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Derived Categories of Toric Varieties

Yujiro Kawamata

1. Introduction

The purpose of this paper is to investigate the structure of the derived category of a toric variety. We shall prove the following result.

Theorem 1.1. *Let X be a projective toric variety with at most quotient singularities, let B be an invariant $\mathbb{Q}$-divisor whose coefficients belong to the set $\left\{\frac{r-1}{r};\ r \in \mathbb{Z}_{>0}\right\}$, and let $\mathcal{X}$ be the smooth Deligne–Mumford stack associated to the pair (X, B) as in* [12]. *Then the bounded derived category of coherent sheaves $D^b(\mathrm{Coh}(\mathcal{X}))$ has a complete exceptional collection consisting of sheaves.*

An object of a triangulated category $a \in T$ is called *exceptional* if

$$\mathrm{Hom}^p(e,e) \cong \begin{cases} \mathbb{C} & \text{for } p = 0, \\ 0 & \text{for } p \neq 0. \end{cases}$$

A sequence of exceptional objects $\{e_1, \dots, e_m\}$ is said to be an *exceptional collection* if

$$\mathrm{Hom}^p(e_i, e_j) = 0 \quad \text{for all } p \text{ and } i > j.$$

The sequence is said to be *strong* if, in addition, $\mathrm{Hom}^p(e_i, e_j) = 0$ for $p \neq 0$ and all i, j; it is called *complete* if T coincides with the smallest triangulated subcategory containing all the e_i (cf. [2]).

It is usually hard to determine the explicit structure of a derived category of a variety. But it is known that some special varieties, such as projective spaces or Grassmann varieties, have strong complete exceptional collections consisting of vector bundles [1; 8; 9; 10]. Such sheaves are useful for further investigation of the derived categories (see e.g. [6; 7; 14; 18]).

We use the minimal model program for toric varieties as developed in [17] (and corrected in [15]) in order to prove the theorem. A special feature of this approach is that, even if we start with the smooth and nonboundary case $B = 0$, we are forced to deal not only with singularities but also with the case $B \neq 0$ because Mori fiber spaces have multiple fibers in general. Thus we are inevitably led to consider the general situation concerning Deligne–Mumford stacks even if we need results for smooth varieties only. The "stacky" sheaves need careful treatment because there exist nontrivial stabilizer groups on the stacks (cf. Remark 5.1).

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We start with the Beilinson theorem for the case of projective spaces and then build up exceptional collections following the procedure of the minimal model program. We use a covering trick to proceed from projective spaces to log Fano varieties (Section 3). Then we proceed by induction on the dimension. First we consider a Mori fiber space in Section 4, where the base space is assumed to have already a complete exceptional collection by the induction hypothesis. Though a Mori fiber space has singular fibers, the associated morphism of stacks is proved to be smooth (Corollary 4.2), and we can define a complete exceptional collection on the total space by using twisted pull-backs. The behavior of derived categories under birational transformations such as divisorial contractions or flips was studied in [12]. We use this result, together with results of Section 4, in Sections 5 and 6. Indeed, the exceptional locus of a divisorial contraction or a flip has the structure of a Mori fiber space itself. The argument of the proof is a generalization of that in [16], which considered the derived categories of projective space bundles and blow-ups of smooth varieties with smooth centers.

2. Toric Minimal Model Program

Let X be a projective toric variety of dimension n that is quasi-smooth (i.e., X has only quotient singularities). We note that a toric variety is quasi-smooth if and only if it is $\mathbb{Q}$-factorial. We consider a $\mathbb{Q}$-divisor B on X whose prime components are invariant divisors with coefficients contained in the set $\left\{\frac{r-1}{r};\ r \in \mathbb{Z}_{>0}\right\}$. Let $\mathcal{X}$ be the smooth Deligne–Mumford stack associated to the pair (X, B) with the natural morphism $\pi_X : \mathcal{X} \to X$ as in [12].

The pair (X, B) has only log terminal singularities. We work on the log minimal model program for (X, B) (see [15; 17]). Let $\phi\colon X \to Y$ be a primitive contraction morphism corresponding to an extremal ray with respect to $K_X + B$. Then Y is also a projective toric variety and ϕ is a toric morphism. If ϕ is a birational morphism, then the boundary divisor C on Y is defined to be the strict transform of B. Otherwise, it will be defined later.

Let N_X be the lattice of 1-parameter subgroups of the torus acting on X, and let Δ_X be the fan in $N_{X,\mathbb{R}}$ corresponding to X. Let $w = \langle v_3, \dots, v_{n+1}\rangle$ be a wall in Δ_X corresponding to an extremal rational curve, where the v_i are primitive vectors in N_X on the edges of w. Let $v_1, v_2 \in N_X$ be two primitive vectors, each of which forms an n-dimensional cone in Δ_X when combined with w. Let D_i be the prime divisors on X corresponding to the v_i, and let $\mathcal{D}_i$ be the corresponding prime divisors on $\mathcal{X}$. Let $\frac{r_i-1}{r_i}$ be the coefficients of the D_i in B. Then the natural morphism $\pi_X : \mathcal{X} \to X$ ramifies along D_i such that $\pi_X^* D_i = r_i \mathcal{D}_i$.

The contraction morphism is described by the equation

$$a_1 v_1 + \cdots + a_{n+1} v_{n+1} = 0, \tag{2.1}$$

where the a_i are integers such that

$$
\begin{gathered}
(a_1, \dots, a_{n+1}) = 1; \\
\begin{aligned}
a_i &> 0 \quad \text{for } 1 \le i \le \alpha, \\
a_i &= 0 \quad \text{for } \alpha + 1 \le i \le \beta, \\
a_i &< 0 \quad \text{for } \beta + 1 \le i \le n + 1;
\end{aligned} \\
2 \le \alpha \le \beta \le n + 1.
\end{gathered}
$$

Note that we use slightly different notation from the literature in which the a_i are rational numbers. Since $K_X + B$ is negative for ϕ, we have

$$\sum_{i=1}^{n+1} \frac{a_i}{r_i} > 0.$$

The following lemma asserts that the set of integers $\{a_i\}$ is *well prepared.*

LEMMA 2.1. *Let i_0 be an integer such that $1 \le i_0 \le \alpha$ or $\beta + 1 \le i_0 \le n + 1$. Then the set of $n + \alpha - \beta$ integers a_i, where $1 \le i \le \alpha$ and $\beta + 1 \le i \le n + 1$ except one $i = i_0$, is coprime for any i_0.*

Proof. Let c be the largest common divisor of these integers, and set $a_i = c\bar{a}_i$. Then we have $(a_{i_0}, c) = 1$. Let x, y be integers such that $a_{i_0}x + cy = 1$. Then

$$\frac{1}{c} v_{i_0} = \frac{1}{c} v_{i_0} - \frac{x}{c} \sum_i a_i v_i = -x \sum_{i \ne i_0} \bar{a}_i v_i + y v_{i_0} \in N_X;$$

hence $c = 1$. □

One of the following cases occurs.

(1) Mori fiber space: $\beta = n + 1$; then $\dim Y = n + 1 - \alpha$.
(2) Divisorial contraction: $\beta = n$.
(3) Small contraction: $\beta < n$.

We treat these cases separately in the following sections. According to the minimal model program, Theorem 1.1 follows from the combination of Corollaries 4.4, 5.3, and 6.2.

3. Fano Case

We start with the case where (X, B) is a log $\mathbb{Q}$-Fano variety with $\rho = 1$. We have $\alpha = \beta = n + 1$. In this case, there are no edges in Δ_X besides $\mathbb{R}_{\ge 0} v_i$. Such a variety X is not necessarily a weighted projective space, as noted in [15]. But it is covered by a weighted projective space via a finite morphism that is étale in codimension 1. Indeed, a weighted projective space is characterized by the property that the divisor class group has no torsion (cf. Lemma 3.1).

Let $N_{X'}$ be the sublattice of N_X generated by the v_i. By equation (2.1), the toric variety X' corresponding to the fan Δ_X in $N_{X',\mathbb{R}}$ (with the lattice $N_{X'}$) is isomorphic to the weighted projective space $\mathbb{P}(a_1, \dots, a_{n+1})$. The natural morphism

$\bar{\sigma}_1 \colon X' \to X$ is étale in codimension 1. Let D'_i be the prime divisors on X' corresponding to the v_i, let $\mathcal{X}'$ be the smooth Deligne–Mumford stack associated to the pair $\left(X', \sum_i \frac{r_i-1}{r_i} D'_i\right)$ with the projections $\pi_{X'} \colon \mathcal{X}' \to X'$ and $\sigma_1 \colon \mathcal{X}' \to \mathcal{X}$, and let $\mathcal{D}'_i$ be the prime divisors on $\mathcal{X}'$ such that $\pi^*_{X'} D'_i = r_i \mathcal{D}'_i$.

Let r be a positive integer such that $a_i r$ is divisible by r_i for any i, and let $N_{\tilde{X}}$ be the sublattice of N_X generated by the vectors $\tilde{v}_i = a_i r v_i$. We have

$$\sum_{i=1}^{n+1} \tilde{v}_i = 0,$$

and the toric variety $\tilde{X}$ corresponding to the fan Δ_X in $N_{\tilde{X},\mathbb{R}}$ (with the lattice $N_{\tilde{X}}$) is isomorphic to the projective space $\mathbb{P}^n$. Let $\sigma_2 \colon \tilde{X} \to \mathcal{X}'$ be the natural morphism, and set $\sigma = \sigma_1 \circ \sigma_2$, $\bar{\sigma}_2 = \pi_{X'} \circ \sigma_2$, and $\bar{\sigma} = \pi_X \circ \sigma$. Let $\tilde{D}_i$ be the prime divisors on $\tilde{X}$ corresponding to the vectors $\tilde{v}_i$. Moreover, let $N_{X''}$ be the sublattice of N_X generated by the $r_i v_i$. Observe that the vectors $r_i v_i$ are not necessarily primitive in this lattice.

LEMMA 3.1. (1) *A divisor $\sum_i k_i \mathcal{D}_i$ is torsion in the divisor class group of $\mathcal{X}$ if and only if*

$$\sum_i \frac{a_i k_i}{r_i} = 0.$$

(2) *The group of torsion divisor classes on $\mathcal{X}$ is dual to the quotient group $N_X/N_{X''}$.*

(3) *The group of torsion Weil divisor classes on X is dual to the quotient group $N_X/N_{X'}$.*

Proof. (1) A divisor $\sum_i k_i \mathcal{D}_i$ is linearly equivalent to 0 if and only if there exists an $m \in M_X = N_X^*$ such that $(m, r_i v_i) = k_i$, because the morphism $\pi_X \colon \mathcal{X} \to X$ is birational. Thus $\sum_i k_i \mathcal{D}_i$ is torsion if and only if there exists $m \in M_{X,\mathbb{R}}$ such that $(m, r_i v_i) = k_i$. The latter condition is equivalent to the equality $\sum_i \frac{a_i k_i}{r_i} = 0$.

(2) For $m \in M_{X,\mathbb{R}}$, we have $(m, r_i v_i) \in \mathbb{Z}$ for all v_i if and only if $m \in M_{X''} = N^*_{X''}$. Therefore, the group of torsion divisor classes is isomorphic to $M_{X''}/M_X$.

(3) is a particular case of (2). □

REMARK 3.2. If $B = 0$ (i.e., if $r_i = 1$ for all i), then the divisor class groups of X and $\mathcal{X}$ are isomorphic.

EXAMPLE 3.3. Torsion divisor classes correspond to étale coverings of the stack. For example, let $X = \mathbb{P}^n$ be the projective space and let $\mathcal{X}$ be the smooth stack associated to the pair $\left(X, \frac{r-1}{r} \sum_{i=1}^{n+1} H_i\right)$, where the H_i are coordinate hyperplanes. Let $\mathcal{H}_i$ be the prime divisors on $\mathcal{X}$ above the H_i so that $\pi_X^* H_i = r\mathcal{H}_i$ for the projection $\pi_X \colon \mathcal{X} \to X$.

Let $\tilde{X} = \mathbb{P}^n$ be another projective space, and let $\sigma \colon \tilde{X} \to \mathcal{X}$ be the Kummer covering with Galois group $(\mathbb{Z}/r)^n$ obtained by taking the rth roots of the coordinates. Then σ is étale, and we have

$$\sigma_*\mathcal{O}_{\tilde{X}} \cong \bigoplus_{l_1,\dots,l_n=0}^{r-1} \mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^n l_i\mathcal{H}_i + \left(-\sum_{i=1}^n l_i\right)\mathcal{H}_{n+1}\right).$$

We note that the direct summands are invertible sheaves on $\mathcal{X}$ corresponding to the torsion divisor classes. In the usual language, if we set $\bar{\sigma} = \pi_X \circ \sigma : \tilde{X} \to X$ then

$$\bar{\sigma}_*\mathcal{O}_{\tilde{X}} \cong \bigoplus_{l_1,\dots,l_n=0}^{r-1} \mathcal{O}_X\left(-\ulcorner \frac{\sum_{i=1}^n l_i}{r}\urcorner H_{n+1}\right),$$

because

$$\pi_{X*}\mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^n l_i\mathcal{H}_i + \left(-\sum_{i=1}^n l_i\right)\mathcal{H}_{n+1}\right) = \mathcal{O}_X\left(-\ulcorner \frac{\sum_{i=1}^n l_i}{r}\urcorner H_{n+1}\right).$$

More generally, we have

$$\sigma_*\mathcal{O}_{\tilde{X}}(-p) \cong \bigoplus_{l_1,\dots,l_n=0}^{r-1} \mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^n l_i\mathcal{H}_i + \left(-p-\sum_{i=1}^n l_i\right)\mathcal{H}_{n+1}\right)$$

and

$$\bar{\sigma}_*\mathcal{O}_{\tilde{X}}(-p) \cong \bigoplus_{l_1,\dots,l_n=0}^{r-1} \mathcal{O}_X\left(-\ulcorner \frac{p+\sum_{i=1}^n l_i}{r}\urcorner H_{n+1}\right).$$

For example, the direct images of the sheaves $\mathcal{O}_{\tilde{X}}(-p)$ for $0 \le p \le n$, which generate the derived category $D^b(\mathrm{Coh}(\tilde{X}))$ (see [1]), have direct summands of the form $\mathcal{O}_X(-q)$ for $0 \le q \le n$ (cf. [11]).

LEMMA 3.4. (1) *Let $G_1 = N_X/N_{X'}$ be the Galois group of the covering $\bar{\sigma}_1 \colon X' \to X$. Then we have the following decomposition into eigenspaces with respect to the G_1-action*:

$$\sigma_{1*}\mathcal{O}_{\mathcal{X}'}\left(\sum_i d_i\mathcal{D}'_i\right) \cong \bigoplus_k \mathcal{O}_{\mathcal{X}}\left(\sum_i (d_i + k_i r_i)\mathcal{D}_i\right),$$

where the sequences of integers $k = (k_i)$ in the summation are determined by the equation $k_i = (m, v_i)$ for the representatives m of the group of torsion Weil divisor classes $M_{X'}/M_X$ of X.

(2) *Let $G_2 = N_{X'}/N_{\tilde{X}}$ be the Galois group of the covering $\bar{\sigma}_2 \colon \tilde{X} \to X'$. Then there is the following decomposition into eigenspaces with respect to the G_2-action*:

$$\sigma_{1*}\mathcal{O}_{\tilde{X}}(-p) \cong \bigoplus_l \mathcal{O}_{\mathcal{X}'}\left(\sum_{1\le i\le n} \llcorner \frac{l_i r_i}{a_i r}\lrcorner \mathcal{D}'_i + \llcorner \frac{(l_{n+1}-p)r_{n+1}}{a_{n+1}r}\lrcorner \mathcal{D}'_{n+1} - \frac{1}{r}\sum_{i=1}^{n+1} l_i\right),$$

where the sequences of integers $l = (l_i)$ in the summation run under the conditions that $0 \le l_i < a_i r$ and $r \mid \sum_{i=1}^{n+1} l_i$.

(3) *Let $G = N_X/N_{\tilde{X}}$ be the Galois group of the covering $\bar{\sigma} \colon \tilde{X} \to X$. Then we have the following decomposition into eigenspaces with respect to the G-action*:

$$\sigma_*\mathcal{O}_{\tilde{X}}(-p) \cong \bigoplus_k \mathcal{O}_{\mathcal{X}}\left(\sum_{1\le i\le n} \llcorner \frac{k_i r_i}{a_i r} \lrcorner \mathcal{D}_i + \llcorner \frac{(k_{n+1}-p)r_{n+1}}{a_{n+1}r} \lrcorner \mathcal{D}_{n+1}\right),$$

where the sequences of integers $k = (k_i)$ satisfy the equation

$$\sum_{i=1}^{n+1} k_i = 0.$$

Proof. (1) is clear.

(2) We have an exact sequence

$$0 \to \mathbb{Z}/r \to \bigoplus_{i=1}^{n+1} \mathbb{Z}/a_i r \to G_2 \to 0,$$

where 1 in the first term is sent to (a_i) in the second term. Thus

$$G_2^* \cong \left\{(l_i) \in \bigoplus_{i=1}^{n+1} \mathbb{Z}/a_i r;\ \sum_{i=1}^{n+1} l_i = 0 \bmod r\right\}.$$

We have $\sigma_1^*\mathcal{D}_i' = \frac{a_i r}{r_i}\tilde{D}_i$ for the prime divisor $\tilde{D}_i$ on $\tilde{X}$ above $\mathcal{D}_i'$. Since $\frac{a_i l_i}{a_i r} = \frac{l_i}{r}$, we obtain the formula. We remark that $\mathcal{O}_{\mathcal{X}'}(1)$ is well-defined because $\mathcal{X}'$ has no torsion divisor classes.

(3) Combining (1) and (2) yields

$$\sigma_*\mathcal{O}_{\tilde{X}}(-p)$$
$$\cong \bigoplus_{k,l} \mathcal{O}_{\mathcal{X}}\left(\sum_{1\le i\le n}\left(k_i r_i + \llcorner \frac{l_i r_i}{a_i r} \lrcorner\right)\mathcal{D}_i + \left(k_{n+1}r_{n+1} + \llcorner \frac{(l_{n+1}-p)r_{n+1}}{a_{n+1}r} \lrcorner\right)\mathcal{D}_{n+1}\right),$$

where the $k = (k_i)$ satisfy

$$\sum_{i=1}^{n+1} a_i k_i = -\frac{1}{r}\sum_{i=1}^{n+1} l_i$$

and where the summation on $l = (l_i)$ is under the restriction that $0 \le l_i < a_i r$ and $r \mid \sum_i l_i$. If we replace $a_i r k_i + l_i$ by k_i, then we obtain our assertion. □

THEOREM 3.5. (1) *An invertible sheaf $\mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^{n+1} k_i\mathcal{D}_i\right)$ on $\mathcal{X}$ is an exceptional object for any sequence of integers $k = (k_i)$ for $1 \le i \le n+1$.*

(2) *If $\sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} > \sum_{i=1}^{n+1} \frac{a_i k_i'}{r_i} > \sum_{i=1}^{n+1} \frac{a_i(k_i-1)}{r_i}$, then*

$$\operatorname{Hom}^q\left(\mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^{n+1} k_i\mathcal{D}_i\right), \mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^{n+1} k_i'\mathcal{D}_i\right)\right) = 0$$

for all q, where $k' = (k_i')$ is another sequence of integers.

(3) *If $\sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} = \sum_{i=1}^{n+1} \frac{a_i k_i'}{r_i}$ and $\sum_{i=1}^{n+1} k_i\mathcal{D}_i \not\sim \sum_{i=1}^{n+1} k_i'\mathcal{D}_i$, then*

$$\operatorname{Hom}^q\left(\mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^{n+1} k_i\mathcal{D}_i\right), \mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^{n+1} k_i'\mathcal{D}_i\right)\right) = 0$$

for all q.

(4) *If* $\sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} \le \sum_{i=1}^{n+1} \frac{a_i k'_i}{r_i}$, *then*

$$\mathrm{Hom}^q\left(\mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^{n+1} k_i \mathcal{D}_i\right), \mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^{n+1} k'_i \mathcal{D}_i\right)\right) = 0$$

for $q \neq 0$.

(5) *The set of invertible sheaves* $\mathcal{O}_{\mathcal{X}}\big(\sum_{i=1}^{n+1} k_i \mathcal{D}_i\big)$ *for*

$$0 \ge \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} > -\sum_{i=1}^{n+1} \frac{a_i}{r_i}$$

generates the triangulated category $D^b(\mathrm{Coh}(\mathcal{X}))$.

Proof. The canonical divisor of $\mathcal{X}$ is given by

$$\omega_{\mathcal{X}} \cong \pi_X^* \omega_X \otimes \mathcal{O}_{\mathcal{X}}\left(\sum_i (r_i - 1)\mathcal{D}_i\right) \cong \mathcal{O}_{\mathcal{X}}\left(-\sum_i \mathcal{D}_i\right).$$

An invertible sheaf $\mathcal{O}_{\mathcal{X}}\big(\sum_i k_i \mathcal{D}_i\big)$ is ample if and only if $\sum_i \frac{a_i k_i}{r_i} > 0$. Therefore, assertions (1)–(4) follow immediately from the vanishing theorem [13].

(5) follows from a similar generalization of the Beilinson resolution theorem [1] as in [11, Sec. 5]. Indeed, the integral functor corresponding to an object e on $\mathcal{X} \times \mathcal{X}$ given by

$$e = \{0 \to [\sigma_* \mathcal{O}_{\tilde{X}}(-n) \boxtimes \sigma_* \Omega^n_{\tilde{X}}(n)]^G \to \cdots \\ \to [\sigma_* \mathcal{O}_{\tilde{X}}(-1) \boxtimes \sigma_* \Omega^1_{\tilde{X}}(1)]^G \to [\sigma_* \mathcal{O}_{\tilde{X}} \boxtimes \sigma_* \mathcal{O}_{\tilde{X}}]^G \to 0\}$$

is isomorphic to the identity functor, where the group G acts diagonally on the tensor products. Thus the derived category $D^b(\mathrm{Coh}(\mathcal{X}))$ is generated by the direct summands of the sheaves $\sigma_* \mathcal{O}_{\tilde{X}}(-p)$ for $0 \le p \le n$ given in Lemma 3.4(3).

Since $\sum_i k_i = 0$, it follows that

$$\sum_{i=1}^{n} \frac{a_i \frac{k_i r_i}{a_i r}}{r_i} + \frac{a_{n+1} \frac{(k_{n+1} - p) r_{n+1}}{a_{n+1} r}}{r_{n+1}} = -\frac{p}{r}.$$

Then we calculate

$$0 \ge \sum_{i=1}^{n} \frac{a_i \llcorner \frac{k_i r_i}{a_i r} \lrcorner}{r_i} + \frac{a_{n+1} \llcorner \frac{(k_{n+1} - p) r_{n+1}}{a_{n+1} r} \lrcorner}{r_{n+1}}$$

$$\ge -\sum_{i=1}^{n+1} \frac{a_i}{r_i}\left(1 - \frac{1}{s_i}\right) - \frac{n}{r} \ge -\sum_{i=1}^{n+1} \frac{a_i}{r_i} + \frac{1}{r},$$

where we set $a_i r = r_i s_i$ for some integers s_i. □

COROLLARY 3.6. *Let* (X, B) *be a* $\mathbb{Q}$*-factorial projective toric variety such that* $-(K_X + B)$ *is ample,* $\rho(X) = 1$, *and the coefficients of* B *belong to the set* $\left\{\frac{r-1}{r};\ r \in \mathbb{Z}_{>0}\right\}$. *Let* $\mathcal{X}$ *be the smooth Deligne–Mumford stack associated to the pair*

(X, B). *Then the derived category* $D^b(\mathrm{Coh}(\mathcal{X}))$ *has a strong complete exceptional collection consisting of invertible sheaves.*

Proof. There is a finite number of isomorphism classes of the set of invertible sheaves $\mathcal{O}_{\mathcal{X}}\big(\sum_i k_i\mathcal{D}_i\big)$ for $0 \geq \sum_i \frac{a_i k_i}{r_i} > -\sum_i \frac{a_i}{r_i}$. □

4. Mori Fiber Space

We consider a toric Mori fiber space $\phi\colon X \to Y$ with respect to $K_X + B$. This fibration is not necessarily locally trivial, because there may be multiple fibers. Yet it becomes locally trivial after taking coverings, as we now show.

LEMMA 4.1. *Let* Y_0 *be an invariant open affine subset of* Y, *and let* $X_0 = \phi^{-1}(Y_0)$. *Then there exist finite surjective toric morphisms* $\tau_{X_0}\colon X_0' \to X_0$ *and* $\tau_{Y_0}\colon Y_0'' \to Y_0$, *with a toric surjective morphism* $\phi_0'\colon X_0' \to Y_0''$, *that satisfy the following conditions*:

(1) τ_{X_0} *is étale in codimension* 1;
(2) $\phi \circ \tau_{X_0} = \tau_{Y_0} \circ \phi_0'$;
(3) X_0' *is isomorphic to the direct product of* Y_0'' *and a weighted projective space, and* ϕ_0' *corresponds to the projection.*

Proof. Let N_Y be the lattice of 1-parameter subgroups of the torus for Y, and let Δ_Y be the fan in $N_{Y,\mathbb{R}}$ corresponding to Y. We take the wall w described in the formula (2.1) such that the corresponding extremal rational curve is contained in X_0. We have

$$N_Y = N_X \Big/ \Big(\bigoplus_{i=1}^{\alpha} \mathbb{R}v_i \cap N_X \Big).$$

Let $h\colon N_X \to N_Y$ be the projection. We write $h(v_i) = s_i\bar{v}_i$ for primitive vectors $\bar{v}_i$ in N_Y and positive integers s_i, where $\alpha + 1 \leq i \leq n + 1$. Then these $\bar{v}_i$ give the set of edges of an $(n + 1 - \alpha)$-dimensional cone σ_0 in Δ_Y corresponding to Y_0. Let E_i be the prime divisors on Y corresponding to the vectors $\bar{v}_i$. Now X_0 coincides with the toric variety corresponding to the fan $\Delta_X \cap h^{-1}(\sigma_0)$ in $N_{X,\mathbb{R}}$.

Let $N_{X_0'}$ be the sublattice of N_X generated by the v_i for $1 \leq i \leq n + 1$, and let $N_{Y_0'}$ (resp. $N_{Y_0''}$) of N_Y be generated by the $\bar{v}_i$ (resp. $h(v_i)$) for $\alpha + 1 \leq i \leq n + 1$. Let X_0' be the toric variety corresponding to the fan $\Delta_X \cap h^{-1}(\sigma_0)$ in $N_{X_0',\mathbb{R}}$, and let Y_0' (resp. Y_0'') be the one corresponding to the cone σ_0 in $N_{Y_0',\mathbb{R}}$ (resp. $N_{Y_0'',\mathbb{R}}$). Then the natural morphisms $\tau_{X_0}\colon X_0' \to X$ and $\tau_{Y_0}'\colon Y_0' \to Y$ are étale in codimension 1, while $\tau_{Y_0}''\colon Y_0'' \to Y_0'$ is not in general. Since

$$\sum_{i=1}^{\alpha} a_i v_i = 0,$$

it follows that X_0' is isomorphic to the product of Y_0'' with a weighted projective space $\mathbb{P}(a_1, \ldots, a_\alpha)$. □

We define the boundary $\mathbb{Q}$-divisor C on Y by assigning coefficients $\frac{r_i s_i - 1}{r_i s_i}$ to the irreducible components E_i, where the s_i are as defined in the proof of Lemma 4.1. We note that, even if we start with the nonboundary case $B = 0$, the naturally defined boundary divisor C on Y is nonzero in general because there may be multiple fibers for ϕ. Let $\mathcal{Y}$ be the smooth Deligne–Mumford stack associated to the pair (Y, C). Then Lemma 4.1 implies the following corollary.

COROLLARY 4.2. *The natural morphism $\psi : \mathcal{X} \to \mathcal{Y}$ is smooth.*

THEOREM 4.3. (1) *The functor $\psi^* : D^b(\mathrm{Coh}(\mathcal{Y})) \to D^b(\mathrm{Coh}(\mathcal{X}))$ is fully faithful. Let $D^b(\mathrm{Coh}(\mathcal{Y}))_k$ denote the full subcategory of $D^b(\mathrm{Coh}(\mathcal{X}))$ defined by*

$$D^b(\mathrm{Coh}(\mathcal{Y}))_k = \psi^* D^b(\mathrm{Coh}(\mathcal{Y})) \otimes \mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^{\alpha} k_i \mathcal{D}_i\right)$$

for a sequence of integers $k = (k_i)$ with $1 \le i \le \alpha$.

(2) *If $\sum_{i=1}^{\alpha} \frac{a_i k_i}{r_i} > \sum_{i=1}^{\alpha} \frac{a_i k_i'}{r_i} > \sum_{i=1}^{\alpha} \frac{a_i (k_i - 1)}{r_i}$, then*

$$\mathrm{Hom}^q(D^b(\mathrm{Coh}(\mathcal{Y}))_k, D^b(\mathrm{Coh}(\mathcal{Y}))_{k'}) = 0$$

for all q, where $k' = (k_i')$ is another sequence of integers.

(3) *If $\sum_{i=1}^{\alpha} \frac{a_i k_i}{r_i} = \sum_{i=1}^{\alpha} \frac{a_i k_i'}{r_i}$ and $\mathcal{O}_{\mathcal{X}}\big(\sum_{i=1}^{\alpha} (k_i - k_i')\mathcal{D}_i\big) \notin \psi^* D^b(\mathrm{Coh}(\mathcal{Y}))$, then*

$$\mathrm{Hom}^q(D^b(\mathrm{Coh}(\mathcal{Y}))_k, D^b(\mathrm{Coh}(\mathcal{Y}))_{k'}) = 0$$

for all q.

(4) *The set of subcategories $D^b(\mathrm{Coh}(\mathcal{Y}))_k$ for*

$$0 \ge \sum_{i=1}^{\alpha} \frac{a_i k_i}{r_i} > -\sum_{i=1}^{\alpha} \frac{a_i}{r_i}$$

generates the triangulated category $D^b(\mathrm{Coh}(\mathcal{X}))$.

Proof. (1) By [4] or [5], it is sufficient to prove the following statement: If A and B are skyscraper sheaves on $\mathcal{Y}$ of length 1, then the natural homomorphism $\mathrm{Hom}^p(A, B) \to \mathrm{Hom}^p(\psi^* A, \psi^* B)$ is bijective. This follows from the facts that (a) X_0' is isomorphic to the product of Y_0'' with a weighted projective space $\mathbb{P}(a_1, \dots, a_\alpha)$ and (b) the natural homomorphism of Galois groups $N_X/N_{X_0'} \to N_Y/N_{Y_0''}$ is surjective .

For (2) and (3), we use a spectral sequence

$$E_2^{p,q} = H^p\left(\mathcal{Y}, \mathcal{H}om(A, B) \otimes R^q \psi_* \mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^{\alpha} k_i \mathcal{D}_i\right)\right) \Longrightarrow \mathrm{Hom}^{p+q}\left(\psi^* A, \psi^* B \otimes \mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^{\alpha} k_i \mathcal{D}_i\right)\right)$$

for invertible sheaves A, B on $\mathcal{Y}$. The direct image sheaves vanish in our case since the relative canonical divisor for ψ is given by

$$\omega_{\mathcal{X}/\mathcal{Y}} \cong \mathcal{O}_{\mathcal{X}}\Big(-\sum_{i=1}^{\alpha} \mathcal{D}_i\Big)$$

and since an invertible sheaf $\mathcal{O}_{\mathcal{X}}\big(\sum_{i=1}^{\alpha} k_i\mathcal{D}_i\big)$ is ψ-ample if and only if $\sum_{i=1}^{\alpha} \frac{a_i k_i}{r_i} > 0$ [13].

(4) In general, a full triangulated subcategory $\mathcal{B}$ of a triangulated category $\mathcal{A}$ is said to be *right* (resp. *left*) *admissible* if $\mathcal{A}$ is generated by $\mathcal{B}$ and $\mathcal{B}^{\perp}$ (resp. $\mathcal{B}$ and ${}^{\perp}\mathcal{B}$), where $\mathcal{B}^{\perp}$ (resp. ${}^{\perp}\mathcal{B}$) denotes the right (resp. left) orthogonal complement of $\mathcal{B}$ in $\mathcal{A}$ [3]. The triangulated subcategory $\mathcal{T}$ of $D^b(\mathrm{Coh}(\mathcal{X}))$ generated by the subcategories $D^b(\mathrm{Coh}(\mathcal{Y}))_k$ is admissible by [3, 1.12, 2.6, 2.11]. Therefore, it is sufficient to prove that the left orthogonal ${}^{\perp}\mathcal{T}$ consists of zero objects.

Let A be an arbitrary skyscraper sheaf of length 1 on $\mathcal{X}$ supported at a point P. Then, by Theorem 3.6, there exists a skyscraper sheaf B of length 1 on $\mathcal{Y}$ supported at $Q = \phi(P)$ such that A is contained in the subcategory generated by the sheaves of the form $\psi^*B \otimes \mathcal{O}_{\mathcal{X}}\big(\sum_{i=1}^{\alpha} k_i\mathcal{D}_i\big)$ for $0 \ge \sum_{i=1}^{\alpha} \frac{a_i k_i}{r_i} > -\sum_{i=1}^{\alpha} \frac{a_i}{r_i}$. Therefore, A is contained in $\mathcal{T}$. Hence ${}^{\perp}\mathcal{T} = 0$, because such A span $D^b(\mathrm{Coh}(\mathcal{X}))$ ([4] or [5]). □

COROLLARY 4.4. *Assume that $D^b(\mathrm{Coh}(\mathcal{Y}))$ has a complete exceptional collection consisting of sheaves. Then so has $D^b(\mathrm{Coh}(\mathcal{X}))$.*

5. Divisorial Contraction

We consider a toric divisorial contraction $\phi\colon X \to Y$. Here $K_X + B$ is negative for ϕ, and $C = \phi_* B$ is the strict transform. Let D be the exceptional divisor of the contraction. Then the restriction $\bar{\phi}\colon D \to F = \phi(D)$ is a Mori fiber space, which was treated in the previous section.

Let $\mathcal{Y}$ be the stack associated to the pair (Y, C). We note that there is no morphism of stacks from $\mathcal{X}$ to $\mathcal{Y}$ in general. But there is still a fully faithful functor $\Phi\colon D^b(\mathrm{Coh}(\mathcal{Y})) \to D^b(\mathrm{Coh}(\mathcal{X}))$ by [12, Thm. 4.2(2)]. Indeed, let $\mathcal{W}$ be the normalization of the fiber product $\mathcal{X} \times_Y \mathcal{Y}$, and let $\mu\colon \mathcal{W} \to \mathcal{X}$ and $\nu\colon \mathcal{W} \to \mathcal{Y}$ be the projections. Then $\Phi = \mu_* \circ \nu^*$ is fully faithful. We regard $D^b(\mathrm{Coh}(\mathcal{Y}))$ as a full subcategory of $D^b(\mathrm{Coh}(\mathcal{X}))$ through this functor.

Let $E_i = \phi_* D_i$ be the prime divisors on Y corresponding to the edges v_i for $1 \le i \le n$. These E_i for $1 \le i \le \alpha$ are the divisors that contain the center F of the blow-up ϕ, and $D = D_{n+1}$ is the exceptional divisor. Let $\mathcal{E}_i$ be the prime divisors on $\mathcal{Y}$ corresponding to the E_i. The following formula is proved in the proof of [12, Thm. 4.2(2)]:

$$\Phi\Big(\mathcal{O}_{\mathcal{Y}}\Big(\sum_{i=1}^{n} k_i\mathcal{E}_i\Big)\Big) \cong \mathcal{O}_{\mathcal{X}}\Big(\sum_{i=1}^{n+1} k_i\mathcal{D}_i\Big),$$

$$k_{n+1} = \llcorner \frac{r_{n+1}}{b_{n+1}} \sum_{i=1}^{n} \frac{a_i k_i}{r_i} \lrcorner$$

for any integers k_i with $1 \le i \le n$, where we put $b_{n+1} = -a_{n+1} > 0$.

Let r be a positive integer such that $a_i r$ is divisible by r_i for $1 \le i \le n+1$. We set

$$|a_i| \cdot r = r_i s_i.$$

Let $s = (s_1, \dots, s_{n+1})$ be the greatest common divisor, and set $s_i = s\bar{s}_i$. Then the fractional part of the rational number

$$\frac{r_{n+1}}{b_{n+1}} \sum_{i=1}^{n} \frac{a_i k_i}{r_i} = \frac{\sum_{i=1}^{n} k_i \bar{s}_i}{\bar{s}_{n+1}}$$

can take an arbitrary value in the set $\left\{0, \frac{1}{\bar{s}_{n+1}}, \dots, \frac{\bar{s}_{n+1}-1}{\bar{s}_{n+1}}\right\}$ when we vary the sequence k, because $(\bar{s}_1, \dots, \bar{s}_{n+1}) = 1$.

The Mori fiber space $\bar{\phi}\colon D \to F$ is described as follows. The lattice of 1-parameter subgroups for D is given by $\bar{N} = N_X/\mathbb{Z}v_{n+1}$. We write $v_i \bmod \mathbb{Z}v_{n+1} = t_i\bar{v}_i$ for $1 \le i \le n$, where the t_i are positive integers and the $\bar{v}_i$ are primitive vectors in $\bar{N}$. Let $t = (a_1t_1, \dots, a_nt_n)$ be the greatest common divisor, and set $a_it_i = t\bar{a}_i$; then

$$\bar{a}_1\bar{v}_1 + \cdots + \bar{a}_n\bar{v}_n = 0.$$

We define a $\mathbb{Q}$-divisor $\bar{B}$ on D by putting coefficients $\frac{r_it_i-1}{r_it_i}$ to the prime divisors $\bar{D}_i = D_i \cap D$ for $1 \le i \le n$. We also define a $\mathbb{Q}$-divisor $\bar{C}$ on the base space of the Mori fiber space F using $\bar{B}$ as in the previous section. Let $\mathcal{D}$ and $\mathcal{F}$ be the smooth stacks associated to the pairs $(D, \bar{B})$ and $(F, \bar{C})$, respectively. Then there are induced morphisms of stacks $\bar{\psi}: \mathcal{D} \to \mathcal{F}$ and $j\colon \mathcal{D} \to \mathcal{X}$. Let $\bar{\mathcal{D}}_i$ be the prime divisors on $\mathcal{D}$ corresponding to the $\bar{D}_i$ for $1 \le i \le n$. Then we have

$$j^*\mathcal{O}_{\mathcal{X}}(\mathcal{D}_i) \cong \mathcal{O}_{\mathcal{D}}(\bar{\mathcal{D}}_i).$$

We note that $D_i|_D = \frac{1}{t_i}\bar{D}_i$ in the usual language.

REMARK 5.1. If $r_{n+1} > 1$, then the action of the stabilizer group at the generic point of $\mathcal{D}_{n+1}$ is nontrivial. Hence $j^*\mathcal{O}_{\mathcal{X}}(k\mathcal{D}_{n+1}) = 0$ on $\mathcal{D}$ if k is not divisible by r_{n+1}. Indeed, we have

$$\mathrm{Hom}(j^*\mathcal{O}_{\mathcal{X}}(k\mathcal{D}_{n+1}), A) \cong \mathrm{Hom}(\mathcal{O}_{\mathcal{X}}(k\mathcal{D}_{n+1}), j_*A) \cong 0$$

for any sheaf A on $\mathcal{D}$ in this case.

For example, let X be an affine line with a point P, and let $\mathcal{X}$ be the stack associated to the pair $\left(X, \frac{r-1}{r}P\right)$ with a point $\mathcal{P}$ above P. Then we have $j^*\mathcal{O}_{\mathcal{X}}(k\mathcal{P}) = 0$ if k is not divisible by r, where $j\colon P \to \mathcal{X}$ is the natural morphism. From a resolution

$$0 \to \mathcal{O}_{\mathcal{X}}((k-1)\mathcal{P}) \to \mathcal{O}_{\mathcal{X}}(k\mathcal{P}) \to \mathcal{O}_{\mathcal{P}}(k\mathcal{P}) \to 0$$

it follows that $L_q j^*\mathcal{O}_{\mathcal{P}}(k\mathcal{P})$ is isomorphic to $\mathcal{O}_P$ if $q = 0$ and $k \equiv 0$ modulo r or if $q = 1$ and $k \equiv 1$ modulo r, and is zero otherwise. Thus

$$\begin{aligned}\mathrm{Hom}^q(\mathcal{O}_{\mathcal{P}}(k\mathcal{P}), \mathcal{O}_{\mathcal{P}}) &\cong \mathrm{Hom}^q(\mathcal{O}_{\mathcal{P}}(k\mathcal{P}), j_*\mathcal{O}_P) \\ &\cong \mathrm{Hom}^q(Lj^*\mathcal{O}_{\mathcal{P}}(k\mathcal{P}), \mathcal{O}_P)\end{aligned}$$

is nonzero if and only if $q = 0$ and $k \equiv 0$ modulo r or $q = 1$ and $k \equiv 1$ modulo r.

THEOREM 5.2. (1) *The functor* $j_*\bar{\psi}^*\colon D^b(\mathrm{Coh}(\mathcal{F})) \to D^b(\mathrm{Coh}(\mathcal{X}))$ *is fully faithful.*

Let $D^b(\mathrm{Coh}(\mathcal{F}))_k$ *denote the full subcategory of* $D^b(\mathrm{Coh}(\mathcal{X}))$ *defined by*

$$D^b(\mathrm{Coh}(\mathcal{F}))_k = j_*\bar{\psi}^* D^b(\mathrm{Coh}(\mathcal{F})) \otimes \mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^{n+1} k_i \mathcal{D}_i\right)$$

for a sequence of integers $k = (k_i)$ *with* $1 \le i \le n+1$.

(2) *If* $0 > \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} \ge -\sum_{i=1}^{n+1} \frac{a_i}{r_i}$, *then*

$$\mathrm{Hom}^q(\Phi(D^b(\mathrm{Coh}(\mathcal{Y}))), D^b(\mathrm{Coh}(\mathcal{F}))_k) = 0$$

for all q.

(3) *If* $\sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} > \sum_{i=1}^{n+1} \frac{a_i k_i'}{r_i} > \sum_{i=1}^{n+1} \frac{a_i (k_i - 1)}{r_i}$, *then*

$$\mathrm{Hom}^q(D^b(\mathrm{Coh}(\mathcal{F}))_k, D^b(\mathrm{Coh}(\mathcal{F}))_{k'}) = 0$$

for all q, *where* $k' = (k_i')$ *is another sequence of integers.*

(4) *If* $\sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} = \sum_{i=1}^{n+1} \frac{a_i k_i'}{r_i}$ *but* $j^*\mathcal{O}_{\mathcal{X}}\big(\sum_{i=1}^{n+1}(k_i - k_i')\mathcal{D}_i\big) = 0$ *or*

$$j^*\mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^{n+1}(k_i - k_i')\mathcal{D}_i\right) \notin \bar{\psi}^* D^b(\mathrm{Coh}(\mathcal{F})),$$

then

$$\mathrm{Hom}^q(D^b(\mathrm{Coh}(\mathcal{F}))_k, D^b(\mathrm{Coh}(\mathcal{F}))_{k'}) = 0$$

for all q.

(5) *The subcategories* $\Phi(D^b(\mathrm{Coh}(\mathcal{Y})))$ *and the* $D^b(\mathrm{Coh}(\mathcal{F}))_k$ *for*

$$0 > \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} \ge -\sum_{i=1}^{n+1} \frac{a_i}{r_i}$$

generate the triangulated category $D^b(\mathrm{Coh}(\mathcal{X}))$.

Proof. (1) It is sufficient to prove that the natural homomorphism

$$\mathrm{Hom}^q(L, L') \to \mathrm{Hom}^q(j_*\bar{\psi}^* L, j_*\bar{\psi}^* L')$$

is bijective for all q and all locally free sheaves L and L' on $\mathcal{F}$, because these sheaves span the category $D^b(\mathrm{Coh}(\mathcal{F}))$.

We have an exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}}(-\mathcal{D}_{n+1}) \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{D}_{n+1}} \to 0$$

with an isomorphism $\mathcal{O}_{\mathcal{D}_{n+1}} \cong j_*\mathcal{O}_{\mathcal{D}}$. Hence

$$L_q j^* j_* \mathcal{O}_{\mathcal{D}} \cong \begin{cases} \mathcal{O}_{\mathcal{D}} & \text{for } q = 0, \\ j^*\mathcal{O}_{\mathcal{X}}(-\mathcal{D}_{n+1}) & \text{for } q = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $j^*\mathcal{O}_{\mathcal{X}}(-\mathcal{D}_{n+1})$ is an invertible sheaf on $\mathcal{D}$ if $r_{n+1} = 1$ and is zero otherwise.

If $r_{n+1} > 1$, then

$$\begin{aligned}\mathrm{Hom}^q(j_*\bar{\psi}^*L, j_*\bar{\psi}^*L') &\cong \mathrm{Hom}^q(Lj^*j_*\bar{\psi}^*L, \bar{\psi}^*L') \\ &\cong \mathrm{Hom}^q(\bar{\psi}^*L, \bar{\psi}^*L') \cong \mathrm{Hom}^q(L, L')\end{aligned}$$

as required. If $r_{n+1} = 1$ then we know that $j^*\mathcal{O}_{\mathcal{X}}(\mathcal{D}_{n+1})$ is negative for $\bar{\psi}$, while $j^*\mathcal{O}_{\mathcal{X}}\big(\sum_{i=1}^{n+1}\mathcal{D}_{n+1}\big)$ is ample for $\bar{\psi}$ because $\sum_{i=1}^{n+1}\frac{a_i}{r_i} > 0$. Since

$$\omega_{\mathcal{D}/\mathcal{F}} \cong \mathcal{O}_{\mathcal{D}}\Bigg(-\sum_{i=1}^{\alpha}\bar{\mathcal{D}}_i\Bigg) \cong j^*\mathcal{O}_{\mathcal{X}}\Bigg(-\sum_{i=1}^{\alpha}\mathcal{D}_i\Bigg),$$

we may calculate

$$\mathrm{Hom}^q(L_1j^*j_*\bar{\psi}^*L, \bar{\psi}^*L') \cong \mathrm{Hom}^q(\bar{\psi}^*L, \bar{\psi}^*L' \otimes j^*\mathcal{O}_{\mathcal{X}}(\mathcal{D}_{n+1})) \cong 0$$

by the relative vanishing theorem for $\bar{\psi}$ [13]. Therefore, we have also our assertion in this case.

(2) It is sufficient to prove

$$\mathrm{Hom}^q\Bigg(\mathcal{O}_{\mathcal{X}}\Bigg(\sum_{i=1}^{n+1}k_i'\mathcal{D}_i\Bigg), j_*\bar{\psi}^*A \otimes \mathcal{O}_{\mathcal{X}}\Bigg(\sum_{i=1}^{n+1}k_i\mathcal{D}_i\Bigg)\Bigg) = 0$$

for all integers q, for all sheaves A on $\mathcal{F}$, and for the sequences (k) and (k') under the additional conditions that

$$k'_{n+1} = \llcorner \frac{r_{n+1}}{b_{n+1}}\sum_{i=1}^{n}\frac{a_ik_i'}{r_i}\lrcorner,$$

$$0 > \sum_{i=1}^{n+1}\frac{a_ik_i}{r_i} \geq -\sum_{i=1}^{n+1}\frac{a_i}{r_i}.$$

By the first condition, we have

$$0 \leq \sum_{i=1}^{n+1}\frac{a_ik_i'}{r_i} < \frac{b_{n+1}}{r_{n+1}};$$

hence

$$0 > \sum_{i=1}^{n+1}\frac{a_i(k_i - k_i')}{r_i} > -\sum_{i=1}^{n}\frac{a_i}{r_i}.$$

By the relative vanishing theorem for $\bar{\psi}$, we have

$$\begin{aligned}&\mathrm{Hom}^q\Bigg(\mathcal{O}_{\mathcal{X}}\Bigg(\sum_{i=1}^{n+1}k_i'\mathcal{D}_i\Bigg), j_*\bar{\psi}^*A \otimes \mathcal{O}_{\mathcal{X}}\Bigg(\sum_{i=1}^{n+1}k_i\mathcal{D}_i\Bigg)\Bigg) \\ &\qquad\cong \mathrm{Hom}^q\Bigg(j^*\mathcal{O}_{\mathcal{X}}\Bigg(\sum_{i=1}^{n+1}(k_i' - k_i)\mathcal{D}_i\Bigg), \bar{\psi}^*A\Bigg) \cong 0.\end{aligned}$$

(3) is similarly proved as in (1). Since $0 > \sum_{i=1}^{n+1}\frac{a_i(k_i'-k_i)}{r_i} > -\sum_{i=1}^{n}\frac{a_i}{r_i} + \frac{b_{n+1}}{r_{n+1}}$, it follows that

$$R\bar{\psi}_* j^* \mathcal{O}_{\mathcal{X}}\Bigg(\sum_{i=1}^{n+1}(k'_i - k_i)\mathcal{D}_i\Bigg) = R\bar{\psi}_* j^* \mathcal{O}_{\mathcal{X}}\Bigg(\mathcal{D}_{n+1} + \sum_{i=1}^{n+1}(k'_i - k_i)\mathcal{D}_i\Bigg) = 0$$

by the relative vanishing theorem for $\bar{\psi}$. Thus

$$\mathrm{Hom}^q\Bigg(j_*\bar{\psi}^* L, j_*\bar{\psi}^* L' \otimes \mathcal{O}_{\mathcal{X}}\Bigg(\sum_{i=1}^{n+1}(k'_i - k_i)\mathcal{D}_i\Bigg)\Bigg) = 0$$

for all q and for all locally free sheaves L and L' on $\mathcal{F}$.

(4) is similar to (3).

(5) We shall prove that the left orthogonal ${}^{\perp}\mathcal{T}$ to the triangulated subcategory $\mathcal{T}$ of $D^b(\mathrm{Coh}(\mathcal{X}))$ generated by these subcategories consists of zero objects as in the proof of Theorem 4.3.

Let A be an arbitrary skyscraper sheaf of length 1 on $\mathcal{X}$ supported at a point P. If $P \notin \mathcal{D}_{n+1}$ then $A \in \mathcal{T}$; otherwise, there is a point $\bar{P}$ on $\mathcal{D}$ such that $P = j(\bar{P})$. Then, by Theorem 3.6, there exists a skyscraper sheaf B of length 1 on $\mathcal{F}$ supported at $\bar{Q} = \bar{\psi}(\bar{P})$ such that A is contained in the subcategory generated by the sheaves of the form $j_*\bar{\psi}^* B \otimes \mathcal{O}_{\mathcal{X}}\big(\sum_{i=1}^{n+1} k_i \mathcal{D}_i\big)$ for

$$\frac{b_{n+1}}{r_{n+1}} > \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} \ge -\sum_{i=1}^{n+1} \frac{a_i}{r_i}.$$

If $\frac{b_{n+1}}{r_{n+1}} > \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} \ge 0$, then $k_{n+1} = \llcorner \frac{r_{n+1}}{b_{n+1}} \sum_{i=1}^{n} \frac{a_i k_i}{r_i} \lrcorner$. Therefore, A is contained in $\mathcal{T}$ and hence ${}^{\perp}\mathcal{T} = 0$. □

COROLLARY 5.3. *Assume that $D^b(\mathrm{Coh}(\mathcal{Y}))$ has a complete exceptional collection consisting of sheaves. Then so has $D^b(\mathrm{Coh}(\mathcal{X}))$.*

6. Log Flip

We consider a toric small contraction $\phi \colon X \to Y$ with the log flip $\phi^+ \colon X^+ \to Y$. Here $K_X + B$ is negative for ϕ and $K_{X^+} + B^+$ is ample for ϕ^+, where $B^+ = (\phi^+_*)^{-1}\phi_* B$ is the strict transform. The argument for log flips in this section is surprisingly similar to that for the divisorial contractions in the previous section.

Let $\mathcal{X}^+$ be the smooth Deligne–Mumford stack associated to the pair (X^+, B^+). Then there is a fully faithful functor $\Phi \colon D^b(\mathrm{Coh}(\mathcal{X}^+)) \to D^b(\mathrm{Coh}(\mathcal{X}))$ by [12, Thm. 4.2(3)]. Indeed, let $\mathcal{W}$ be the normalization of the fiber product $\mathcal{X} \times_Y \mathcal{X}^+$, and let $\mu \colon \mathcal{W} \to \mathcal{X}$ and $\nu \colon \mathcal{W} \to \mathcal{X}^+$ be the projections. Then $\Phi = \mu_* \circ \nu^*$ is fully faithful. We regard $D^b(\mathrm{Coh}(\mathcal{X}^+))$ as a full subcategory of $D^b(\mathrm{Coh}(\mathcal{X}))$ through this functor.

Let $D_i^+ = (\phi^+_*)^{-1}\phi_* D_i$ be the prime divisors on X^+ corresponding to the edges v_i for $1 \le i \le n+1$, and let $\mathcal{D}_i^+$ be the corresponding prime divisors on $\mathcal{X}^+$. The following formula is proved in the proof of [12, Thm. 4.2(3)]:

$$\Phi\Bigg(\mathcal{O}_{\mathcal{X}^+}\Bigg(\sum_{i=1}^{n+1} k_i \mathcal{D}_i^+\Bigg)\Bigg) \cong \mathcal{O}_{\mathcal{X}}\Bigg(\sum_{i=1}^{n+1} k_i \mathcal{D}_i\Bigg)$$

if

$$0 \le \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} < \sum_{i=\beta+1}^{n+1} \frac{b_i}{r_i},$$

where we put $b_i = -a_i$ for $\beta + 1 \le i \le n+1$.

Let D be the exceptional locus of the contraction ϕ. Then we have $D = \bigcap_{i=\beta+1}^{n+1} D_i$, and the restriction $\bar{\phi}\colon D \to F = \phi(D)$ is a Mori fiber space described as follows. The lattice of 1-parameter subgroups for D is given by $\bar{N} = N_X / \bigoplus_{i=\beta+1}^{n+1} \mathbb{Z} v_i$. We write $v_i \bmod \bigoplus_{i=\beta+1}^{n+1} \mathbb{Z} v_i = t_i \bar{v}_i$ for $1 \le i \le \beta$, where the t_i are positive integers and the $\bar{v}_i$ are primitive vectors in $\bar{N}$. Let $t = (a_1 t_1, \dots, a_\beta t_\beta)$ be the greatest common divisor and let $a_i t_i = t \bar{a}_i$. Then

$$\bar{a}_1 \bar{v}_1 + \cdots + \bar{a}_\beta \bar{v}_\beta = 0.$$

We define a $\mathbb{Q}$-divisor $\bar{B}$ on D by putting coefficients $\frac{r_i t_i - 1}{r_i t_i}$ to the prime divisors $\bar{D}_i = D_i \cap D$ for $1 \le i \le \beta$. We also define a $\mathbb{Q}$-divisor $\bar{C}$ on the base space of the Mori fiber space F using $\bar{B}$ as before. Let $\mathcal{D}$ and $\mathcal{F}$ be the smooth Deligne–Mumford stacks associated to the pairs $(D, \bar{B})$ and $(F, \bar{C})$, respectively. Then there are induced morphisms of stacks $\bar{\psi} : \mathcal{D} \to \mathcal{F}$ and $j : \mathcal{D} \to \mathcal{X}$. Let $\bar{\mathcal{D}}_i$ be the prime divisors on $\mathcal{D}$ corresponding to the $\bar{D}_i$ for $1 \le i \le \beta$. Then

$$j^* \mathcal{O}_{\mathcal{X}}(\mathcal{D}_i) \cong \mathcal{O}_{\mathcal{D}}(\bar{\mathcal{D}}_i).$$

We note that $D_i|_D = \frac{1}{t_i} \bar{D}_i$ in the usual language.

THEOREM 6.1. (1) *The functor* $j_* \bar{\psi}^*\colon D^b(\mathrm{Coh}(\mathcal{F})) \to D^b(\mathrm{Coh}(\mathcal{X}))$ *is fully faithful.*

Let $D^b(\mathrm{Coh}(\mathcal{F}))_k$ *denote the full subcategory of* $D^b(\mathrm{Coh}(\mathcal{X}))$ *defined by*

$$D^b(\mathrm{Coh}(\mathcal{F}))_k = j_* \bar{\psi}^* D^b(\mathrm{Coh}(\mathcal{F})) \otimes \mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^{n+1} k_i \mathcal{D}_i \right)$$

for a sequence of integers $k = (k_i)$ *with* $1 \le i \le n+1$.

(2) *If* $0 > \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} \ge -\sum_{i=1}^{n+1} \frac{a_i}{r_i}$, *then*

$$\mathrm{Hom}^q(\Phi(D^b(\mathrm{Coh}(\mathcal{X}^+))), D^b(\mathrm{Coh}(\mathcal{F}))_k) = 0$$

for all q.

(3) *If* $\sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} > \sum_{i=1}^{n+1} \frac{a_i k_i'}{r_i} > \sum_{i=1}^{n+1} \frac{a_i (k_i - 1)}{r_i}$, *then*

$$\mathrm{Hom}^q(D^b(\mathrm{Coh}(\mathcal{F}))_k, D^b(\mathrm{Coh}(\mathcal{F}))_{k'}) = 0$$

for all q, *where* $k' = (k_i')$ *is another sequence of integers.*

(4) *If* $\sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} = \sum_{i=1}^{n+1} \frac{a_i k_i'}{r_i}$ *but* $j^* \mathcal{O}_{\mathcal{X}}\big(\sum_{i=1}^{n+1} (k_i - k_i') \mathcal{D}_i \big) = 0$ *or*

$$j^* \mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^{n+1} (k_i - k_i') \mathcal{D}_i \right) \notin \bar{\psi}^* D^b(\mathrm{Coh}(\mathcal{F})),$$

then

$$\mathrm{Hom}^q(D^b(\mathrm{Coh}(\mathcal{F}))_k, D^b(\mathrm{Coh}(\mathcal{F}))_{k'}) = 0$$

for all q.

(5) *The subcategories* $\Phi(D^b(\mathrm{Coh}(\mathcal{X}^+)))$ *and the* $D^b(\mathrm{Coh}(\mathcal{F}))_k$ *for*

$$0 > \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} \geq -\sum_{i=1}^{n+1} \frac{a_i}{r_i}$$

generate the triangulated category $D^b(\mathrm{Coh}(\mathcal{X}))$.

Proof. (1) We shall prove that the natural homomorphism

$$\mathrm{Hom}^q(L, L') \to \mathrm{Hom}^q(j_*\bar{\psi}^*L, j_*\bar{\psi}^*L')$$

is bijective for all q and for all locally free sheaves L and L' on $\mathcal{F}$.

We have an exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}}\Big(-\sum_{i=\beta+1}^{n+1} \mathcal{D}_i\Big) \to \cdots \to \bigwedge^2\Big(\bigoplus_{i=\beta+1}^{n+1} \mathcal{O}_{\mathcal{X}}(-\mathcal{D}_i)\Big)$$
$$\to \bigoplus_{i=\beta+1}^{n+1} \mathcal{O}_{\mathcal{X}}(-\mathcal{D}_i) \to \mathcal{O}_{\mathcal{X}} \to j_*\mathcal{O}_{\mathcal{D}} \to 0.$$

Hence

$$L_q j^* j_* \mathcal{O}_{\mathcal{D}} \cong \bigwedge^q\Big(\bigoplus_{i=\beta+1}^{n+1} j^*\mathcal{O}_{\mathcal{X}}(-\mathcal{D}_i)\Big).$$

The sheaf $j^*\mathcal{O}_{\mathcal{X}}\big(-\sum_{i\in I} \mathcal{D}_i\big)$ for any subset $I \subset \{\beta+1, \dots, n+1\}$ is either invertible or zero, and it is negative for $\bar{\psi}$ if it is not a zero sheaf.

Since

$$\omega_{\mathcal{D}/\mathcal{F}} \cong \mathcal{O}_{\mathcal{D}}\Big(-\sum_{i=1}^{\alpha} \bar{\mathcal{D}}_i\Big) \cong j^*\mathcal{O}_{\mathcal{X}}\Big(-\sum_{i=1}^{\alpha} \mathcal{D}_i\Big),$$

we may calculate

$$\mathrm{Hom}^q(L_p j^* j_* \bar{\psi}^* L, \bar{\psi}^* L')$$
$$\cong \mathrm{Hom}^q\Big(\bar{\psi}^*L, \bar{\psi}^*L' \otimes \bigwedge^p\Big(\bigoplus_{i=\beta+1}^{n+1} j^*\mathcal{O}_{\mathcal{X}}(-\mathcal{D}_i)\Big)\Big) \cong 0$$

for $p > 0$ and for any q by the relative vanishing theorem for $\bar{\psi}$, because $\sum_{i=1}^{n+1} \frac{a_i}{r_i} > 0$. Hence

$$\mathrm{Hom}^q(j_*\bar{\psi}^*L, j_*\bar{\psi}^*L') \cong \mathrm{Hom}^q(j^*j_*\bar{\psi}^*L, \bar{\psi}^*L')$$
$$\cong \mathrm{Hom}^q(\bar{\psi}^*L, \bar{\psi}^*L') \cong \mathrm{Hom}^q(L, L')$$

for any q, as required.

(2) It is sufficient to prove

$$\mathrm{Hom}^q\Big(\mathcal{O}_{\mathcal{X}}\Big(\sum_{i=1}^{n+1} k_i'\mathcal{D}_i\Big), j_*\bar{\psi}^* A \otimes \mathcal{O}_{\mathcal{X}}\Big(\sum_{i=1}^{n+1} k_i\mathcal{D}_i\Big)\Big) = 0$$

for all integers q, for all sheaves A on $\mathcal{F}$, and for the sequences (k) and (k') under the additional conditions that

$$0 \le \sum_{i=1}^{n+1} \frac{a_i k_i'}{r_i} < \sum_{i=\beta+1}^{n+1} \frac{b_i}{r_i},$$

$$0 > \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} \ge -\sum_{i=1}^{n+1} \frac{a_i}{r_i}.$$

It follows that

$$0 > \sum_{i=1}^{n+1} \frac{a_i(k_i - k_i')}{r_i} > -\sum_{i=1}^{\alpha} \frac{a_i}{r_i}.$$

By the relative vanishing theorem for $\bar{\psi}$, we have

$$\mathrm{Hom}^q\Big(\mathcal{O}_{\mathcal{X}}\Big(\sum_{i=1}^{n+1} k_i'\mathcal{D}_i\Big), j_*\bar{\psi}^* A \otimes \mathcal{O}_{\mathcal{X}}\Big(\sum_{i=1}^{n+1} k_i\mathcal{D}_i\Big)\Big)$$
$$\cong \mathrm{Hom}^q\Big(j^*\mathcal{O}_{\mathcal{X}}\Big(\sum_{i=1}^{n+1}(k_i' - k_i)\mathcal{D}_i\Big), \bar{\psi}^* A\Big) \cong 0.$$

(3) is similarly proved as in (1). Since $0 > \sum_{i=1}^{n+1} \frac{a_i(k_i'-k_i)}{r_i} > -\sum_{i=1}^{\alpha} \frac{a_i}{r_i} + \sum_{i=\beta+1}^{n+1} \frac{b_i}{r_i}$, it follows that

$$R\bar{\psi}_* j^*\mathcal{O}_{\mathcal{X}}\Big(\sum_{i\in I}\mathcal{D}_i + \sum_{i=1}^{n+1}(k_i' - k_i)\mathcal{D}_i\Big) = 0$$

for any subset $I \subset \{\beta+1, \dots, n+1\}$ by the relative vanishing theorem for $\bar{\psi}$. Thus

$$\mathrm{Hom}^q\Big(j_*\bar{\psi}^* L, j_*\bar{\psi}^* L' \otimes \mathcal{O}_{\mathcal{X}}\Big(\sum_{i=1}^{n+1}(k_i' - k_i)\mathcal{D}_i\Big)\Big) = 0$$

for all q and for all locally free sheaves L and L' on $\mathcal{F}$.

(4) is similar to (3).

(5) We shall prove that the left orthogonal ${}^{\perp}\mathcal{T}$ to the triangulated subcategory $\mathcal{T}$ of $D^b(\mathrm{Coh}(\mathcal{X}))$ generated by these subcategories consists of zero objects.

Let A be an arbitrary skyscraper sheaf of length 1 on $\mathcal{X}$ supported at a point P. If P is not above a point in D then $A \in \mathcal{T}$; otherwise, there is a point $\bar{P}$ on $\mathcal{D}$ such that $P = j(\bar{P})$. Then, by Theorem 3.6, there exists a skyscraper sheaf B of length 1 on $\mathcal{F}$ supported at $\bar{Q} = \bar{\psi}(\bar{P})$ such that A is contained in the subcategory generated by the sheaves of the form $j_* \bar{\psi}^* B \otimes \mathcal{O}_{\mathcal{X}}\big(\sum_{i=1}^{n+1} k_i \mathcal{D}_i\big)$ for

$$\sum_{i=\beta+1}^{n+1} \frac{b_i}{r_i} > \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} \geq -\sum_{i=1}^{n+1} \frac{a_i}{r_i}.$$

Therefore, A is contained in $\mathcal{T}$ and hence ${}^{\perp}\mathcal{T} = 0$. □

COROLLARY 6.2. *Assume that* $D^b(\mathrm{Coh}(\mathcal{X}^+))$ *has a complete exceptional collection consisting of sheaves. Then so has* $D^b(\mathrm{Coh}(\mathcal{X}))$.

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Department of Mathematical Sciences
University of Tokyo
Komaba, Meguro
Tokyo, 153-8914
Japan

kawamata@ms.u-tokyo.ac.jp