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Quasihyperbolic Growth Conditions and Compact Embeddings of Sobolev Spaces

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1. Introduction

In this paper, Ω will always denote an open and connected subset of the Euclidean space $\mathbb{R}^n$, $n \geq 2$. Recall that the Sobolev space $W^{1,p}(\Omega)$ consists of the functions in $L^p(\Omega)$ whose first-order distributional derivatives all belong to $L^p(\Omega)$ also. One of the fundamental inequalities associated with Sobolev functions is the Sobolev–Poincaré inequality, according to which

$$\left(\int_\Omega |u - u_\Omega|^{np/(n-p)}\,dx\right)^{(n-p)/np} \leq C\left(\int_\Omega |\nabla u|^p\,dx\right)^{1/p} \tag{1}$$

holds for $1 \leq p < n$ whenever $u \in W^{1,p}(\Omega)$ and Ω is sufficiently nice—say, when Ω is bounded and satisfies the cone condition. In such a domain it follows that $W^{1,p}(\Omega) \subset L^{np/(n-p)}(\Omega)$, and one has a compact embedding of $W^{1,p}(\Omega)$ into $L^q(\Omega)$ for all $1 \leq q < np/(n-p)$. This is called the Rellich–Kondrachov compact embedding theorem (cf. [7]) and it, together with its analogues, are of fundamental importance in the study of certain partial differential equations (e.g., for the Neumann problem for second-order elliptic equations). In fact, the discreteness of the spectrum is equivalent to the weaker conclusion that the embedding of $W^{1,2}(\Omega)$ into $L^2(\Omega)$ is compact; see [10]. This compact embedding is usually called the Rellich lemma [12].

One is then led to ask for minimal regularity conditions on Ω that would allow for the compactness of the embedding of $W^{1,p}(\Omega)$ into $L^p(\Omega)$. It is essentially due to Nikodym [11] that we must require some additional condition (besides the necessary boundedness assumption) to be posed on Ω. In this paper we consider conditions given in terms of the quasihyperbolic metric, defined by setting

$$k_\Omega(x, y) = \inf_\gamma \int_\gamma \frac{ds}{d(z, \partial\Omega)}$$

for each pair of points $x, y \in \Omega$, where the infimum is taken over all rectifiable curves γ in Ω joining x to y. In 1985, Axler and Shields [1] asked if the growth condition

$$k_\Omega(x, x_0) \leq C_1 \log\left(\frac{1}{d(x, \partial\Omega)}\right) + C_2 \tag{2}$$

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for a simply connected planar Ω implies that

$$\int_{\Omega} |u - u_{\Omega}|^2 \, dx \leq C \int_{\Omega} |\nabla u|^2 \, dx \tag{3}$$

for functions in $W^{1,2}(\Omega)$. We can easily check that this Poincaré inequality is necessary for the compactness of the embedding of $W^{1,2}(\Omega)$ into $L^2(\Omega)$ (cf. [17]). In 1990, Smith and Stegenga [14] answered this question in the positive by proving that (2) implies the Poincaré inequality (3) even without the assumption of simple connectivity. They also established the corresponding Poincaré inequality for $W^{1,n}(\Omega)$ in higher dimensions, where the exponent 2 is replaced by n. Moreover, they verified that (2) guarantees the compactness of the embedding of $W^{1,n}(\Omega)$ into $L^n(\Omega)$. In the subsequent paper [15], Smith and Stegenga relaxed the growth condition (2) essentially to

$$k_{\Omega}(x, x_0) \leq C_1 \left(\log\left(\frac{1}{d(x, \partial\Omega)} \right) \right)^{3/2} + C_2$$

without altering their conclusions (cf. Corollary 4 and its proof in [15]). Notice that this growth order is slower than any root of $1/d(x, \partial\Omega)$. In this paper we prove the following sharp result.

THEOREM 1.1. *Suppose that $k_{\Omega}(x, x_0) \leq C_0 d(x, \partial\Omega)^{-\alpha}$ for all $x \in \Omega$, where $x_0 \in \Omega$ is a fixed basepoint. If $\alpha < \frac{n}{2n-1}$, then the embedding of $W^{1,n}(\Omega)$ into $L^n(\Omega)$ is compact. Moreover, there is a bounded domain Ω for which the preceding growth order holds with $\alpha = \frac{n}{2n-1}$ and for which the embedding of $W^{1,n}(\Omega)$ into $L^n(\Omega)$ fails to be compact.*

For simply connected planar domains, Theorem 1.1 could be reformulated in terms of univalent multipliers of the Dirichlet space; see [1].

In the course of our proof of Theorem 1.1, we shall establish an essentially sharp Sobolev–Poincaré type inequality in the domains under consideration. Notice that our result applies only to the borderline case $p = n$. For exponents $p < n$, the correct assumption appears to be (2). Indeed, it follows from [8] that (2) implies the compactness of the embedding of $W^{1,p}(\Omega)$ into $L^p(\Omega)$ when $p > \max\{1, n - n/C_1\}$, but this may fail when $1 \leq p = n - n/C_1$. See also [2] for weaker results.

The paper is organized as follows. In Section 2, we give definitions and results related to the quasihyperbolic metric and Whitney decompositions that will be needed later on. Section 3 contains the proof of the first part of Theorem 1.1 together with some generalizations. In the final section, Section 4, we give the example referred to in Theorem 1.1.

2. Definitions and Preliminary Results

When A is a subset of the n-dimensional Euclidean space $\mathbb{R}^n$, we denote by $|A|$ the n-dimensional Lebesgue measure of A. By C we will denote various constants, which may vary from expression to expression.

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a proper subdomain. The quasihyperbolic metric k_Ω in Ω is defined by

$$k_\Omega(x, y) = \inf_\gamma \int_\gamma \frac{ds}{d(z, \partial\Omega)},$$

where the infimum is taken over all rectifiable curves γ in Ω joining x to y. This metric was introduced by Gehring and Palka in [4]. Given any two points $x, y \in \Omega$, there exists (by results of Gehring and Osgood [3]) a quasihyperbolic geodesic γ joining x to y. We fix such a geodesic for every pair $x, y \in \Omega$ and denote this by $\gamma_{x,y}$.

In this paper, our main assumption for the domain $\Omega \subset \mathbb{R}^n$ is that it satisfies the quasihyperbolic growth condition

$$k_\Omega(x, x_0) \leq C_0 d(x, \partial\Omega)^{-\alpha} \tag{4}$$

for all $x \in \Omega$, where $\alpha > 0$, $x_0 \in \Omega$ is a fixed basepoint, and $C_0 > 0$ is a constant.

Let $\mathcal{W} = \mathcal{W}(\Omega)$ be a Whitney decomposition of Ω—that is, a collection of dyadic cubes $Q \subset \Omega$ having pairwise disjoint interiors and satisfying the condition

$$\operatorname{diam}(Q) \leq d(Q, \partial\Omega) \leq 4 \operatorname{diam}(Q)$$

(see [16] for the existence and properties of Whitney decompositions). We denote by c_Q the center of a cube $Q \in \mathcal{W}$. We also fix a central cube $Q_0 \in \mathcal{W}$ and assume that x_0 is the center of Q_0. For $j \in \mathbb{N}$ we then set

$$\mathcal{W}_j = \{Q \in \mathcal{W} : j \leq k_\Omega(c_Q, x_0) < j + 1\}.$$

The quasihyperbolic metric k_Ω and the Whitney decomposition $\mathcal{W}(\Omega)$ have close relations. In fact, when $x, y \in \Omega$ and $|x - y| \geq d(x, \partial\Omega)/2$, we have

$$N(x, y)/C \leq k_\Omega(x, y) \leq CN(x, y), \tag{5}$$

where $N(x, y)$ is the number of cubes $Q \in \mathcal{W}$ intersecting the quasihyperbolic geodesic $\gamma_{x,y}$.

For $Q \in \mathcal{W}$ we set $P(Q) = \{Q' \in \mathcal{W} : Q' \cap \gamma_{c_Q, x_0} \neq \emptyset\}$ and define the *shadow* of a cube $Q \in \mathcal{W}$ by

$$S(Q) = \bigcup_{\substack{\tilde{Q} \in \mathcal{W} \\ Q \in P(\tilde{Q})}} \tilde{Q}.$$

Next, we prove some preliminary results that relate the quasihyperbolic metric to the Whitney decomposition and to the shadows of the Whitney cubes.

LEMMA 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain and let γ be a quasihyperbolic geodesic in Ω starting at the fixed basepoint $x_0 \in \Omega$. Then there is a constant $C = C(n) >$ 0 such that, for each $j \geq 1$,*

$$\#\{Q \in \mathcal{W}_j : Q \cap \gamma \neq \emptyset\} \leq C.$$

Proof. Denote

$$a_j := \#\{Q \in \mathcal{W}_j : Q \cap \gamma \neq \emptyset\}.$$

Let $Q_1, \dots, Q_{a_j} \in \mathcal{W}_j$ be the cubes intersecting γ, ordered so that if $k < l$ then there exist $x_k \in Q_k \cap \gamma$ such that, for every $x \in Q_l \cap \gamma$, we have $k_\Omega(x_k, x_0) \le k_\Omega(x, x_0)$. We may assume that $d(x_1, x_{a_j}) \ge d(x_1, \partial\Omega)/2$; thus, by (5), $k_\Omega(x_1, x_{a_j}) \ge \frac{a_j}{C}$. Since $k_\Omega(x_l, c_{Q_l}) \le 1$ for all $l = 1, \dots, a_j$, we calculate

$$\begin{aligned}\frac{a_j}{C} &\le k_\Omega(x_1, x_{a_j}) = k_\Omega(x_{a_j}, x_0) - k_\Omega(x_1, x_0) \\ &\le k_\Omega(c_{a_j}, x_0) + k_\Omega(c_{a_j}, x_{a_j}) - k_\Omega(c_1, x_0) + k_\Omega(c_1, x_1) \\ &\le (j+1) + 1 - j + 1 = 3.\end{aligned}$$

Hence $a_j \le 3C$, where C is a constant depending only on n. □

LEMMA 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a domain. Then there is a constant $C = C(n) > 0$ such that, for each $j \ge 1$,*

$$\sum_{Q \in \mathcal{W}_j} \chi_{S(Q)}(x) \le C$$

whenever $x \in \Omega$.

Proof. Let $x \in \Omega$. The number of Whitney cubes containing x is bounded, so we may assume that there is a unique cube $Q(x) \in \mathcal{W}$ such that $x \in Q(x)$. Let γ be the fixed geodesic joining $c_{Q(x)}$ to x_0. Then $x \in S(Q)$ for $Q \in \mathcal{W}_j$ if and only if $\gamma \cap Q \neq \emptyset$. By Lemma 2.1, the number of such cubes $Q \in \mathcal{W}_j$ is bounded by a constant that is independent of j. This proves the lemma. □

The next lemma, which we use in the proof of our main theorem, is an integral version of Lemma 2.2.

LEMMA 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain. Then, for each $s \ge 1$ and for every measurable subset $E \subset \Omega$,*

$$\sum_{Q \in \mathcal{W}_j} |S(Q) \cap E|^s \le C|E|^s, \tag{6}$$

where $C = C(n, s) > 0$.

Proof. The case $s = 1$ follows directly from Lemma 2.2:

$$\sum_{Q \in \mathcal{W}_j} |S(Q) \cap E| = \int_E \sum_{Q \in \mathcal{W}_j} \chi_{S(Q)}(x)\, dx \le \int_E C.$$

But now (6) holds also for $s > 1$, since

$$\sum_{Q \in \mathcal{W}_j} \left(\frac{|S(Q) \cap E|}{C|E|}\right)^s \le \sum_{Q \in \mathcal{W}_j} \frac{|S(Q) \cap E|}{C|E|} \le 1. \qquad \square$$

When a domain Ω satisfies the condition (4) for some exponent $0 < \alpha < 1$, we obtain the following estimate for the diameter of the shadow $S(Q)$ for cubes $Q \in \mathcal{W}_j$.

LEMMA 2.4. *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the quasihyperbolic growth condition* (4) *with an exponent $0 < \alpha < 1$. Then there exists a constant $C > 0$, independent of j, such that*

$$\operatorname{diam}(S(Q)) \le Cj^{(\alpha-1)/\alpha} \tag{7}$$

for each Whitney cube $Q \in \mathcal{W}_j$.

Proof. Let $j \in \mathbb{N}$ and let Q be a cube in $\mathcal{W}_j$. Let $\tilde{Q} \subset S(Q)$ and let γ be the fixed geodesic joining x_0 to $c_{\tilde{Q}}$. Then, by the definition of the shadow, there exists a point $x_Q \in \gamma \cap Q$. From condition (4) and properties of the Whitney decomposition it follows that, for each $Q \in \mathcal{W}_i$,

$$\operatorname{diam}(Q) \le d(c_Q, \partial\Omega) \le (C_0 i)^{-1/\alpha}.$$

By Lemma 2.1, γ intersects a bounded number of cubes from each $\mathcal{W}_i$, $i \ge j$. Thus

$$\begin{aligned} d(c_Q, c_{\tilde{Q}}) &\le d(c_Q, x_Q) + d(x_Q, c_{\tilde{Q}}) \\ &\le \operatorname{diam}(Q) + \operatorname{diam}(\gamma \cap S(Q)) \\ &\le \operatorname{diam}(Q) + \sum_{i \ge j} \sum_{\substack{Q' \in \mathcal{W}_i \\ Q' \cap \gamma \ne \emptyset}} \operatorname{diam}(Q') \\ &\le C \sum_{i \ge j} i^{-1/\alpha} \le C j^{-1/\alpha + 1}. \end{aligned} \tag{8}$$

Now we can take the supremum over all cubes $\tilde{Q} \subset S(Q)$ in (8) and use the triangle inequality to obtain the lemma. □

Note in particular that a domain Ω satisfying condition (4) with an exponent $0 < \alpha < 1$ must be bounded. This is a special case of results of Gotoh [5].

Finally, we say that a domain of finite volume $\Omega \subset \mathbb{R}^n$ is a *(q, p)-Poincaré domain,* $1 \le p \le q < \infty$, if there exist a constant $M_{q,p} > 0$ such that

$$\left(\int_\Omega |u - u_\Omega|^q \, dx \right)^{1/q} \le M_{q,p} \left(\int_\Omega |\nabla u|^p \, dx \right)^{1/p} \tag{9}$$

for all $u \in C^\infty(\Omega)$. Here we use the notation $u_\Omega = \fint_\Omega u \, dx = |\Omega|^{-1} \int_\Omega u \, dx$. Let us record the following well-known result due to Maz'ja [9; 10]. For a simple proof see [6, Thm. 5].

Lemma 2.5. *If $\Omega \subset \mathbb{R}^n$ is a (q, p)-Poincaré domain, then the embedding of $W^{1,p}(\Omega)$ into $L^s(\Omega)$ is compact for all $1 \le s < q$.*

3. Proof of the First Part of Theorem 1.1

We begin with the following theorem, in which we obtain for the planar case the essentially sharp quasihyperbolic growth condition needed to guarantee that a domain $\Omega \subset \mathbb{R}^2$ is a $(q, 2)$-Poincaré domain for some $q \ge 2$.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^2$ satisfy the quasihyperbolic growth condition* (4) *with an exponent $0 < \alpha < \frac{2}{3}$. Then Ω is a $(q, 2)$-Poincaré domain for all $2 \le q < 4\frac{1-\alpha}{\alpha}$.*

Proof. By a capacity-type characterization of Poincaré domains due to Maz'ja ([9; 10]; see also [6, Thm. 1]), it suffices to show that

$$|E|^{2/q} \le C \int_\Omega |\nabla u|^2 \tag{10}$$

whenever $E \subset \Omega$ is a compact subset such that $E \cap Q_0 = \emptyset$ and $u \in C^\infty(\Omega)$ satisfies $u|_{Q_0} = 0$ and $u|_E \ge 1$.

Let $E \subset \Omega$ and $u \in C^\infty(\Omega)$ be as before. We may assume that, for each $x \in E$, there is a cube $Q(x) \in \mathcal{W}$ such that $x \in Q(x)$ and $u_{Q(x)} \ge \frac{1}{2}$. Indeed, if this is not the case then we denote

$$E_g = \{x \in E : x \in Q \in \mathcal{W},\ u_Q \le \tfrac{1}{2}\}$$

and use Poincaré's inequality on cubes to obtain

$$\begin{aligned} |E_g|^{2/q} &\le \sum_{Q \in \mathcal{W}} |E_g \cap Q|^{2/q} \\ &\le 4 \sum_{\substack{Q \in \mathcal{W} \\ Q \cap E_g \ne \emptyset}} \left(\int_Q |u - u_Q|^q \right)^{2/q} \\ &\le C \sum_{\substack{Q \in \mathcal{W} \\ Q \cap E_g \ne \emptyset}} \left(\int_Q |\nabla u|^2 \right) \le C \int_\Omega |\nabla u|^2. \end{aligned}$$

Thus (10) holds for the set E_g.

Using a straightforward chaining argument as in [14, Lemma 8] with the assumption $u_{Q(x)} \ge \frac{1}{2}$, it now follows that, for every $x \in E$,

$$1 \le C \sum_{Q \in P(Q(x))} \operatorname{diam}(Q) \fint_Q |\nabla u(y)|\, dy. \tag{11}$$

Integration of (11) over E and a simple use of Hölder's inequality give

$$|E| \le C \int_E \sum_{Q \in P(Q(x))} \operatorname{diam}(Q) \left(\fint_Q |\nabla u(y)|^2\, dy \right)^{1/2} dx. \tag{12}$$

We can now interchange the order of summation and integration in (12) and then use Schwarz's inequality to obtain

$$\begin{aligned} |E| &\le C \sum_{Q \in \mathcal{W}} |S(Q) \cap E| \left(\int_Q |\nabla u(y)|^2\, dy \right)^{1/2} \\ &\le C \left(\sum_{Q \in \mathcal{W}} |S(Q) \cap E|^2 \right)^{1/2} \left(\sum_{Q \in \mathcal{W}} \int_Q |\nabla u(y)|^2\, dy \right)^{1/2}. \end{aligned} \tag{13}$$

Next, we estimate the first part of the product in (13) using Lemmas 2.3 and 2.4. If $0 < s \le 1$, we calculate

$$\begin{aligned}\sum_{Q\in\mathcal{W}} |S(Q)\cap E|^2 &\le \sum_{j=1}^{\infty} \max_{Q\in\mathcal{W}_j} (|S(Q)\cap E|^s) \sum_{Q\in\mathcal{W}_j} |S(Q)\cap E|^{2-s} \\ &\le C|E|^{2-s} \sum_{j=1}^{\infty} \max_{Q\in\mathcal{W}_j} (\operatorname{diam}(S(Q))^{2s}) \\ &\le C|E|^{2-s} \sum_{j=1}^{\infty} j^{2s(\alpha-1)/\alpha}. \end{aligned} \tag{14}$$

Here the sum converges if $s > \frac{\alpha}{2(1-\alpha)}$. Thus, by (13) and (14) we have that

$$|E| \le C|E|^{(2-s)/2}\left(\int_\Omega |\nabla u(y)|^2\,dy\right)^{1/2}$$

and hence

$$|E|^s \le C\int_\Omega |\nabla u(y)|^2\,dy$$

for $s > \frac{\alpha}{2(1-\alpha)}$. We can now choose $s = \frac{2}{q}$, and so (10) holds for E and the proof is complete. $\square$

From the proof of Theorem 3.1 we obtain the following corollary.

COROLLARY 3.2. *If* $\Omega \subset \mathbb{R}^2$, $|\Omega| < \infty$, *and there is a constant* $C > 0$ *such that*

$$\sum_{Q\in\mathcal{W}} |S(Q)\cap E|^2 \le C|E|^{2(q-1)/q} \tag{15}$$

whenever $E \subset \Omega$ *is a compact subset with* $E\cap Q_0 = \emptyset$, *then* Ω *is a* $(q,2)$*-Poincaré domain.*

REMARKS. (i) From the estimates in (14) we see that (15) holds if $|S(Q)|^{2/q} \le Cj^{-t}$ for some $t > 1$ for all $Q \in \mathcal{W}_j$. On the other hand, the estimate $|S(Q)|^{2/q} \le j^{-1}$ is known to be necessary but, in general, not sufficient for a simply connected Ω to be a $(q,2)$-Poincaré domain (cf. [13]).

(ii) A similar argument was used in [8] for $W^{1,p}$ $(p < n)$ under the growth condition (2).

From Theorem 3.1 and Lemma 2.5 we obtain the next corollary, which proves the first part of Theorem 1.1 in the case $n = 2$.

COROLLARY 3.3. *Let* $\Omega \subset \mathbb{R}^2$ *satisfy the quasihyperbolic growth condition* (4) *with an exponent* $0 < \alpha < \frac{2}{3}$. *Then, for each* $2 \le q < 4\frac{1-\alpha}{\alpha}$, *the embedding* $W^{1,2}(\Omega) \subset L^q(\Omega)$ *is compact.*

In $\mathbb{R}^n$ $(n \ge 3)$ we obtain results similar to Theorem 3.1. Theorem 3.4 also finishes the proof of the first part of Theorem 1.1.

THEOREM 3.4. *Let* $\Omega \subset \mathbb{R}^n$ *satisfy the quasihyperbolic growth condition* (4) *with an exponent* $0 < \alpha < \frac{n}{2n-1}$. *Then* Ω *is a* (q, n)*-Poincaré domain for all*

$$n \le q < \frac{n^2}{n-1} \cdot \frac{1-\alpha}{\alpha},$$

and the embedding $W^{1,n}(\Omega) \subset L^q(\Omega)$ *is compact for all* q *as above.*

Proof. The proof of the (q, n)-Poincaré inequality is the same as the proof of Theorem 3.1 with following minor changes. We use Hölder's inequality with exponent n in (12) and with exponents $\frac{n}{n-1}$ and n in (13). In (14) we take $0 < s \le \frac{1}{n-1}$ and the sum converges when $s > \frac{\alpha}{n(1-\alpha)}$. For these s we obtain

$$|E|^{s(n-1)} \le C \int_\Omega |\nabla u(y)|^n \, dy.$$

We can now choose $s = \frac{n}{n-1}\frac{1}{q}$ and hence, again by results of Maz'ja [10], Ω is a (q, n)-Poincaré domain.

The compactness of the Sobolev embedding follows from the first part of the theorem and Lemma 2.5. □

4. Construction for the Second Part of Theorem 1.1

In this section, we construct examples that prove the second part of Theorem 1.1: the upper bound for the exponent α given in Theorem 1.1 is the best possible. We use the notation $A \lesssim B$ if there exists a constant $C > 0$ such that $A \le CB$ and the notation $A \approx B$ if $A \lesssim B \lesssim A$.

Consider first the case $n = 2$. We show that, given $0 < \alpha \le \frac{2}{3}$, there exists a bounded domain $\Omega \subset \mathbb{R}^2$ such that Ω satisfies the quasihyperbolic growth condition (4) with exponent α; but for $q = 4\frac{1-\alpha}{\alpha}$ the embedding $W^{1,2}(\Omega) \subset L^q(\Omega)$ is not compact. This example proves the essential sharpness of Corollary 3.3. By Lemma 2.5, such a domain cannot be a $(p, 2)$-Poincaré domain for any $p > q$, so this example proves also the essential sharpness of Theorem 3.1, although here the case $q = 4\frac{1-\alpha}{\alpha}$ remains unknown.

For $\lambda > 1$ and $0 < L \le 1$ we define

$$T_L^\lambda = \{(t, y) \in \mathbb{R}^2 : 0 < t \le L,\ 0 < y < t^\lambda\}.$$

We fix $0 < L \le 1$, choose $t_0 = (L/2)$, and define $t_k = t_{k-1} - t_{k-1}^\lambda$ for each $k \ge 1$. Let $\tilde{T}_{t_k}^\lambda$ be a copy of $T_{t_k}^\lambda$ rotated 90° counterclockwise around the point $(t_k, 0)$. Define

$$\hat{D}_L^\lambda = T_L^\lambda \cup \bigcup_{k=0}^\infty \tilde{T}_{t_k}^\lambda$$

and

$$D_j^\lambda = \hat{D}_{2^{-j}}^\lambda + 2^{-j}(-1, 1),$$

where (as usual)

$$A + b = \{a + b \in \mathbb{R}^2 : a \in A\}$$

for $A \subset \mathbb{R}^2$ and $b \in \mathbb{R}^2$. Now we can define our domain Ω^λ by setting

$$\Omega^\lambda = \Omega_0 \cup \bigcup_{j=0}^{\infty} D_j^\lambda,$$

where $\Omega_0 = [0,1] \times [0,2]$. By construction, it is clear that Ω^λ satisfies condition (4) for $\alpha = \frac{\lambda-1}{\lambda}$, so we need only show that the embedding $W^{1,2}(\Omega^\lambda) \subset L^q(\Omega^\lambda)$ is not compact for $\lambda = \frac{4+q}{q}$.

Define functions $u_j : \Omega^\lambda \to \mathbb{R}$ as

$$u_j(x,y) = \begin{cases} 1 & \text{if } x \in D_j^\lambda \cap \{(x,y) \in \mathbb{R}^2 : x \le -2^{-(j+1)}\}, \\ -2^{j+1}x & \text{if } x \in D_j^\lambda \cap \{(x,y) \in \mathbb{R}^2 : -2^{-(j+1)} < x < 0\}, \\ 0 & \text{if } x \in \Omega^\lambda \setminus D_j^\lambda, \end{cases}$$

and denote $E_j^\lambda = \{u_j \equiv 1\}$. Then u_j is a Lipschitz function in D_j^λ for each $j \in \mathbb{N}$ and is locally Lipschitz in Ω^λ; moreover,

$$\int_{\Omega^\lambda} |\nabla u_j|^2 \approx 2^{(1-\lambda)j}. \tag{16}$$

We also have

$$\int_{\Omega^\lambda} |u_j|^2 \approx |E_j^\lambda| \approx \sum_{k=0}^{\infty} |T_{t_k}^\lambda| \approx \sum_{k=0}^{\infty} t_k^{\lambda+1}, \tag{17}$$

where the sequence $(t_k)_k$ now corresponds to $L = 2^{-j}$. For $z_k = (t_k, t_k^\lambda/2)$ we have $k_\Omega(z_k, z_{k+1}) \approx 1$ for all $k \in \mathbb{N}$. Denote $\kappa_0 = k_\Omega(z_0, x_0)$. Then

$$k + \kappa_0 \approx k_\Omega(z_k, x_0) \approx t_k^{1-\lambda}$$

and so, by (17),

$$\int_{\Omega^\lambda} |u_j|^2 \approx \sum_{k=0}^{\infty} (k+\kappa_0)^{(\lambda+1)/(1-\lambda)} \approx \kappa_0^{2/(1-\lambda)} \approx t_0^2 \approx 2^{-2j}. \tag{18}$$

Since $\lambda = \frac{4+q}{q} \le 3$, it follows that $2^{-j} \le 2^{j(1-\lambda)/2}$, and we obtain from (16) and (18) that

$$\|u_j\|_{1,2} = \|u_j\|_2 + \|\nabla u_j\|_2 \lesssim 2^{j(1-\lambda)/2}.$$

Now define $v_j = 2^{j(\lambda-1)/2} u_j$; then the sequence (v_j) is bounded in $W^{1,2}(\Omega^\lambda)$ and, because $q = \frac{4}{\lambda-1}$,

$$\|v_j\|_q \approx 2^{j(\lambda-1)/2} |E_j^\lambda|^{1/q} \approx 2^{j(\lambda-1)/2} 2^{j(-2/q)} = 1$$

for all $j \in \mathbb{N}$. Since the supports of functions v_i and v_j are disjoint when $i \neq j$, this implies that there does not exist a subsequence of (v_j) that converges in $L^q(\Omega^\lambda)$ to a function $v \in L^q(\Omega^\lambda)$, and thus the embedding of $W^{1,2}(\Omega^\lambda)$ to $L^q(\Omega^\lambda)$ is not compact.

We can construct similar examples in $\mathbb{R}^n$, $n \ge 3$. Define

$$T_L^{\lambda,n} = \{(t,y) \in \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} : 0 < t \le L,\ 0 < |y| < t^\lambda,\ y_1 > 0\}$$

and

$$\tilde{T}_{t_k}^{\lambda,n} = \{(t,y) \in \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} : t_{k+1} < t < t_k, \\ 0 < |y| < t_k - (t_k - t)^{1/\lambda},\ y_1 < 0\},$$

and let $\Omega^{\lambda,n}$ be an appropriate union of translations of sets $T_L^{\lambda,n}$ and $\tilde{T}_{t_k}^{\lambda,n}$, following the ideas presented in the case $n = 2$. Then one can show the following: (i) if

$$0 < \alpha \leq \frac{n}{2n-1}$$

then the domain $\Omega^{\lambda,n} \subset \mathbb{R}^n$, where $\lambda = \frac{1}{1-\alpha}$, satisfies the quasihyperbolic boundary condition (4) with exponent α; but (ii) the embedding

$$W^{1,n}(\Omega^{\lambda,n}) \subset L^q(\Omega^{\lambda,n})$$

is not compact for

$$q = \frac{n^2}{n-1} \cdot \frac{1-\alpha}{\alpha}.$$

This proves the essential sharpness of Theorem 3.4 and finishes the proof of Theorem 1.1.

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