Hubbard's conjectures concerning the dynamics of  $f_c(x, y)$ .

- (1) In the case  $c = 2$ , the saddle  $(-1, -1)$  is isolated.
- (2) Let  $U := \{c \in \mathbf{C} : \text{the set } K(f_c) \cap \Gamma_c \text{ consists of three points}\}$ . Then  $U$  is a nonempty set containing 2 in its interior.

We give affirmative answers to these conjectures. Let

$$D(z_0, r) = \{z \in \mathbf{C} : |z_0 - z| < r\}.$$

**THEOREM 4.1.** *There exists a small positive number  $\delta$  such that, for any  $c \in D(2, \delta)$ , a saddle  $(-c + 1, -c + 1)$  is isolated in the non-wandering set of  $f_c$ .*

**THEOREM 4.2.** *There exists a small positive number  $\delta$  such that, if  $c = 2 + \varepsilon \in D(2, \delta)$ , then*

$$\begin{aligned} &K(f_c) \cap \Gamma_c \\ &= \left\{ \left(-1 - \frac{\varepsilon}{2}, -1 - \frac{\varepsilon}{2}\right), \left(\left(-1 - \frac{\varepsilon}{2}\right)\omega, \left(-1 - \frac{\varepsilon}{2}\right)\omega^2\right), \left(\left(-1 - \frac{\varepsilon}{2}\right)\omega^2, \left(-1 - \frac{\varepsilon}{2}\right)\omega\right) \right\}, \end{aligned}$$

where  $\omega$  is a cubic root of unity.

Theorems 4.1 and 4.2 imply the following.

**THEOREM 4.3.** *There exists a small positive number  $\delta$  such that, if  $c \in D(2, \delta)$ , then any periodic point other than the three saddles*

$$(-c + 1, -c + 1), (\omega(-c + 1), \omega^2(-c + 1)), (\omega^2(-c + 1), \omega(-c + 1))$$

*is repelling.*

We first prove Theorem 4.1 and then use its proof to show Theorem 4.2. Given both theorems, we then prove Theorem 4.3.

Theorem 4.2 tells us that, when  $c$  is near 2, the set  $K(f_c) \cap \Gamma_c$  consists essentially of the critical point  $-\frac{c}{2}$  of  $q_c(z)$  in (1.1). See Lemma 2.4. Hence the dynamics of  $f_c(x, y)$  is similar to that of the function  $q_c(z)$ .

In Proposition 2.2, we showed that the fixed point  $P_1 = (-c + 1, -c + 1)$  is a saddle when  $|c - 2| < 1$ . Its eigenvalues are  $-c + 2$  and  $-3c + 2$  and its respective eigenvectors are  $(1, 1)$  and  $(-1, 1)$ . In the special case  $c = 2$  we have that  $f_2$  restricted to  $(x = y)$  is  $q_2(x) = x^2 + 2x$ . Then the stable manifold of  $(-1, -1)$  includes the attractive basin of  $q_2$  at the fixed point  $-1$ . Clearly  $q_2(x)$  is conjugate to  $p_0(z) = z^2$ . Hence the attractive basin of  $q_2(x)$  is a unit disk  $|x + 1| < 1$  and repelling periodic points are dense on the boundary. The eigenvalues of  $Df_2(z, z)$  are  $2z + 2$  and  $2z - 2$ , and their respective eigenvectors are  $(1, 1)$  and  $(-1, 1)$ . Therefore, in  $\mathbf{C}^2$ , the stable manifold of the saddle  $(-1, -1)$  is a unit disk on  $(x = y)$ .

The iterated preimages of a repelling fixed point  $(0, 0)$  under the map  $f_2(x, y)$  are depicted in Figure 1.2.



We begin with the proof of Theorem 4.1. We change the coordinate system  $(x, y)$  into another coordinate system  $(\xi, \eta)$  so that the saddle  $(-c + 1, -c + 1)$  becomes the origin  $(0, 0)$  in the  $(\xi, \eta)$  coordinate; then  $\xi$  and  $\eta$  correspond to the directions of the eigenvectors  $(1, 1)$  and  $(-1, 1)$ . We use the abbreviation  $f$  for  $f_c$  in this section.

Set  $c = 2 + \varepsilon$  with  $|\varepsilon| < \delta$ . Set

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \varphi \begin{pmatrix} x \\ y \end{pmatrix},$$

where

$$\varphi \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x + 1 + \varepsilon \\ y + 1 + \varepsilon \end{pmatrix}. \tag{4.1}$$

Then

$$\varphi \circ f \circ \varphi^{-1} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\xi^2 + \eta^2) - \varepsilon\xi \\ \sqrt{2}\xi\eta - 4\eta - 3\varepsilon\eta \end{pmatrix}.$$

Thus we define the endomorphism  $k$  on  $\mathbf{C}^2$  in this coordinate by

$$k(\xi, \eta) = \begin{pmatrix} \frac{1}{\sqrt{2}}(\xi^2 + \eta^2) - \varepsilon\xi, \sqrt{2}\xi\eta - 4\eta - 3\varepsilon\eta \end{pmatrix}. \tag{4.2}$$

In this section, we assume that  $|\varepsilon|$  is sufficiently small.

From (4.1) it follows that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\xi + \eta) - 1 - \varepsilon \\ \frac{1}{\sqrt{2}}(\xi - \eta) - 1 - \varepsilon \end{pmatrix}.$$

We set

$$\xi = \xi_1 + i\xi_2, \quad \eta = \eta_1 + i\eta_2, \quad \varepsilon = \varepsilon_1 + i\varepsilon_2,$$

where  $\xi_j, \eta_j,$  and  $\varepsilon_j$  are real. Then

$$|x|^2 + |y|^2 = (\xi_1 + \sqrt{2}(-1 - \varepsilon_1))^2 + (\xi_2 - \sqrt{2}\varepsilon_2)^2 + \eta_1^2 + \eta_2^2.$$

Therefore,

$$|\eta|^2 > 2(|c| + 1)^2 \implies \max\{|x|, |y|\} > |c| + 1. \tag{4.3}$$

If (4.3) holds then  $|x|$  and  $|y|$  satisfy the condition in Lemma 3.1. As a result,

$$\|f^n(x, y)\| \rightarrow \infty \quad (n \rightarrow \infty).$$

We consider the dynamics in the  $(\xi, \eta)$  coordinate. Let

$$B((a, b), r) = \{(\xi, \eta) \in \mathbf{C}^2 : |\xi - a|^2 + |\eta - b|^2 < r^2\}.$$

To show Theorem 4.1, it suffices to prove the following proposition.

**PROPOSITION 4.4.** *Let  $(\xi, \eta)$  be an element in  $\overline{B((0, 0), 0.1)}$ . If  $\eta \neq 0$ , then there exists a positive integer  $n$  satisfying*

$$|\eta_n| > \sqrt{2}(|c| + 1), \quad \text{where } (\xi_n, \eta_n) := k^n(\xi, \eta).$$

*If  $\eta = 0$ , then  $k^n(\xi, 0) \rightarrow (0, 0)$  ( $n \rightarrow \infty$ ).*

Before starting the proof, we describe the stable manifold and the unstable manifold of the saddle  $(0, 0)$ . The local stable manifold is included in the line  $\eta = 0$ ; the local unstable manifold is orthogonal to the line  $\eta = 0$ . We shall show that any orbit  $\{k^n(x, y)\}$  of a point  $(x, y)$  in the local unstable manifold tends to infinity.

To prove this proposition, we need four lemmas.

LEMMA 4.5. *Let*

$$V_1 = \{(\xi, \eta) \in \overline{B((0, 0), 0.1)} : |\eta| \leq \frac{1}{50} \text{ or } 4.3|\eta| \leq |\xi|\}.$$

Assume that  $(\xi, \eta) \in V_1$ . If  $\eta \neq 0$ , then there exists a positive integer  $n$  such that

$$k^n(\xi, \eta) \in \overline{B((0, 0), 0.1)} \setminus V_1.$$

If  $\eta = 0$ , then  $(\xi, 0)$  lies in the stable manifold of the saddle  $(0, 0)$ .

*Proof.* We first consider the case  $\eta \neq 0$ . Set  $(\xi_1, \eta_1) = k(\xi, \eta)$ . Then by (4.2) we have

$$|\xi_1| < 0.8 \times (0.1)^2 \quad \text{and} \quad 3.85|\eta| < |\eta_1| < 4.15|\eta|.$$

If  $|\eta| \leq \frac{1}{50}$ , then

$$|\xi_1|^2 + |\eta_1|^2 < (0.1)^2 \left( 0.8^2 \times 0.1^2 + \left( \frac{4.15}{5} \right)^2 \right) < (0.1)^2.$$

Similarly, we have  $(\xi_1, \eta_1) \in B((0, 0), 0.1)$  for the case where  $4.3|\eta| \leq |\xi|$ . Since  $|\eta_1| > 3.85|\eta|$ , we get an integer  $n$  satisfying  $k^n(\xi, \eta) \notin V_1$ .

For the case  $\eta = 0$ ,  $(\xi, 0)$  lies on the line  $(\eta = 0)$ . Since  $|\xi| \leq 0.1$ , it follows that  $\xi$  is contained in the attractive basin of a fixed point  $0$  on the line. □

LEMMA 4.6. *If a point  $(\xi, \eta)$  is in  $\overline{B((0, 0), 0.1)} \setminus V_1$  then  $k(\xi, \eta)$  is in the interior  $(V_2)^\circ$ , where*

$$V_2 := \{(\xi, \eta) \in \overline{B((0, 0), 0.44)} : |\eta| \geq 2.7|\xi| \text{ and } |\eta| \geq \frac{1}{50}\}.$$

*Proof.* Set  $(\xi_1, \eta_1) := k(\xi, \eta)$ . Then, by (4.2),

$$|\xi_1| \leq \frac{1}{\sqrt{2}}(4.3^2 + 1)|\eta|^2 + 4.3|\varepsilon||\eta| < (1.3782 + 4.3|\varepsilon|)|\eta|.$$

Hence

$$\frac{|\eta_1|}{2.7} > \frac{3.85}{2.7}|\eta| > (1.3782 + 4.3|\varepsilon|)|\eta| > |\xi_1|.$$

Other properties can be easily verified. □

LEMMA 4.7. *If a point  $(\xi, \eta)$  is in  $V_2$  then  $k(\xi, \eta) \in (V_{16})^\circ$ , where*

$$V_{16} := \{(\xi, \eta) \in \mathbf{C}^2 : |\eta|^2 \geq 16|\xi| \text{ and } \eta \neq 0\}.$$

*Proof.* Set  $(\xi_1, \eta_1) := k(\xi, \eta)$ . Clearly,

$$|\xi_1| \leq \frac{1}{\sqrt{2}} \left( \frac{1}{2.7^2} + 1 \right) |\eta|^2 + \frac{|\eta|}{2.7} |\varepsilon|.$$

Since  $0.44^2 \geq |\xi|^2 + |\eta|^2$  and  $|\eta| \geq 2.7|\xi|$ , we have  $|\xi| < 0.1529$ . Thus

$$|\eta_1| > |\eta|(4 - \sqrt{2}|\xi| - 3|\varepsilon|) > 3.77|\eta| \geq 3.77 \times \frac{1}{50} \geq 0.0754.$$

Therefore,

$$\frac{1}{16}|\eta_1|^2 > \frac{3.77^2}{16}|\eta|^2 \geq 0.888|\eta|^2 > \frac{1}{\sqrt{2}}\left(\frac{1}{2.7^2} + 1\right)|\eta|^2 + \frac{|\varepsilon|}{2.7}|\eta| \geq |\xi_1|. \quad \square$$

LEMMA 4.8. *If a point  $(\xi, \eta)$  is in  $V_{16}$ , then there exists a positive integer  $n$  satisfying*

$$|\eta_n| > \sqrt{2}(|c| + 1), \quad \text{where } (\xi_n, \eta_n) := k^n(\xi, \eta).$$

*Proof.* Let

$$V_3 := \left\{ (\xi, \eta) \in \mathbf{C}^2 : |\xi| \leq \frac{|\eta|^2}{16}, 0 < |\eta| \leq 2 \right\}.$$

We first demonstrate the following claim.

*Claim.* If  $(\xi, \eta) \in V_3$ , then  $k(\xi, \eta) \in (V_{16})^\circ$  and  $|\eta_1| > 3|\eta|$ .

Indeed, set  $(\xi_j, \eta_j) := k^j(\xi, \eta)$ . Then

$$\begin{aligned} |\xi_1| &\leq \frac{1}{256\sqrt{2}}|\eta|^4 + \frac{1}{\sqrt{2}}|\eta|^2 + \frac{|\varepsilon|}{16}|\eta|^2, \\ |\eta_1| &\geq |\eta|\left(4 - \frac{\sqrt{2}}{16}|\eta|^2 - 3|\varepsilon|\right). \end{aligned}$$

To prove this claim, it suffices to prove the inequality

$$\frac{1}{16}\left(4 - \frac{\sqrt{2}}{16}|\eta|^2 - 3|\varepsilon|\right)^2 \geq \frac{|\eta|^2}{256\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{|\varepsilon|}{16}. \tag{4.4}$$

See Figure 4.1. Clearly, if  $0 < |\eta| \leq 2$  then the inequality (4.4) holds for small  $|\varepsilon|$ . The inequality  $|\eta_1| > 3|\eta|$  is obvious.

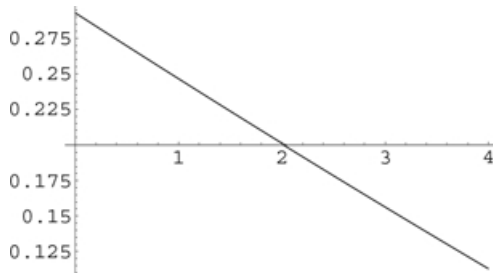


Figure 4.1  $\frac{t^2}{2008} - \left(\frac{\sqrt{2}}{32} + \frac{1}{256\sqrt{2}}\right)t + \left(1 - \frac{1}{\sqrt{2}}\right), 0 \leq t \leq 4$

From the claim we see that there exists a smallest positive integer  $q$  such that

$$k^q(\xi, \eta) \in V_{16} \setminus V_3.$$

If  $|\eta_q| > \sqrt{2}(|c| + 1)$ , then Lemma 4.8 follows. Next we consider the case

$$(\xi_q, \eta_q) \in V_{16} \cap \{(\xi, \eta) \in \mathbf{C}^2 : 2 \leq |\eta| \leq 5\}.$$

Since

$$|\xi_q| \leq \frac{|\eta_q|^2}{16},$$

it follows that

$$|\eta_{q+1}| \geq |\eta_q| \left( 4 - \frac{\sqrt{2}}{16} |\eta_q|^2 - 3|\varepsilon| \right). \tag{4.5}$$

Finally, we consider the real function

$$Q(x) = x \left( 4 - \frac{\sqrt{2}}{16} x^2 \right).$$

If  $2 \leq x \leq 5$ , then  $Q(x) \geq 7$ . Hence, if  $2 \leq |\eta_q| \leq 5$  then by (4.5) we have

$$|\eta_{q+1}| \geq 6 > \sqrt{2}(|c| + 1). \quad \square$$

This completes the proof of Proposition 4.4 and so that of Theorem 4.1. We note that, when  $\varepsilon = 0$ , the conditions  $|\eta| \leq \frac{1}{50}$  and  $|\eta| \geq \frac{1}{50}$  in the definitions of  $V_1$  and  $V_2$  (respectively) are not needed.

Now we begin the proof of Theorem 4.2. We consider the dynamics of the map in the  $(x, y)$  coordinate system. The key observation is that, when  $c = 2$ , the critical set  $xy = 1$  passes through the saddle point  $(-1, -1)$  and is near the unstable manifold of the saddle. In the statement of Theorem 4.2, the point  $(-1 - \frac{\varepsilon}{2}, -1 - \frac{\varepsilon}{2})$  corresponds to the point  $(-\frac{\varepsilon}{2}t, -\frac{\varepsilon}{2}t)$  with  $t = 1$ . This point is included in the stable manifold of the saddle  $(-1 - \varepsilon, -1 - \varepsilon)$ . Observe that  $B((0, 0), 0.1)$  in the  $(\xi, \eta)$  coordinate is equivalent to  $B((-1 - \varepsilon, -1 - \varepsilon), 0.1)$  in the  $(x, y)$  coordinate.

LEMMA 4.9. *We assume that  $c = 2$ . The set*

$$\left\{ \left( -t, -\frac{1}{t} \right) : t \in D(1, 0.068) \right\}$$

*is included in the ball  $B((-1, -1), 0.1)$ . Hence, for any element  $t$  in the set with  $t \neq 1$ ,*

$$\|(u_n(t), v_n(t))\| \rightarrow \infty \quad (n \rightarrow \infty).$$

*Proof.* Set  $t = 1 + re^{i\theta}$  with  $0 \leq r < 0.068$ . Then

$$d\left( (-1, -1), \left( -t, -\frac{1}{t} \right) \right)^2 = r^2 + \frac{r^2}{1 + r^2 + 2r \cos \theta},$$

where  $d(\cdot, \cdot)$  is the Euclidean distance. Hence,

$$d\left( (-1, -1), \left( -t, -\frac{1}{t} \right) \right) \leq d\left( (-1, -1), \left( r - 1, \frac{1}{r - 1} \right) \right).$$

If  $0 \leq r < 0.068$ , then

$$d\left((-1, -1), \left(r - 1, \frac{1}{r - 1}\right)\right) < d\left((-1, -1), \left(-0.932, -\frac{1}{0.932}\right)\right) < 0.1.$$

The intersections of the critical set  $xy = 1$  and the line  $x = y$  are  $(-1, -1)$  and  $(1, 1)$ . Hence, in the ball  $B((-1, -1), 0.1)$ , the only intersection is  $(-1, -1)$ . The second assertion then follows from Lemma 3.1 and Proposition 4.4.  $\square$

We note that this lemma also holds when  $c = 2 + \varepsilon$  with  $|\varepsilon|$  sufficiently small.

In the  $t$ -plane we consider the following closed domain  $\Omega$ . Let

$$A = \left\{ t \in \mathbb{C} : \frac{1}{4} \leq |t| \leq 1, 0 \leq \arg t \leq \frac{2\pi}{3} \right\}.$$

We set

$$\Omega = A - D(1, 0.068) - D(\omega, 0.068).$$

If  $|t| < \frac{1}{4}$  then, by Lemma 3.1,

$$\left\| f^n\left(-\frac{c}{2}t, -\frac{c}{2t}\right) \right\| \rightarrow \infty.$$

From Lemma 4.9 we also know that any point  $t$  in  $D(1, 0.068)$  with  $t \neq 1$  satisfies this property.

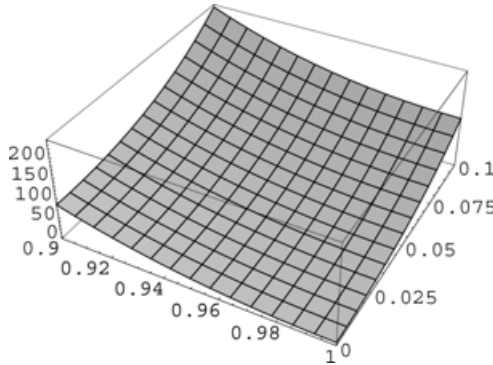
When  $c = 2$ , we consider the potential function

$$P(t) = |u_3(t)|^2 + |v_3(t)|^2,$$

where

$$(u_3(t), v_3(t)) = f^3\left(-t, -\frac{1}{t}\right).$$

See Figure 4.2.



**Figure 4.2** The potential function  $P(t)$

**PROPOSITION 4.10.** Assume that  $c = 2$ . If  $t \in \Omega$ , then

$$P(t) \geq 20.$$

From this proposition and Lemma 3.1 it follows that, if  $t \in \Omega$ , then

$$\left\| f^n\left(-t, -\frac{1}{t}\right) \right\| \rightarrow \infty.$$

First, we show a symmetry. Let  $t = re^{i\theta}$ . Then

$$P(r \exp(i\theta)) = P\left(r \exp\left(\left(\frac{2\pi}{3} - \theta\right)i\right)\right), \quad (4.6)$$

since

$$P(r \exp(i\theta)) = P(\omega r \exp(i\theta)) \quad \text{and} \quad P(r \exp(i\theta)) = P(r \exp(-i\theta)).$$

For the proof of Proposition 4.10, we need two lemmas.

LEMMA 4.11. *Assume that  $t = re^{i\theta} \in \Omega$ . Then the following statements hold:*

$$\begin{aligned} 0 \leq \theta \leq \frac{\pi}{3} &\implies \frac{\partial P}{\partial \theta} \geq 0; \\ \frac{\pi}{3} \leq \theta \leq \frac{2\pi}{3} &\implies \frac{\partial P}{\partial \theta} \leq 0. \end{aligned}$$

*Proof.* By symmetry, it suffices to prove

$$t = re^{i\theta} \in \Omega \wedge 0 \leq \theta \leq \frac{\pi}{3} \implies \frac{\partial P}{\partial \theta} \geq 0. \quad (4.7)$$

Direct calculation yields

$$\frac{\partial P}{\partial \theta} = \frac{96}{r^{13}} \sin 3\theta \times (a_0 T^3 + a_1 T^2 + a_2 T + a_3),$$

where  $T = \cos 3\theta$  and

$$\begin{aligned} a_0 &= -76r^9(1 + r^8), \\ a_1 &= 12r^6(7 + 38r^6 + 38r^8 + 7r^{14}), \\ a_2 &= -2r^3(11 + r^6(205 + 418r^2 + 418r^6 + 205r^8 + 11r^{14})), \\ a_3 &= 1 + r^6(67 + 266r^2 + 502r^6 + 502r^8 + 266r^{12} + 67r^{14} + r^{20}). \end{aligned}$$

Set  $p(T, r) := a_0 T^3 + a_1 T^2 + a_2 T + a_3$ . Proving (4.7) requires only that we establish the following claims.

*Claim 1.* If  $0 \leq r \leq 1$  and  $-1 \leq T \leq 1$ , then  $p(T, r) > 0$ .

To prove Claim 1, it suffices to prove that

$$p_1 := \frac{\partial p}{\partial T} < 0 \quad \text{and} \quad p(1, r) > 0.$$

*Claim 2.* If  $0 \leq r \leq 1$  and  $-1 \leq T \leq 1$ , then  $p_1(T, r) < 0$ .

To prove Claim 2, it suffices to prove that

$$p_2 := \frac{\partial p_1}{\partial T} > 0 \quad \text{and} \quad p_1(1, r) < 0.$$

*Claim 3.* If  $0 \leq r \leq 1$  and  $-1 \leq T \leq 1$ , then  $p_2(T, r) > 0$ .

We begin with the proof of Claim 3. By direct calculation, we have

$$p_2(T, r) = 96(-456r^9(1 + r^8)T + 24r^6(7 + 38r^6 + 38r^8 + 7r^{14})).$$

Then we can prove Claim 3 easily. Hence, to prove Claim 1, it suffices to show that  $p(1, r) > 0$  and  $p_1(1, r) < 0$ . These inequalities can be verified by direct calculation.  $\square$

From Lemma 4.11, we know that  $P(t)$  has its minimum value on the boundary  $\partial\Omega$ . Thus, to prove Proposition 4.10 it suffices to prove

$$t \in \partial\Omega \wedge 0 \leq \arg t \leq \frac{\pi}{3} \implies P(t) \geq 20. \tag{4.8}$$

To prove this, we modify the boundary  $\partial\Omega$ . In place of a part of the boundary

$$\{t \in \mathbf{R} : \frac{1}{4} \leq t \leq 0.932\} \cup (\Omega \cap \partial D(1, 0.068)),$$

we consider the following curves:

$$\begin{aligned} \gamma_0 &= \{t \in \mathbf{R} : \frac{1}{4} \leq t \leq 0.95\}, \\ \gamma_1 &= \{0.95 \exp(i\theta) : 0 \leq \theta \leq \sigma, \sigma = 0.0472885\}, \\ \gamma_2 &= \{r \exp(i\sigma) : 0.95 \leq r \leq 1\}. \end{aligned}$$

Observe that, in  $\Omega$ , the intersection of the circles  $\partial D(0, 0.95)$  and  $\partial D(1, 0.068)$  is the point  $0.95 \exp(i \times 0.0472885 \dots)$ . Hence, by Lemma 4.11, to prove (4.8) we need only prove the following lemma.

LEMMA 4.12. *If  $t \in \gamma_0 \cup \gamma_2$ , then  $P(t) \geq 20$ .*

*Proof.*

*Case 1:  $t \in \gamma_0$ .* By direct calculation, we have following properties:

- (1) if  $\frac{1}{4} \leq t \leq 0.75$ , then  $u_3(t) > 16$ ;
- (2) if  $0.75 \leq t \leq 0.935$ , then  $u_3(t)^2 + v_3(t)^2 > 30$ ;
- (3) if  $0.935 \leq t \leq 0.95$ , then  $u_3(t)^2 > 12$  and  $v_3(t)^2 > 8$ .

*Case 2:  $t \in \gamma_2$ .* We shall prove that, if  $0.95 \leq r \leq 1$ , then

$$\frac{dP(re^{i\sigma})}{dr} \leq 0 \tag{4.9}$$

and so  $P(re^{i\sigma}) \geq P(e^{i\sigma}) > 23$ .

First we note that  $dP(re^{i\sigma})/dr$  is divisible by  $(r^2 - 1)$ . We set

$$q(r) = \frac{dP(re^{i\theta})}{dr} \times \frac{r^{17}}{16(r^2 - 1)};$$

here  $q(r)$  is a polynomial of degree 32 with real coefficients. Set

$$q^{(j)}(r) = \frac{d^j q(r)}{dr^j} \quad (j = 1, \dots, 15).$$

Then it can be easily proved that, if  $0.95 \leq r \leq 1$ , then  $q^{(j)}(r)$  is positive and monotone increasing for any  $j$  ( $1 \leq j \leq 15$ ). Hence we have (4.9). Direct computation then yields

$$P(e^{i\sigma}) = 23.9245 \dots \quad \square$$

This completes the proof of Proposition 4.10 when  $c = 2$ . When  $c = 2 + \varepsilon$ , where  $|\varepsilon|$  is sufficiently small, we can prove that  $P(t) \geq 19$ . Hence, in this case

$$P(t) \geq 2(|c| + 1)^2.$$

We can therefore extend Proposition 4.10 to the case  $c = 2 + \varepsilon$  provided  $|\varepsilon|$  is sufficiently small. As noted previously, the point  $(-\frac{c}{2}, -\frac{c}{2})$  in  $\Gamma_c$  lies in the stable manifold of the saddle  $(-1 - \varepsilon, -1 - \varepsilon)$ .

Combining Lemma 2.4 and the extended versions of Lemma 4.9 and Proposition 4.10, we obtain Theorem 4.2.

Finally, we prove Theorem 4.3. We begin by making three key observations.

(i) The postcritical set

$$P(f_c) = \overline{\bigcup_{n>0} f_c^n(\Gamma_c)}$$

is controlled.

(ii) Let  $W$  be a set of points  $(x, y)$  such that  $f_c^n(x, y)$  converges to one of the three saddles

$$(-c + 1, -c + 1), (\omega(-c + 1), \omega^2(-c + 1)), (\omega^2(-c + 1), \omega(-c + 1))$$

or to the line at infinity as  $n \rightarrow \infty$ . Then

$$f_c^{-2}((D(0, |c| + 1) \times D(0, |c| + 1)) \setminus W) \cap P(f_c) = \emptyset.$$

(iii) We may define the infinitesimal Kobayashi metric on  $f_c^{-2}((D(0, |c| + 1) \times D(0, |c| + 1)) \setminus W)$  and then prove that  $f_c$  is strictly expanding.

We next define several sets via the  $(x, y)$  coordinate:

$$\begin{aligned} EX &:= \mathbf{C}^2 \setminus \overline{(D(0, |c| + 1) \times D(0, |c| + 1))}, \\ V &:= \overline{B((-c + 1, -c + 1), 0.1)} \cup V_2 \cup V_{16}, \\ V(\omega) &:= \{(x\omega, y\omega^2) : (x, y) \in V\}, \\ V(\omega^2) &:= \{(x\omega^2, y\omega) : (x, y) \in V\}, \\ W &:= \overline{EX} \cup V \cup V(\omega) \cup V(\omega^2). \end{aligned}$$

Note that  $W$  is a closed set. We assume that  $c = 2 + \varepsilon$  with  $|\varepsilon|$  sufficiently small, and we use  $f$  and  $\Gamma$  to abbreviate  $f_c$  and  $\Gamma_c$  (respectively).

LEMMA 4.13. *Let notation be as before.*

- (1)  $f(EX \cup V \cup V(\omega) \cup V(\omega^2)) \subset (EX \cup V \cup V(\omega) \cup V(\omega^2))^\circ$ .
- (2)  $f^2(\partial EX) \subset EX$ ; hence  $f^2(W) \subset (W)^\circ$ .

*Proof.* To prove assertion (1), it suffices to prove the following properties:

- (1-1)  $f(EX) \subset EX$ ,
- (1-2)  $f(V) \subset (EX \cup V)^\circ$ ,



(1-3)  $f(V(\omega)) \subset (V(\omega^2) \cup EX)^\circ,$

(1-4)  $f(V(\omega^2)) \subset (V(\omega) \cup EX)^\circ.$

Property (1-1) follows from Lemma 3.1. To prove (1-2), we assume that  $(x, y) \in V$ . Then, by Lemmas 4.5–4.8, we have

$$f(x, y) \in (EX \cup V)^\circ.$$

Clearly  $(\omega(-1 - \varepsilon), \omega^2(-1 - \varepsilon))$  and  $(\omega^2(-1 - \varepsilon), \omega(-1 - \varepsilon))$  are periodic saddle points of period 2. Hence we can prove (1-3) and (1-4) similarly.

Next we prove assertion (2). Set

$$(x_n, y_n) := f^n(x, y).$$

If  $|x| = |c| + 1$  and  $|y| < |c| + 1$ , then  $|x_1| > |c| + 1$ . Hence we assume that

$$x = e^{i\alpha}(|c| + 1) \quad \text{and} \quad y = e^{i\beta}(|c| + 1).$$

Let  $c = e^{i\theta}|c|$ . Here  $0 \leq \alpha, \beta, \theta < 2\pi$ . If  $\alpha \neq \theta$  or  $\beta \neq \theta$ , then  $\max\{|x_1|, |y_1|\} > |c| + 1$ . Therefore,  $(x_1, y_1) \in EX$ . If  $\alpha = \beta = \theta$ , then  $\max\{|x_2|, |y_2|\} > |c| + 1$ . □

Lemma 4.13 yields the following statement.

PROPOSITION 4.14. *Let  $(x, y) \in W$ . Then  $f^n(x, y)$  converges to one of the points*

$$(-1 - \varepsilon, -1 - \varepsilon), (\omega(-1 - \varepsilon), \omega^2(-1 - \varepsilon)), (\omega^2(-1 - \varepsilon), \omega(-1 - \varepsilon))$$

*or to the line at infinity.*

Next we show that the postcritical set  $P(f)$  is controlled.

PROPOSITION 4.15. *For any  $n \geq 3$ ,*

$$f^n(\Gamma) \subset (EX \cup V \cup V(\omega) \cup V(\omega^2))^\circ.$$

*Proof.* Let  $\phi(t) = (-\frac{\varepsilon}{2}t, -\frac{\varepsilon}{2}\frac{1}{t})$ . By Proposition 4.10, if  $n \geq 3$  then

$$f^n(\phi(\Omega)) \subset EX.$$

Similarly,

$$f^3(\phi(\omega\Omega)) \subset EX \quad \text{and} \quad f^3(\phi(\omega^2\Omega)) \subset EX.$$

Set  $\Gamma^* := \phi(\overline{D(1, 0.068)} \cap \overline{D(0, 1)})$ . Then it follows from the extended version of Lemma 4.9 that

$$\Gamma^* \subset B((-1 - \varepsilon, -1 - \varepsilon), 0.1) \subset V.$$

Hence, from Lemma 4.13, we deduce that

$$f^n(\Gamma^*) \subset (EX \cup V)^\circ \quad \text{for } n \geq 1.$$

Now set

$$\Gamma^*(\omega) = \phi(\overline{D(\omega, 0.068)} \cap \overline{D(0, 1)}),$$

$$\Gamma^*(\omega^2) = \phi(\overline{D(\omega^2, 0.068)} \cap \overline{D(0, 1)}).$$

Then

$$\Gamma^*(\omega) \subset \{(x\omega, y\omega^2) : (x, y) \in B((-1 - \varepsilon, -1 - \varepsilon), 0.1)\} \subset V(\omega)$$

and  $\Gamma^*(\omega^2) \subset V(\omega^2)$ . Hence, by Lemma 4.13, we have

$$f(\Gamma^*(\omega) \cup \Gamma^*(\omega^2)) \subset (EX \cup V(\omega) \cap V(\omega^2))^\circ$$

and so

$$f^n(\Gamma^*(\omega) \cup \Gamma^*(\omega^2)) \subset (EX \cup V(\omega) \cup V(\omega^2))^\circ \quad \text{for } n \geq 1.$$

Since

$$\phi(\{t \in \mathbf{C} : \frac{1}{4} \leq |t| \leq 1\}) = \Gamma^* \cup \Gamma^*(\omega) \cup \Gamma^*(\omega^2) \cup \phi(\Omega) \cup \phi(\omega\Omega) \cup \phi(\omega^2\Omega),$$

it follows that

$$f^n(\phi(\{t \in \mathbf{C} : \frac{1}{4} \leq |t| \leq 1\})) \subset (EX \cup V \cup V(\omega) \cup V(\omega^2))^\circ \quad \text{for } n \geq 3.$$

As a result,

$$f^n(\Gamma) \subset (EX \cup V \cup V(\omega) \cup V(\omega^2))^\circ \quad \text{for } n \geq 3. \quad \square$$

Combining Lemma 4.13 and Proposition 4.15 now yields the following lemma.

LEMMA 4.16.

$$f^2(W \cup f(\Gamma) \cup f^2(\Gamma)) \subset (W)^\circ.$$

Now consider the set  $G$  defined by

$$G := (D(0, |c| + 1) \times D(0, |c| + 1)) \setminus W.$$

We aim to show that  $f$  is strictly expanding on  $f^{-2}(G)$ . From Lemma 4.13, it follows that  $f^{-2}(G) \subset G$ .

LEMMA 4.17. (1) *Let  $P(f)$  be the postcritical set. Then*

$$f^{-2}(G) \cap P(f) = \emptyset.$$

(2) *Let  $p$  be any periodic point of  $f$  other than the three saddles*

$$(-c + 1, -c + 1), (\omega(-c + 1), \omega^2(-c + 1)), (\omega^2(-c + 1), \omega(-c + 1)).$$

*Then  $p$  lies in the set  $f^{-2}(G)$ .*

*Proof.* By Lemma 4.15, the set

$$\bigcup_{n=1}^{\infty} f^n(\Gamma)$$

is included in the closed set

$$W \cup f(\Gamma) \cup f^2(\Gamma).$$

Then part (1) follows from Lemma 4.16. For part (2), note that the periodic point  $p$  does not lie in the set  $W$ . Hence,  $f^2(p) \notin W$ .  $\square$

Let  $G_0$  be any component of  $f^{-2}(G)$ .

LEMMA 4.18. *The domain  $G_0$  is Kobayashi hyperbolic.*

*Proof.* Since

$$G_0 \subset G \subset D(0, |c| + 1) \times D(0, |c| + 1),$$

$G_0$  is a bounded domain of  $\mathbf{C}^2$  and hence  $G_0$  is Kobayashi hyperbolic. □

LEMMA 4.19. (1) *With notation as before, we have*

$$G_0 \subset\subset G.$$

(2) *Let  $G_n$  be any component of  $f^{-n}(G_0)$ , and let  $G_{n+1}$  be any component  $f^{-1}(G_n)$ . Then*

$$G_{n+1} \subset\subset G_n.$$

*Proof.* Part (1) follows from  $f^2(\partial W) \subset (W)^\circ$ , and part (2) follows from (1). □

To complete the proof of Theorem 4.3, we use Fornæss and Sibony’s argument in [FS2, Thm. 3.8].

*Proof of Theorem 4.3.* Since  $G_0$  is Kobayashi hyperbolic, we can define the infinitesimal Kobayashi metric  $K_0(x, \xi)$  at a point  $x$  in  $G_0$  and a tangent vector  $\xi$ . Consider the map

$$f : G_{n+1} \rightarrow G_n.$$

From Lemma 4.17 we know that  $f$  is a holomorphic unramified covering map. Let  $K_m$  denote the infinitesimal Kobayashi metric for  $G_m$ . Then, for  $z \in G_{n+1}$ ,

$$K_{n+1}(z, \xi) = K_n(f(z), Df(z)\xi).$$

By Lemma 4.19,

$$K_{n+1}(z, \xi) \geq (1 + a)K_n(z, \xi) \quad \text{for a constant } a > 0$$

and hence

$$K_n(f(z), Df(z)\xi) \geq (1 + a)K_n(z, \xi). \tag{4.10}$$

By Lemma 4.17(2), we know that any periodic point  $p$  of period  $l$  satisfying the condition of Theorem 4.3 lies in a component of  $f^{-2}(G)$ . Hence we may use (4.10) to conclude that all eigenvalues of  $Df^l(p)$  have modulus greater than 1. □

We now extend Theorem 4.3. Let

$$U := \{c \in \mathbf{C} : \text{the set } K(f_c) \cap \Gamma_c \text{ consists of three points}\}.$$

Let  $U_0$  be the connected component of  $U \cap D(2, 1)$  containing the value 2. Theorem 4.2 guarantees that  $U_0$  is not empty.

COROLLARY 4.20. *For any  $c \in U_0$ , any periodic point other than*

$$(-c + 1, -c + 1), (\omega(-c + 1), \omega^2(-c + 1)), (\omega^2(-c + 1), \omega(-c + 1))$$

*is repelling.*

*Proof.* We consider three cases.

(1) Suppose that, for some  $c$  in  $U_0$ ,  $f_c$  has another saddle periodic point  $p$ . Any hyperbolic periodic point attracts a point from a critical set of  $f_c$ , so there is a point  $q$  in  $\Gamma_c$  such that  $f_c^n(q) \rightarrow p$  ( $n \rightarrow \infty$ ). Then  $q \in \Gamma_c \cap K(f_c)$ . When  $c \in D(2, 1)$ , the three periodic points displayed in our statement of the corollary are saddles. Each saddle attracts a point in  $\Gamma_c$  and so  $\Gamma_c \cap K(f_c)$  contains four points—and a contradiction follows.

(2) Suppose that, for some  $c$  in  $U_0$ ,  $f_c$  has an attracting periodic point. By the same argument as in (1), a contradiction follows.

(3) Suppose that, for some  $c$  in  $U_0$ ,  $f_c$  has a nonhyperbolic periodic point  $p$ . Let the eigenvalues of  $Df_c(p)$  be  $\lambda_1$  and  $\lambda_2$ , and assume that  $|\lambda_1| = 1$ . Let  $V$  be a small neighborhood of the point  $c$  included in  $U_0$ . Then  $\lambda_1$  is a holomorphic function of  $z$  in  $V$  and does not vanish there. When  $\lambda_1(z)$  is not a constant function,  $|1/\lambda_1(z)|$  has the maximum on the boundary  $\partial V$ . Then case (1) or (2) occurs. When  $\lambda_1(z)$  is constant, a contradiction follows from Theorem 4.3.  $\square$

Uchimura [U4] shows that, if  $c \in D(2, \delta)$  for some small positive number  $\delta$ , then the following statements hold:

- (1) the second Julia set of  $f_c(x, y)$  is connected;
- (2)  $f_c(x, y)$  is an Axiom A endomorphism of  $\mathbf{C}^2$  with  $f^{-1}S_2 = S_2$ .

For more details, see [U4].

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## References

- [BCSh] J.-Y. Briend, S. Cantat, and M. Shishikura, *Linearity of the exceptional set for maps of  $\mathbf{P}_k(\mathbf{C})$* , Math. Ann. 330 (2004), 39–43.
- [DH] M. Denker and S. Heinemann, *Jordan tori and polynomial endomorphisms in  $\mathbf{C}^2$* , Fund. Math. 157 (1998), 139–159.
- [FS1] J. Fornæss and N. Sibony, *Complex dynamics in higher dimension I*, Astérisque 222 (1994), 201–231.
- [FS2] ———, *Dynamics of  $\mathbf{P}^2$  (Examples)*, Contemp. Math., 269, pp. 47–85, Amer. Math. Soc., Providence, RI, 2001.
- [J] M. Jonsson, *Dynamics of polynomial skew products on  $\mathbf{C}^2$* , Math. Ann. 314 (1999), 403–447.
- [L1] A. O. Lopes, *Dynamics of real polynomials on the plane and triple point phase transition*, Math. Comput. Modelling 13 (1990), 17–32.
- [L2] ———, *On the dynamics of real polynomials on the plane*, Comput. Graphics 16 (1992), 15–23.
- [M] J. Milnor, *Dynamics in one complex variable*, Vieweg, Weisbaden, 1999.
- [U1] K. Uchimura, *The dynamical systems associated with Chebyshev polynomials in two variables*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 6 (1996), 2611–2618.
- [U2] ———, *Attractive basins of certain non-holomorphic maps*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 8 (1998), 1517–1526.

- [U3] ———, *The sets of points with bounded orbits for generalized Chebyshev mappings*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 11 (2001), 91–107.
- [U4] ———, *Axiom A endomorphisms of  $\mathbf{C}^2$  each of which has the connected  $J_2$  set and three saddles*, preprint.
- [Ue] T. Ueda, *Critically finite maps on the complex projective space*, Research on complex dynamical systems: Current state and prospects (Kyoto, 1998), RIMS Kokyuroku 1087 (1999), 132–138.

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