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Degenerations and Fundamental Groups Related to Some Special Toric Varieties

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1. Introduction

Let X be a projective algebraic surface embedded in a projective space $\mathbb{CP}^N$. Take a general linear subspace V in $\mathbb{CP}^N$ of dimension $N-3$. Then the projection centered at V to $\mathbb{CP}^2$ defines a finite map $f: X \to \mathbb{CP}^2$. Let $B \subset \mathbb{CP}^2$ be the branch curve of f, and let $\pi_1(\mathbb{CP}^2 \setminus B)$ be *the fundamental group of the complement of the branch curve*. This group is an invariant of the surface. Closely related to this group is the affine part $\pi_1(\mathbb{C}^2 \setminus B)$.

In this work we compute the groups just defined as they relate to four toric varieties. The first surface is $X_1 := F_1 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$, the Hirzebruch surface of degree 1 in $\mathbb{CP}^6$ embedded by the line bundle with the class $s + 3g$, where s is the negative section and g is a general fiber. The second surface is $X_2 := F_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$, the Hirzebruch surface of degree 0 in $\mathbb{CP}^7$ embedded by $\mathcal{O}(1,3)$; we generalize the results to the case where X_2 is embedded in $\mathbb{CP}^{2n+1}$ by $\mathcal{O}(1,n)$. The third is $X_3 := F_2 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$ in $\mathbb{CP}^5$ embedded by the class $s + 3g$, and the fourth is a singular toric surface X_4 with one A_1 singular point embedded in $\mathbb{CP}^6$. Here A_1-singularity is an isolated normal singularity of dimension 2 whose resolution consists of one (-2)-curve (i.e., a nonsingular rational curve on a surface with -2 as its self-intersection number). For the first three cases, we use different triangulations of tetragons from those treated in [24] and [25].

This work fits into the program initiated by Moishezon and Teicher to study complex surfaces via braid monodromy techniques. They defined the generators of a braid group from a line arrangement in $\mathbb{CP}^2$, which is the branch curve of a generic projection from a union of projective planes [24]—namely, degeneration. In order to explain the process of such a degeneration, they used schematic figures consisting of triangulations of triangles and tetragons [20; 23; 24]. Moishezon and Teicher studied the cases where X is the projective plane embedded by $\mathcal{O}(3)$

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[24] and where the X are Hirzebruch surfaces $F_k(a,b)$ for a,b relatively prime [19]. Later works compute the group $\pi_1(\mathbb{CP}^2 \setminus B)$ related to $K3$ surfaces [2]; $\mathbb{CP}^1 \times T$, where T is a complex torus [3; 4]; $T \times T$ [5; 6]; and the Hirzebruch surface $F_1(2,2)$ [9]. An interesting and helpful work concerning degenerations, braid monodromy, and fundamental groups was written by Auroux–Donaldson–Katzarkov–Yotov [11].

We consult the foregoing works and give a geometric meaning to these schematic figures from the point of view of toric geometry [14; 27]. The work is done along the following lines. First we degenerate X into a union X_0 of planes; then X_0 is composed of $n = \deg(X_0)$ planes. Note that B_0 is the union of the intersection lines $1, 2, \dots, m$ (as depicted in Figures 1, 5, 7, and 8). The lines are numerated for future use. It is quite complicated to obtain a presentation of $\pi_1(\mathbb{C}^2 \setminus B)$ directly, so we use the regeneration rules of [25] to derive a braid monodromy factorization of B from the one of B_0. Then we can use the van Kampen theorem [31] to get a finite presentation of $\pi_1(\mathbb{C}^2 \setminus B)$ with generators $\Gamma_1, \Gamma_{1'}, \dots, \Gamma_m, \Gamma_{m'}$ ($2m$ is the degree of B). A presentation of $\pi_1(\mathbb{CP}^2 \setminus B)$ is obtained by adding the projective relation $\Gamma_{m'}\Gamma_m \cdots \Gamma_{1'}\Gamma_1 = e$. The reader might want to check [4; 6; 7; 9] in order to get a sense of the type of presentations we are dealing with.

Artin [10] defined the braid group $\mathcal{B}_n$ with $n-1$ generators $\{\sigma_i\}$ and with the relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for } |i-j| > 1, \tag{1}$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}. \tag{2}$$

The main results in this work that are related to X_1, X_2, X_3 appear in Theorems 15, 17, and 20:

- $\pi_1(\mathbb{CP}^2 \setminus B_1) \cong \mathcal{B}_5/\langle \Gamma_4^2\Gamma_3\Gamma_2\Gamma_1^2\Gamma_2\Gamma_3\rangle$;
- $\pi_1(\mathbb{CP}^2 \setminus B_2) \cong \mathcal{B}_6/\langle \Gamma_3\Gamma_4\Gamma_5^2\Gamma_4\Gamma_3\Gamma_2\Gamma_1^2\Gamma_2\rangle$;
- $\pi_1(\mathbb{CP}^2 \setminus B_3) \cong \mathcal{B}_4/\langle \Gamma_2\Gamma_3^2\Gamma_2\Gamma_1^2\rangle$.

REMARK 1. The groups $\pi_1(\mathbb{CP}^2 \setminus B_i)$ are in fact the braid group of points on the sphere. A general geometric interpretation is as follows. The surfaces X_i ($i = 1,2,3$) are ruled surfaces, and if p is any point of $\mathbb{CP}^2$ outside the branch curve, then its N preimages in X_i ($N = 5,6,4$) project to distinct points of $\mathbb{CP}^1$; this gives a homomorphism from $\pi_1(\mathbb{CP}^2 \setminus B_i)$ to $B_N(\mathbb{CP}^1)$.

The result related to X_4 appears in Theorem 24:

- $\pi_1(\mathbb{CP}^2 \setminus B_4)$ *is isomorphic to a quotient of the group* $\tilde{\mathcal{B}}_6 = \mathcal{B}_6/\langle [X,Y]\rangle$ (X,Y *are transversal*) *by* (92).

In this work we are also interested in two important quotient groups. The first one, $\Pi_{(B)} = \pi_1(\mathbb{CP}^2 \setminus B)/\langle \Gamma_i^2, \Gamma_{i'}^2\rangle$, is defined to be a quotient of $\pi_1(\mathbb{CP}^2 \setminus B)$ by the normal subgroup generated by the squares of the generators. This group is a key ingredient in studying invariants of X and in particular $\pi_1(\mathbb{CP}^2 \setminus B)$. The braid monodromy technique of Moishezon–Teicher enables us to compute

$\pi_1(X_{\text{Gal}})$, the fundamental group of a Galois cover X_{Gal} of X, from $\Pi_{(B)}$. In particular, they showed that there is a natural map from $\Pi_{(B)}$ to the symmetric group S_n, where n is the degree of X, and that $\pi_1(X_{\text{Gal}})$ is the kernel of this homomorphism. Moishezon–Teicher proved in [23] that, for $X = \mathbb{CP}^1 \times \mathbb{CP}^1$, the group $\pi_1(X_{\text{Gal}})$ is a finite abelian group on $n-2$ generators each of order g.c.d.(a,b) (a and b are the parameters of the embedding). In [4] the treated surface is $X = \mathbb{CP}^1 \times T$ (T is a complex torus) and $\pi_1(X_{\text{Gal}}) = \mathbb{Z}^{10}$; in [3] the same surface was embedded in $\mathbb{CP}^{2n-1}$ and $\pi_1(X_{\text{Gal}}) = \mathbb{Z}^{4n-2}$. In [7] and [8] the surface $X = T \times T$ was studied, and $\pi_1(X_{\text{Gal}})$ is nilpotent of class 3. In [9] this group was computed for the Hirzebruch surface $F_1(2,2)$, and this group is $\mathbb{Z}_2^{10}$.

It turns out in this paper (Theorems 15, 17, 20, and 24) that:

- *The group* $\Pi_{(B_i)}$ *is isomorphic to* S_5, S_6, S_4, S_6 *for* $i = 1,2,3,4$, *respectively.*

Hence we have the following corollary.

COROLLARY 2. *The fundamental group* $\pi_1((X_i)_{\text{Gal}})$ *is trivial for* $i = 1,2,3,4$.

The second group is a Coxeter group $C = \Pi_{(B)}/\langle \Gamma_i = \Gamma_{i'} \rangle$ defined as a quotient of $\Pi_{(B)}$ under identification of pairs of generators; see [29]. It is still unclear whether C, introduced here, is an invariant of the surface or of the branch curve. It might be conjectured that there exists a dependence on the choice of a pairing between geometric generators Γ_j and $\Gamma_{j'}$ (and hence on the choice of a degeneration to a union of planes). It turns out that C is isomorphic to a symmetric group S_n for Hirzebruch surfaces [9; 19] and $\mathbb{CP}^1 \times \mathbb{CP}^1$ [20; 23]. The cases of $\mathbb{CP}^1 \times T$ [4] and $T \times T$ [7] are the first examples in which C is a larger group—namely, $C \cong \mathbb{Z}_5 \rtimes S_6$ and $C \cong K_C \rtimes S_{18}$ (K_C is a central extension of $\mathbb{Z}^{34}$ by $\mathbb{Z}$), respectively.

As a result we obtain the following.

COROLLARY 3. *The group* C_i *is isomorphic to* S_5, S_6, S_4, S_6 *for* $i = 1,2,3,4$, *respectively.*

The paper is organized as follows. In Section 2 we study degeneration of toric varieties. In Section 3 we compute the requested groups related to the toric varieties X_1, X_2, X_3, and in Section 4 we compute the ones related to X_4.

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2. Degeneration of Toric Surfaces

In their process for calculating the braid monodromy, Moishezon and Teicher studied the projective degeneration of $V_3 = (\mathbb{CP}^2, \mathcal{O}(3))$ [24] and Hirzebruch surfaces [19]. Since $\mathbb{CP}^2$ and the Hirzebruch surfaces are toric surfaces, we shall describe the projective degeneration of toric surfaces in this section.

2.1. Basic Notions

We outline definitions needed in toric geometry and refer to [14] and [27] for further statements and proofs.

DEFINITION 4 (Toric variety). A *toric variety* is a normal algebraic variety X that contains an algebraic torus $T = (\mathbb{C}^*)^n$, as a dense open subset, together with an algebraic action $T \times X \to X$ of T on X that is an extension of the natural action of T on itself.

Let M be a free $\mathbb{Z}$-module of rank n ($n \geq 1$) and let $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ be the extension of the coefficients to the real numbers. Let $T := \operatorname{Spec} \mathbb{C}[M]$ be an algebraic torus of dimension n. Then M is considered as the character group of T; that is, $M = \operatorname{Hom}_{\text{gr}}(T, \mathbb{C}^*)$. We denote an element $m \in M$ by $e(m)$ as a function on T, which is also a rational function on X. Let L be an ample line bundle on X. Then

$$H^0(X, L) \cong \bigoplus_{m \in P \cap M} \mathbb{C}e(m), \tag{3}$$

where P is an integral convex polytope in $M_{\mathbb{R}}$ that is defined as the convex hull $\operatorname{Conv}\{m_0, m_1, \ldots, m_r\}$ of a finite subset $\{m_0, m_1, \ldots, m_r\} \subset M$. Conversely, we can construct a pair (X, L) of a polarized toric variety from an integral convex polytope P so that the preceding isomorphism holds (see [14, Sec. 3.5] or [27, Sec. 2.4]). If an affine automorphism φ of M transforms P to P_1, then φ induces an isomorphism of polarized toric varieties (X, L) to (X_1, L_1), where (X_1, L_1) corresponds to P_1.

EXAMPLE 5. Let $M = \mathbb{Z}^2$. Then $V_3 = (\mathbb{CP}^2, \mathcal{O}(3))$ corresponds to the integral convex polytope $P_3 := \operatorname{Conv}\{(0,0), (3,0), (0,3)\}$.

EXAMPLE 6. The Hirzebruch surface $F_d = \mathbb{P}(\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(d))$ of degree d has generators s, g in the Picard group consisting of the negative section $s^2 = -d$ and general fiber $g^2 = 0$. A line bundle L with $[L] = as + bg$ in $\operatorname{Pic}(F_d)$ is *ample* if $a > 0$ and $b > ad$. Then this pair (F_d, L) corresponds to $P_{d(a,b)} := \operatorname{Conv}\{(0,0), (b - ad, 0), (b, a), (0, a)\}$.

Next we consider degenerations of toric surfaces defined by Moishezon–Teicher. We recall the definition from [24].

DEFINITION 7 (Projective degeneration). A degeneration of X is a proper surjective morphism with connected fibers $\pi : V \to \mathbb{C}$ from an algebraic variety V

such that the restriction $\pi : V \setminus \pi^{-1}(0) \to \mathbb{C} \setminus \{0\}$ is smooth and $\pi^{-1}(t) \cong X$ for $t \neq 0$.

Let X be projective with an embedding $k\colon X \hookrightarrow \mathbb{CP}^n$. Then a degeneration $\pi : V \to \mathbb{C}$ of X is called a *projective degeneration* of k if there exists a morphism $F\colon V \to \mathbb{CP}^n \times \mathbb{C}$ such that (i) the restriction $F_t = F|_{\pi^{-1}(t)}\colon \pi^{-1}(t) \to \mathbb{CP}^n \times t$ is an embedding of $\pi^{-1}(t)$ for all $t \in \mathbb{C}$ and (ii) $F_1 = k$ under the identification of $\pi^{-1}(1)$ with X.

Moishezon and Teicher used the triangulation of P_3 consisting of nine standard triangles as a schematic figure of a union of nine projective planes [24]. In the theory of toric varieties, however, the lattice points $P_3 \cap M$ correspond to rational functions of degree 3 on $V_3 \cong \mathbb{CP}^2$. Let

$$m_0 = (0,0),\ m_1 = (1,0),\ m_2 = (0,1),\ \dots,\ m_9 = (0,3) \in \mathbb{Z}^2.$$

Then we may write

$$e(m_0) = x_0^3,\ e(m_1) = x_0^2 x_1,\ e(m_2) = x_0^2 x_2,\ \dots,\ e(m_9) = x_2^3$$

with a suitable choice of the homogeneous coordinates of $\mathbb{CP}^2$. The Veronese embedding $V_3 \hookrightarrow \mathbb{CP}^9$ is given by $z_i = e(m_i)$ for $i = 0, 1, \dots, 9$ with the homogeneous coordinates $[z_0 : z_1 : \cdots : z_9]$ of $\mathbb{CP}^9$. Let $P_1 := \mathrm{Conv}\{(0,0), (1,0), (0,1)\}$, which corresponds to $(\mathbb{CP}^2, \mathcal{O}(1))$. The subset $P_1 \subset P_3$ corresponds to the linear subspace $\{z_3 = \cdots = z_9 = 0\} \subset \mathbb{CP}^9$. Thus, a triangulation of P_3 into a union of nine standard triangles means that the subvariety of dimension 2, consisting of the union of nine projective planes in $\mathbb{CP}^9$ and each standard triangle, defines a linear subspace of dimension 2 with corresponding coordinates.

2.2. *Constructing the Degeneration of Toric Surfaces*

We now construct a semistable degeneration of toric surfaces according to Hu [15]. Let $M = \mathbb{Z}^2$, and let P be a convex polyhedron in $M_{\mathbb{R}}$ corresponding to a polarized toric surface (X, L). The lattice points $P \cap M$ define the embedding $\varphi_L\colon X \to \mathbb{P}(\Gamma(X, L))$. Let Γ be a triangulation of P consisting of standard triangles with vertices in $P \cap M$. Let $h\colon P \cap M \to \mathbb{Z}_{>0}$ be a function on the lattice points in P with values in positive integers. Let $\tilde{M} = M \oplus \mathbb{Z}$ and let $\tilde{P} = \mathrm{Conv}\{(x,0), (x, h(x));\ x \in P \cap M\}$ be the integral convex polytope in $\tilde{M}_{\mathbb{R}}$. We want to choose an h that satisfies two conditions: $(x, h(x))$ for $x \in P \cap M$ are vertices of $\tilde{P}$; and, for each edge in Γ joining x and $y \in P \cap M$, there is an edge joining $(x, h(x))$ and $(y, h(y))$ as a face of $\partial\tilde{P}$. We say that $\tilde{P}$ *realizes* the triangulation Γ if these conditions are satisfied. Now we assume that $\tilde{P}$ realizes the triangulation Γ. Then $\tilde{P}$ defines a polarized toric 3-fold $(\tilde{X}, \tilde{L})$. From the construction, $\tilde{X}$ has a fibration $p\colon \tilde{X} \to \mathbb{CP}^1$ with $p^{-1}(t) \cong X$ $(t \neq 0)$ and with $p^{-1}(0)$ a union of projective planes. Furthermore, we see that $p^{-1}(\mathbb{CP}^1 \setminus \{0\}) \cong \mathbb{C} \times X$. Hence the flat family $p\colon \tilde{X} \to \mathbb{CP}^1$ yields a degeneration of X into a union of projective planes with the configuration diagram Γ. Hu treats only nonsingular toric varieties of any dimension. The difficulty of this construction is finding a triangulation Γ. Here we restrict ourselves to toric surfaces; then we can find a triangulation for any integral convex polygon P.

EXAMPLE 8. Let

$$m_0 = (0,0),\ m_1 = (1,0),\ m_2 = (0,1),\ m_3 = (1,1) \in M = \mathbb{Z}^2,$$

and let $P = \mathrm{Conv}\{m_0, m_1, m_2, m_3\}$. Then P defines the polarized surface $(X = \mathbb{CP}^1 \times \mathbb{CP}^1, \mathcal{O}(1,1))$. Let Γ be the triangulation of P defined by adding the edge connecting m_1 and m_2. Define $h(m_0) = h(m_3) = 1$ and $h(m_1) = h(m_2) = 2$. Let $\tilde{M} := M \oplus \mathbb{Z}$. Set $m_i = (m_i, 0)$ and $m_i^+ = (m_i, h(m_i))$ for $i = 0, \dots, 3$ and set $m_4 = (1,0,1)$ and $m_5 = (0,1,1)$ in $\tilde{M}$. Then the integral convex polytope $\tilde{P} := \mathrm{Conv}\{m_0, \dots, m_3, m_0^+, \dots, m_3^+\}$ in $\tilde{M}$ defines the polarized toric 3-fold $(\tilde{X}, \tilde{L})$. By definition, $\tilde{X}$ has a fibration $p\colon \tilde{X} \to \mathbb{CP}^1$. The global sections of $\tilde{L}$ define an embedding of $\tilde{X}$ as follows. Let $[z_0 : \cdots : z_9]$ be the homogeneous coordinates of $\mathbb{CP}^9$. The equations $z_i = e(m_i)$ for $i = 0, \dots, 5$ and $z_{6+j} = e(m_j^+)$ for $j = 0, \dots, 3$ define the embedding $\tilde{X} \to \mathbb{CP}^9$. The fiber $p^{-1}(\infty)$ is given by $\{z_0 z_3 = z_1 z_2, z_4 = \cdots = z_9 = 0\}$, which is isomorphic to

$$X \subset \mathbb{P}(\Gamma(X, \mathcal{O}(1,1))) \cong \mathbb{CP}^3 = \{z_4 = \cdots = z_9 = 0\},$$

and the fiber $p^{-1}(0)$ is given by $\{z_6 z_9 = 0, z_0 = \cdots = z_5 = 0\}$, which is a union of two projective planes in $\mathbb{CP}^3 \cong \{z_0 = \cdots = z_5 = 0\}$.

LEMMA 9. *The line bundle $\tilde{L}$ on $\tilde{X}$ is very ample.*

Proof. Let $m_1, m_2, m_3 \in P \cap M$ be three vertices of a standard triangle in the triangulation Γ of P. Set $m_i^- = (m_i, 0)$ and $m_i^+ = (m_i, h(m_i))$ in $\tilde{M}_{\mathbb{R}}$ for $i = 1, 2, 3$. Denote by $Q = \mathrm{Conv}\{m_i^\pm;\ i = 1, 2, 3\}$ the integral convex polytope with vertices $\{m_i^\pm;\ i = 1, 2, 3\}$. Then we divide $\tilde{P}$ into a union of triangular prisms like Q. We can divide Q into a union of standard 3-simplices. We may assume that $h(m_1) \geq h(m_2) \geq h(m_3)$ by renumbering m_i if necessary. Then we can divide Q into a union of $Q_0 = \mathrm{Conv}\{m_1^+, m_2^+, m_3^+, (m_1, h(m_1) - 1)\}$ and $Q_1 = \mathrm{Conv}\{m_1^-, (m_1, h(m_1) - 1), m_2^\pm, m_3^\pm\}$. Here Q_0 is a standard 3-simplex and Q_1 has a similar shape to Q but less volume than that of Q. Thus we obtain a division of $\tilde{P}$ into a union of standard 3-simplices. This is not always a triangulation of $\tilde{P}$, but it does give a covering of $\tilde{P}$ that consists of standard 3-simplices. From the theory of polytopal semigroup rings (see e.g. [13; 30]), we see that $\tilde{L}$ is simply generated and hence very ample. □

We claim that $\tilde{X}$ also defines a projective degeneration of (X, L). Denote by $\Phi := \varphi_{\tilde{L}}\colon \tilde{X} \to \mathbb{P}(\Gamma(\tilde{X}, \tilde{L})) =: \mathbb{P}$ the morphism defined by global sections of $\tilde{X}$. We see that $p^{-1}(t) \cong X$ for $t \neq 0$ with $[1 : t] \in \mathbb{CP}^1$ and that $p^{-1}(\infty) \cong X$ and $p^{-1}(0)$ are T-invariant reduced divisors. Hence the restriction maps $\Gamma(\tilde{X}, \tilde{L}) \to \Gamma(p^{-1}(\infty), \tilde{L}|_{p^{-1}(\infty)}) \cong \Gamma(X, L)$ and $\Gamma(\tilde{X}, \tilde{L}) \to \Gamma(p^{-1}(0), \tilde{L}|_{p^{-1}(0)})$

are surjective. From the construction of $\tilde{P}$, it follows that $\dim\Gamma(X,L) = \dim\Gamma(p^{-1}(0),\tilde{L}|_{p^{-1}(0)})$. Since $p^{-1}(\mathbb{CP}^1 \setminus \{0\}) \cong X \times \mathbb{C}$, we have $\tilde{L}|_{p^{-1}(t)} \cong L$ for $t \neq 0$. Then $F := \Phi \times p\colon \tilde{X} \to \mathbb{P} \times \mathbb{CP}^1$ is a projective degeneration of $k\colon X \to \mathbb{P}(\Gamma(X,L)) \hookrightarrow \mathbb{P}$.

THEOREM 10. *Let P be an integral convex polyhedron of dimension 2 corresponding to a polarized toric surface (X,L), and let Γ be a triangulation of P consisting of standard triangles with vertices in M. Assume that $\tilde{P}$ is an integral convex polytope in $\tilde{M}_{\mathbb{R}}$ realizing the triangulation Γ. Then $\tilde{P}$ defines a polarized toric 3-fold $(\tilde{X},\tilde{L})$ that gives a projective degeneration of (X,L) to a union of projective planes.*

2.3. Degeneration of the Four Toric Surfaces

In this paper we study four degenerations of polarized toric surfaces, each one of which is defined by an integral convex polygon P. We choose a triangulation Γ for each P and define a function $h\colon P \cap M \to \mathbb{Z}_{\geq 0}$ such that the integral convex polytope $\tilde{P}$ of dimension 3 should realize the triangulation Γ of P.

The first surface is the Hirzebruch surface $X_1 := F_1$ of degree 1 embedded in $\mathbb{CP}^6$ by the very ample line bundle L_1 whose class is $s + 3g$, where s is the negative section and g is a general fiber. We mentioned this surface as a polarized toric surface in Example 6, which corresponds to the integral convex polygon $P_{1(1,3)}$ in $M = \mathbb{Z}^2$. Let $m_i = (i,0)$ for $i = 0,1,2,3$ and $m_j = (j-3,1)$ for $j = 4,5,6$. Then $P_{1(1,3)} = \mathrm{Conv}\{m_0,m_3,m_4,m_6\}$. Let Γ_1 be the triangulation of $P_{1(1,3)}$ obtained by adding the edges $\overline{m_1 m_4}$, $\overline{m_2 m_4}$, $\overline{m_2 m_5}$, and $\overline{m_3 m_5}$ (see Figure 1). This triangulation is slightly different from the one treated in [24]. We define a function $h_1\colon P_{1(1,3)} \cap M \to \mathbb{Z}_{>0}$ as $h_1(m_0) = h_1(m_6) = 1$, $h_1(m_1) = h_1(m_3) = h_1(m_4) = h_1(m_5) = 3$, and $h_1(m_2) = 4$. Then we can define an integral convex polytope $\tilde{P}$ in $\tilde{M} = M \oplus \mathbb{Z}$ that realizes the triangulation Γ_1 of $P_{1(1,3)}$. Hence we have a projective degeneration of $\varphi_1 := \varphi_{L_1}\colon F_1 \hookrightarrow \mathbb{CP}^6$.

The second surface is $X_2 := \mathbb{CP}^1 \times \mathbb{CP}^1$ embedded in $\mathbb{CP}^7$ by $\mathcal{O}(3,1)$. This embedded toric surface corresponds to the convex polygon $P_{3,1} := \mathrm{Conv}\{(0,0),(3,0),(0,1),(3,1)\}$ in $M = \mathbb{Z}^2$. Let $m_i = (i,0)$ for $i = 0,1,2,3$ and $m_j = (j-4,1)$ for $j = 4,5,6,7$. Then $P_{3,1} = \mathrm{Conv}\{m_0,m_3,m_4,m_7\}$. Let Γ_2 be the triangulation of $P_{3,1}$ obtained by adding the edges $\overline{m_0 m_5}$, $\overline{m_1 m_5}$, $\overline{m_1 m_6}$, $\overline{m_2 m_6}$, and $\overline{m_2 m_7}$ (see Figure 5). We define a function $h_2\colon P_{3,1} \cap M \to \mathbb{Z}_{>0}$ as $h_2(m_4) = 1$, $h_2(m_0) = h_2(m_3) = 3$, $h_2(m_5) = h_2(m_7) = 4$, and $h_2(m_1) = h_2(m_2) = h_2(m_6) = 5$. Then we have a projective degeneration of $\varphi_2 := \varphi_{\mathcal{O}(3,1)}\colon \mathbb{CP}^1 \times \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^7$ corresponding to the triangulation Γ_2.

The third surface is the Hirzebruch surface $X_3 := F_2$ of degree 2 embedded in $\mathbb{CP}^5$ by the ample line bundle L_2 whose class is $s + 3g$. The corresponding polygon is $P_{2(1,3)}$. Let $m_i = (i,0)$ for $i = 0,1,2,3$ and $m_j = (j-3,1)$ for $j = 4,5$ in $M = \mathbb{Z}^2$. Then $P_{2(1,3)} = \mathrm{Conv}\{m_0,m_3,m_4,m_5\}$ up to an affine automorphism of M. Let Γ_3 be the triangulation of $P_{2(1,3)}$ obtained by adding the edges $\overline{m_1 m_4}$,

$\overline{m_1 m_5}$, and $\overline{m_2 m_5}$ (see Figure 7). We define a function $h_3 \colon P_{2(1,3)} \cap M \to \mathbb{Z}_{>0}$ as $h_3(m_0) = h_3(m_3) = 1$, $h_3(m_4) = 3$, and $h_3(m_1) = h_3(m_2) = h_3(m_5) = 4$. Then we have a projective degeneration of $\varphi_3 := \varphi_{L_2} \colon F_2 \hookrightarrow \mathbb{CP}^5$ corresponding to the triangulation Γ_3.

The last surface is a singular toric surface X_4 embedded in $\mathbb{CP}^6$ corresponding to the polygon $P_4 := \operatorname{Conv}\{(0,0),(2,0),(0,1),(1,2),(2,1)\}$. Let $m_i = (i,0)$ for $i = 0,1,2$, $m_j = (j-3,1)$ for $j = 3,4,5$, and $m_6 = (1,2)$. Let Γ_4 be the triangulation of P_4 obtained by adding the edges $\{\overline{m_i m_4}, \overline{m_1 m_j};\ i = 1,3,5,6$ and $j = 3,5\}$ (see Figure 8). We define a function $h_4 \colon P_4 \cap M \to \mathbb{Z}_{>0}$ as $h_4(m_0) = h_4(m_2) = 1$, $h_4(m_1) = h_4(m_3) = h_4(m_5) = h_4(m_6) = 3$, and $h_4(m_4) = 4$. Then we have a projective degeneration of $\varphi_4 \colon X_4 \hookrightarrow \mathbb{CP}^6$ corresponding to the triangulation Γ_4.

3. The Surfaces X_1, X_2, and X_3

In this section we compute the groups $\pi_1(\mathbb{C}^2 \setminus B_i)$, $\pi_1(\mathbb{CP}^2 \setminus B_i)$, and $\Pi_{(B_i)}$ for $i = 1,2,3$. Zariski [33] investigated indirectly complements of the types of curves as B_1, B_2, and B_3. We compare our methods and results to those of Zariski.

Using degenerations of toric varieties, such as those that we have here, makes these special cases of a more general theory rather than isolated examples. Having the degenerations of X_1, X_2, and X_3, we project them onto $\mathbb{CP}^2$ and obtain line arrangements. By the regeneration lemmas of Moishezon–Teicher [22], the diagonal lines regenerate to conics that are tangent to the lines with which they intersect. When the rest of the lines regenerate, each tangency (the point of tangency of line and conic) regenerates to three cusps. We end up with cuspidal curves B_i, $i = 1,2,3$. The existence of nodes in these curves depends on the existence of the "parasitic intersections" (projecting the degenerations onto $\mathbb{CP}^2$ causes extra intersections). By the braid monodromy techniques and regeneration rules of Moishezon–Teicher [22; 25], we have the related braid monodromy factorizations (by [21], each braid of a parasitic intersection, say Z_{ij}^2, regenerates to $Z_{ii',jj'}^2$ in the factorizations); see Notation 12. We do not use properties of braid groups but instead use the definition of the factorization [21], from which the van Kampen theorem [31] for cuspidal curves gives a complete set of relations for the fundamental groups $\pi_1(\mathbb{C}^2 \setminus B_i)$.

Zariski [33] derives a collection of local relations without using degeneration and regeneration, as follows. He uses properties of curves to establish relations for certain groups, called the Poincaré groups (contemporary fundamental groups). He defines the class of Poincaré groups G_n, which practically coincides with the Artin braid groups [10]. A group of type G_n is also a group of automorphism classes of a sphere with n holes (the points $P_1, \dots, P_n$ are removed); see [16]. For generators $g_1, \dots, g_{n-1}$ (g_1 connects P_1 and P_2, g_2 connects P_2 and $P_3, \dots$), Zariski proves that

$$g_i g_j = g_j g_i, \quad |i-j| \neq 1, \tag{4}$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad \text{and} \tag{5}$$

$$g_1 g_2 \cdots g_{n-2} g_{n-1}^2 g_{n-2} \cdots g_2 g_1 = e \tag{6}$$

constitute a complete set of generating relations of G_n. He denotes a rational curve with degree n and k cusps as (n,k). He shows how the individual generating relations of G_n correspond to the singularities of a maximal cuspidal curve $(2n-2, 3(n-2))$ with $2(n-2)(n-3)$ nodes. The $(n-2)(n-3)/2$ commutativity relations (4) are the typical relations at nodes, while the $n-2$ relations (5) are the typical cusp relations [32].

The cuspidal curves B_1, B_2, and B_3 ((8,9), (10,12), and (6,6), respectively) fulfill the previous statements and are maximal. Therefore, Zariski obtains the groups G_5, G_6, and G_4, respectively. Here the results related to X_1, X_2, and X_3 turn out to be the ones of Zariski; that is, $\pi_1(\mathbb{CP}^2 \setminus B_i)$ is a braid group of points on a sphere.

Because we use (unlike Zariski) the degeneration on toric varieties, it would be worthwhile to give a proof for the groups related to X_1. Those related to X_2 and X_3 are computed in a similar way and so we omit the proofs.

Remark 11. A braid monodromy factorization Δ^2 should normally be written as a product of factors in an actual order (see [25]). Since our goal is to compute fundamental groups, the order of the factors does not matter. Here we list the monodromies in an unmeaningful order and concentrate on finding the relations in the groups by applying the van Kampen theorem on the monodromies.

3.1. The Surface X_1

Let $X_1 = F_1(3,1)$ be the Hirzebruch surface as defined in Section 2. The construction of the degeneration of Hirzebruch surfaces of type $F_1(p,q)$ (for $p > q \geq 2$) appears in [17] and [18]. Section 6.2 in [11] is dedicated to constructing the degeneration of F_1 surfaces and presenting the fundamental groups of complements of branch curves.

The degeneration of X_1 into a union of five planes $(X_1)_0$ is embedded in $\mathbb{CP}^6$. The numeration of lines is fixed according to the numeration of the vertices in Section 2; see Figure 1. Note that each of the points m_2, m_4, m_5 is contained in three distinct planes, while each of m_1, m_3 is contained in two planes.

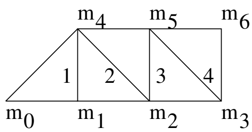

Figure 1 Degeneration of X_1

Take a generic projection $f_1\colon X_1 \to \mathbb{CP}^2$. The union of the intersection lines is the ramification locus R_0 in $(X_1)_0$ of $f_1^0\colon (X_1)_0 \to \mathbb{CP}^2$. Let $(B_1)_0 = f_1^0(R_0)$ be the degenerated branch curve. It is a line arrangement, $(B_1)_0 = \bigcup_{j=1}^4 L_j$.

Denote the singularities of $(B_1)_0$ as $f_1^0(m_i) = m_i$, $i = 1, \ldots, 5$. (The points m_0, m_6 do not lie on numerated lines and so are not singularities of $(B_1)_0$.) The points m_1, m_3 (resp. m_2, m_4, m_5) are called 1-points (resp. 2-points); they were studied in [4; 9; 20; 25]. Other singularities may be the parasitic intersections.

The regeneration of $(X_1)_0$ induces a regeneration of $(B_1)_0$ in such a way that each point on the typical fiber, say c, is replaced by two close points c, c'. The regeneration occurs as follows. We regenerate in a neighborhood of m_1, m_3 to get conics. Now, by the regeneration lemmas of [22], in a neighborhood of m_2, m_4, m_5 the diagonal line regenerates to a conic that is tangent to the line with which it intersects [25, Lemma 1]. See Figure 2 for the regeneration around m_2. When the line regenerates, the tangency regenerates into three cusps (see [22]).

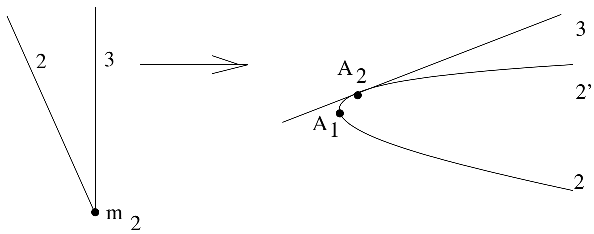

Figure 2 Regeneration around the point m_2

The resulting curve B_1 has degree 8 and nine cusps. The intersection points of the curve with a typical fiber are $\{1, 1', \dots, 4, 4'\}$. We are interested in the braid monodromy factorization of B_1 as well as the groups $\pi_1(\mathbb{C}^2 \setminus B_1)$, $\pi_1(\mathbb{CP}^2 \setminus B_1)$, and $\Pi_{(B_1)}$.

NOTATION 12. We denote by $Z_{i\,j}$ the counterclockwise half-twist of i and j along a path below the real axis. Denote by $Z^2_{i,j\,j'}$ the product $Z^2_{i\,j'} \cdot Z^2_{i\,j}$ and by $Z^2_{i\,i',j\,j'}$ the product $Z^2_{i',j\,j'} \cdot Z^2_{i,j\,j'}$. Likewise, $Z^3_{i,j\,j'}$ denotes the product $(Z^3_{i\,j})^{Z_{j\,j'}} \cdot (Z^3_{i\,j}) \cdot (Z^3_{i\,j})^{Z^{-1}_{j\,j'}}$. Conjugation of braids is defined as $a^b = b^{-1}ab$.

THEOREM 13. *The braid monodromy factorization of the curve B_1 is the product of*

$$\varphi_{m_1} = Z_{1\,1'}, \tag{7}$$

$$\varphi_{m_2} = Z^3_{2',3\,3'} \cdot Z_{2\,2'}^{Z^2_{2',3\,3'}}, \tag{8}$$

$$\varphi_{m_3} = Z_{4\,4'}, \tag{9}$$

$$\varphi_{m_4} = Z^3_{1\,1',2} \cdot Z_{2\,2'}^{Z^2_{1\,1',2}}, \tag{10}$$

$$\varphi_{m_5} = Z^3_{3\,3',4} \cdot Z_{4\,4'}^{Z^2_{3\,3',4}}, \tag{11}$$

and the parasitic intersections braids

$$Z^2_{1\,1',3\,3'},\ Z^2_{1\,1',4\,4'},\ Z^2_{2\,2',4\,4'}. \tag{12}$$

Figure 3 The braids of φ_{m_2}

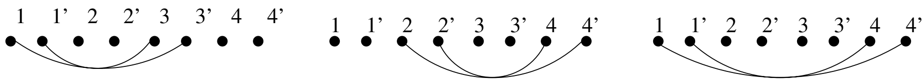

Figure 4 Parasitic intersections braids in the factorization of B_1

Proof. The monodromies (7) and (9) are derived from the regenerations around 1-points, and the ones related to 2-points are (8), (10), and (11); see, for example, the braids of φ_{m_2} in Figure 3. The parasitic intersections were formulated in [21]. These are the intersections of the lines L_1 and L_3, L_1 and L_4, and L_2 and L_4. See Figure 4.

Summing the degrees of the braids gives 56. Since the degree of the factorization is 56 [21, Cor. V.2.3], no other braids are involved. □

NOTATION 14. $\Gamma_{ii'}$ stands for Γ_i or $\Gamma_{i'}$. Also, $\langle \Gamma_a, \Gamma_b \rangle = e$ will be used to signify $\Gamma_a \Gamma_b \Gamma_a = \Gamma_b \Gamma_a \Gamma_b$.

THEOREM 15. *The group $\pi_1(\mathbb{C}^2 \setminus B_1)$ is generated by $\{\Gamma_j\}_{j=1}^4$ subject to the relations*

$$\langle \Gamma_i, \Gamma_{i+1} \rangle = e \quad \textit{for } i = 1, 2, 3, \tag{13}$$

$$[\Gamma_1, \Gamma_i] = e \quad \textit{for } i = 3, 4, \tag{14}$$

$$[\Gamma_2, \Gamma_4] = e, \tag{15}$$

$$[\Gamma_4, \Gamma_3 \Gamma_2 \Gamma_1^2 \Gamma_2 \Gamma_3] = e. \tag{16}$$

The group $\pi_1(\mathbb{CP}^2 \setminus B_1)$ is isomorphic to $\mathcal{B}_5 / \langle \Gamma_4^2 \Gamma_3 \Gamma_2 \Gamma_1^2 \Gamma_2 \Gamma_3 \rangle$, and the group $\Pi_{(B_1)}$ is isomorphic to S_5.

Proof. The group $\pi_1(\mathbb{C}^2 \setminus B_1)$ is generated by the elements $\{\Gamma_j, \Gamma_{j'}\}_{j=1}^4$, where Γ_j and $\Gamma_{j'}$ are loops in $\mathbb{C}^2$ around j and j', respectively.

By the van Kampen theorem, the braids with two branch points give the following relations:

$$\Gamma_i = \Gamma_{i'} \quad \text{for } i = 1, 4. \tag{17}$$

From the monodromies φ_{m_2}, φ_{m_4}, and φ_{m_5} we produce relations (18)–(19), (20)–(21), and (22)–(23), respectively (e.g., from Figure 3 we have (18)–(19)):

$$\langle \Gamma_{2'}, \Gamma_{33'} \rangle = \langle \Gamma_{2'}, \Gamma_3^{-1} \Gamma_{3'} \Gamma_3 \rangle = e, \tag{18}$$

$$\Gamma_{3'} \Gamma_3 \Gamma_{2'} \Gamma_3^{-1} \Gamma_{3'}^{-1} = \Gamma_2; \tag{19}$$

$$\langle \Gamma_{11'}, \Gamma_2 \rangle = \langle \Gamma_1^{-1} \Gamma_{1'} \Gamma_1, \Gamma_2 \rangle = e, \tag{20}$$

$$\Gamma_2 \Gamma_{1'} \Gamma_1 \Gamma_2 \Gamma_1^{-1} \Gamma_{1'}^{-1} \Gamma_2^{-1} = \Gamma_{2'}; \tag{21}$$

$$\langle \Gamma_{33'}, \Gamma_4 \rangle = \langle \Gamma_3^{-1}\Gamma_{3'}\Gamma_3, \Gamma_4 \rangle = e, \tag{22}$$

$$\Gamma_4\Gamma_{3'}\Gamma_3\Gamma_4\Gamma_3^{-1}\Gamma_{3'}^{-1}\Gamma_4^{-1} = \Gamma_{4'}. \tag{23}$$

The parasitic intersections braids contribute the commutative relations

$$[\Gamma_{11'}, \Gamma_{ii'}] = e \quad \text{for } i = 3, 4, \tag{24}$$

$$[\Gamma_{22'}, \Gamma_{44'}] = e. \tag{25}$$

Using (17), (20), and (22), relations (21) and (23) can be rewritten as $\Gamma_1^{-2}\Gamma_2\Gamma_1^2 = \Gamma_{2'}$ and $\Gamma_4^{-2}\Gamma_3\Gamma_4^2 = \Gamma_{3'}$, respectively. Using that $\langle \Gamma_2, \Gamma_{3'} \rangle = \langle \Gamma_1^2\Gamma_{2'}\Gamma_1^{-2}, \Gamma_{3'} \rangle = 1$, we can rewrite (19) as $\Gamma_2^{-1}\Gamma_3\Gamma_{2'}\Gamma_3^{-1}\Gamma_2 = \Gamma_{3'}$. Substituting these three relations into one another yields (16), and substituting them in (18), (20), and (22) (resp., in (24) and (25)) yields (13) (resp., (14) and (15)).

To get $\pi_1(\mathbb{CP}^2 \setminus B_1)$, we add the projective relation $\Gamma_{4'}\Gamma_4\Gamma_{3'}\Gamma_3\Gamma_{2'}\Gamma_2\Gamma_{1'}\Gamma_1 = e$, which is transformed to $\Gamma_4^2\Gamma_3\Gamma_2\Gamma_1^2\Gamma_2\Gamma_3 = e$. Therefore, relation (16) is omitted and we have $\pi_1(\mathbb{CP}^2 \setminus B_1) \cong \mathcal{B}_5/\langle \Gamma_4^2\Gamma_3\Gamma_2\Gamma_1^2\Gamma_2\Gamma_3 \rangle$ and $\Pi_{(B_1)} \cong S_5$. □

3.2. The Surface X_2

In [20], Moishezon and Teicher embed the surface $X_2 = \mathbb{CP}^1 \times \mathbb{CP}^1$ into a big projective space by the linear system $(\mathcal{O}(i), \mathcal{O}(j))$, where $i \geq 2$ and $j \geq 3$. They use its degeneration to compute the fundamental group of the Galois cover corresponding to the generic projection of the surface onto $\mathbb{CP}^2$.

In this paper, the embedding is by the linear system $(\mathcal{O}(3), \mathcal{O}(1))$. The degeneration of X_2 is a union of six planes embedded in $\mathbb{CP}^7$, as depicted in Figure 5.

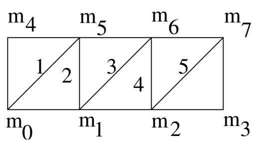

Figure 5 Degeneration of X_2

Now we explain what happens in the regeneration of the branch curve $(B_2)_0$. Each diagonal line regenerates to a conic. This means that in neighborhoods of m_0 and m_7 we have only conics, while in neighborhoods of m_1, m_2, m_5, m_6 the conics are tangent to the lines with which they intersect (the vertical lines in the figure). Then each of these lines regenerates, causing a regeneration of each tangency to three cusps. We end up with the curve B_2 with degree 10 and with twelve cusps.

THEOREM 16. *The braid monodromy factorization of the curve B_2 is the product of*

$$\varphi_{m_0} = Z_{1\,1'}, \tag{26}$$

$$\varphi_{m_1} = Z^3_{2\,2',3} \cdot Z_{3\,3'}^{Z^2_{2\,2',3}}, \tag{27}$$

$$\varphi_{m_2} = Z^3_{4\,4',5} \cdot Z_{5\,5'}^{Z^2_{4\,4',5}}, \tag{28}$$

$$\varphi_{m_5} = Z^3_{1',2\,2'} \cdot Z_{1\,1'}^{Z^2_{1',2\,2'}}, \tag{29}$$

$$\varphi_{m_6} = Z^3_{3',4\,4'} \cdot Z_{3\,3'}^{Z^2_{3',4\,4'}}, \tag{30}$$

$$\varphi_{m_7} = Z_{5\,5'}, \tag{31}$$

and the parasitic intersections braids

$$Z^2_{1\,1',3\,3'},\ \overset{(3)(3')}{Z^2_{1\,1',4\,4'}},\ \overset{(3)(3')}{Z^2_{2\,2',4\,4'}}, Z^2_{1\,1',5\,5'}, Z^2_{2\,2',5\,5'}, Z^2_{3\,3',5\,5'}. \tag{32}$$

Proof. The monodromies φ_{m_0} and φ_{m_7} are braids of branch points of the conics there. The monodromies $\varphi_{m_1}, \varphi_{m_2}$ (resp. $\varphi_{m_5}, \varphi_{m_6}$) are similar to the monodromies (10) and (11) (resp. (8)). According to this similarity of braids (modifying only the indices in Figure 3), we depict only the parasitic intersections braids in Figure 6. □

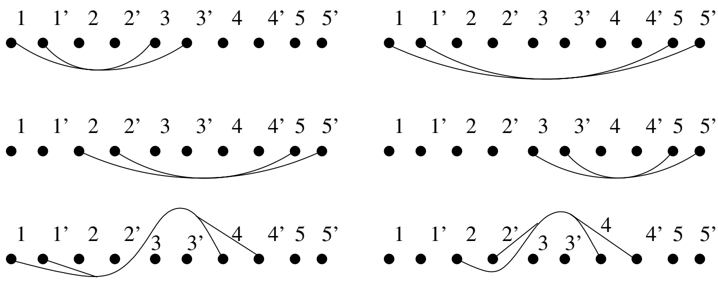

Figure 6 Parasitic intersections braids in the factorization of B_2

We apply the van Kampen theorem to the braids in Figure 6 and obtain a presentation for $\pi_1(\mathbb{C}^2 \setminus B_2)$. Omitting the generators $\Gamma_{i'}$ $(i = 1, \dots, 5)$ and simplifying the relations, as is done in the proof of Theorem 15, yields the following result.

THEOREM 17. *The fundamental group $\pi_1(\mathbb{C}^2 \setminus B_2)$ is generated by $\{\Gamma_j\}_{j=1}^5$ subject to the relations*

$$\langle \Gamma_i, \Gamma_{i+1} \rangle = e \quad \text{for } i = 1, 2, 3, 4, \tag{33}$$

$$[\Gamma_1, \Gamma_i] = e \quad \text{for } i = 3, 4, 5, \tag{34}$$

$$[\Gamma_2, \Gamma_i] = e \quad \text{for } i = 4, 5, \tag{35}$$

$$[\Gamma_3, \Gamma_5] = e, \tag{36}$$

$$\Gamma_2^{-1}\Gamma_1^{-2}\Gamma_2^{-1}\Gamma_3\Gamma_2\Gamma_1^2\Gamma_2 = \Gamma_4^{-1}\Gamma_5^{-2}\Gamma_4^{-1}\Gamma_3\Gamma_4\Gamma_5^2\Gamma_4. \tag{37}$$

The group $\pi_1(\mathbb{CP}^2 \setminus B_2)$ is isomorphic to $\mathcal{B}_6/\langle \Gamma_3\Gamma_4\Gamma_5^2\Gamma_4\Gamma_3\Gamma_2\Gamma_1^2\Gamma_2 \rangle$, and the group $\Pi_{(B_2)}$ is isomorphic to S_6.

One can easily generalize this result. Take $X_2 := \mathbb{CP}^1 \times \mathbb{CP}^1$ embedded in $\mathbb{CP}^{n+1}$ by $\mathcal{O}(n, 1)$. This embedded toric surface corresponds to the convex polygon $P_{n,1} := \mathrm{Conv}\{(0, 0), (n, 0), (0, 1), (n, 1)\}$.

COROLLARY 18. *The groups* $\Pi_{(B)}$ *and* C *are isomorphic to* S_{2n}, *and* $\pi_1((X_2)_{\mathrm{Gal}})$ *is trivial.*

3.3. The Surface X_3

The degeneration of X_3 is a union of four planes embedded in $\mathbb{CP}^5$, as illustrated in Figure 7.

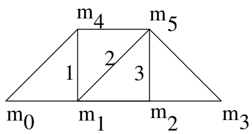

Figure 7 Degeneration of X_3

The branch curve $(B_3)_0$ in $\mathbb{CP}^2$ is a line arrangement. Regenerating it, the diagonal line regenerates to a conic that is tangent to the lines 1 and 3. When the lines regenerate, each tangency regenerates into three cusps. We obtain the branch curve B_3 with degree 6 and with six cusps.

THEOREM 19. *The braid monodromy factorization related to* B_3 *is the product of*

$$\varphi_{m_1} = Z^3_{1\,1',2} \cdot Z_{2\,2'}^{Z^2_{1\,1',2}}, \tag{38}$$

$$\varphi_{m_5} = Z^3_{2',3\,3'} \cdot Z_{2\,2'}^{Z^2_{2',3\,3'}}, \tag{39}$$

$$\varphi_{m_2} = Z_{3\,3'}, \tag{40}$$

$$\varphi_{m_4} = Z_{1\,1'}, \tag{41}$$

and the parasitic intersections braids

$$Z^2_{1\,1',3\,3'}. \tag{42}$$

Proof. Similar to the proof of Theorem 13. □

We apply the van Kampen theorem to the braids of (42) to obtain a presentation for $\pi_1(\mathbb{C}^2 \setminus B_3)$. Once again simplifying the relations and omitting generators, we have the following theorem.

THEOREM 20. *The fundamental group* $\pi_1(\mathbb{C}^2 \setminus B_3)$ *is generated by* $\Gamma_1, \Gamma_2, \Gamma_3$ *subject to the relations*

$$\langle \Gamma_i, \Gamma_{i+1} \rangle = e \quad \text{for } i = 1, 2, \tag{43}$$

$$[\Gamma_1, \Gamma_3] = e, \tag{44}$$

$$\Gamma_1^{-2}\Gamma_2\Gamma_1^2 = \Gamma_3^{-2}\Gamma_2\Gamma_3^2. \tag{45}$$

The group $\pi_1(\mathbb{CP}^2 \setminus B_3)$ *is isomorphic to* $\mathcal{B}_4/\langle \Gamma_2\Gamma_3^2\Gamma_2\Gamma_1^2 \rangle$, *and the group* $\Pi_{(B_3)}$ *is isomorphic to* S_4.

4. The Surface X_4

The degeneration $(X_4)_0$ of X_4 is a union of six planes embedded in $\mathbb{CP}^6$ (see Figure 8).

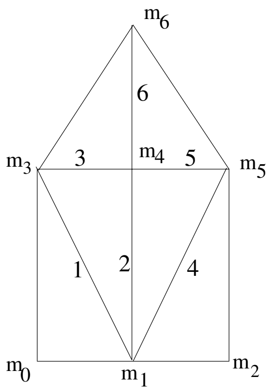

Figure 8 Degeneration of X_4

The regeneration of $(X_4)_0$ induces a regeneration on the branch curve $(B_4)_0$ (line arrangement, composed of six lines). Observe that X_4 has an A_1 singularity as explained in the Introduction. This means that the regeneration of the top vertex m_6 should yield a node in the branch curve that involves components labeled 6 and $6'$ (so that the double cover possesses an ordinary double point). The vertices m_3 and m_5 are 2-points and so the regeneration around them is already known: the line 1 (resp. 4) regenerates to a conic that is tangent to the line 3 (resp. 5). When these lines regenerate, each tangency regenerates to three cusps. The vertex m_4 is a 4-point (see e.g. [2]). The regeneration is as follows. The lines 3 and 5 regenerate to a hyperbola, and each line among 2 and 6 regenerates to a pair of parallel lines. The hyperbola is then tangent to the lines $2, 2', 6, 6'$; see Figure 9. The hyperbola doubles and thus we have four branch points; furthermore, each tangency regenerates to three cusps.

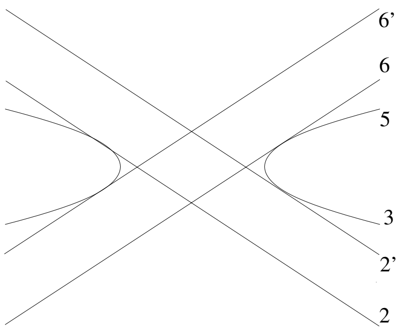

Figure 9 Regeneration around the 4-point m_4

However, the vertex m_1 is of a new type. The regeneration can be done as follows. Line 4 regenerates to a conic, while 1 is still unregenerated; Figure 10 describes this step. The points P_1 and P_2 are the intersections of 1 with the conic (they are complex). The intersection of lines 1 and 2 can then be locally considered as a 2-point; this means that line 1 regenerates to a conic that is tangent to line 2. At this point P_1 and P_2 are doubled. Line 2 then regenerates to a pair of parallel lines 2 and 2′, and each tangency regenerates to three cusps. Note that keeping a parabola, which we get in the regeneration around m_1 as in our depiction of the affine part of the conics, we have possibly another branch point farther away—perhaps at infinity. We shall prove the existence of these two extra branch points, which contribute two half-twists to the braid monodromy factorization.

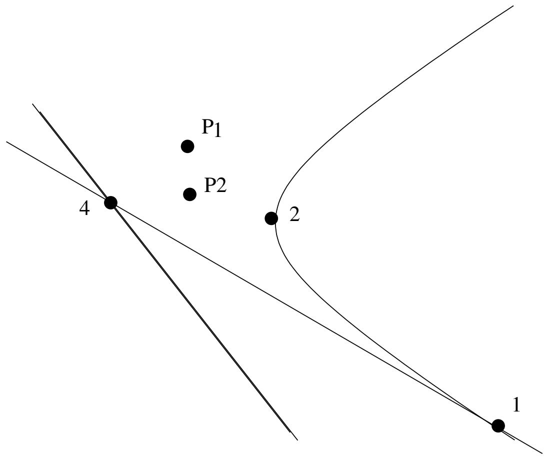

Figure 10 Regeneration around m_1

The parasitic intersections are fixed by Figure 8, and this time they are the intersections in $\mathbb{CP}^2$ of line 1 with lines 5 and 6 and of line 4 with lines 3 and 6.

We thus have the following result.

THEOREM 21. *The braid monodromies derived from the regeneration around* m_1, m_3, m_4, m_5, m_6 *are*

$$\varphi_{m_1} = Z^3_{2\,2',4} \cdot (Z_{4\,4'})^{Z^2_{2\,2',4}} \cdot (Z^2_{1\,1',4'} \cdot (Z^2_{1\,1',4})^{Z^2_{2\,2',4}}) \cdot (Z^3_{1',2\,2'} \cdot (Z_{1\,1'})^{Z^2_{1',2\,2'}}), \quad (46)$$

$$\varphi_{m_3} = Z^3_{1',3\,3'} \cdot (Z_{1\,1'})^{Z^2_{1',3\,3'}}, \quad (47)$$

$$\varphi_{m_5} = Z^3_{4',5\,5'} \cdot (Z_{4\,4'})^{Z^2_{4',5\,5'}}, \quad (48)$$

$$\varphi_{m_6} = Z^2_{6\,6'}, \quad \textit{and} \quad (49)$$

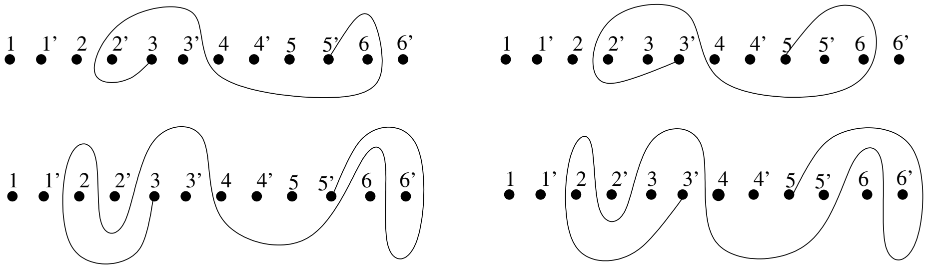

Figure 11 The braids h_1, h_2, h_3, h_4

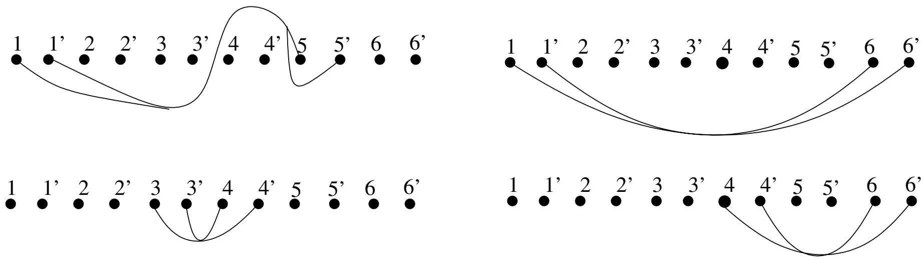

Figure 12 Parasitic intersections braids in the factorization of B_4

$$\varphi_{m_4} = (Z^3_{2',3\,3'} \cdot Z^3_{5\,5',6} \cdot h_1 \cdot h_2 \cdot (Z^2_{2'\,6})^{Z^2_{2',3\,3'}} \cdot Z^2_{2\,6}) \cdot (Z^3_{2,3\,3'} \cdot (Z^3_{5\,5',6'})^{Z^{-2}_{6\,6'}} \cdot h_3 \cdot h_4 \cdot (Z^2_{2\,6'})^{Z^2_{2,3\,3'} Z^{-2}_{6\,6'}} \cdot (Z^2_{2'\,6'})^{Z^{-2}_{2\,2'} Z^{-2}_{6\,6'}}), \quad (50)$$

where h_1, h_2 are the upper braids and h_3, h_4 are the lower braids in Figure 11.

The parasitic intersections braids (*Figure 12*) *are*

$$(Z^2_{1\,1',5\,5'})^{Z^{-2}_{4\,4',5\,5'}},\ Z^2_{1\,1',6\,6'},\ Z^2_{3\,3',4\,4'},\ Z^2_{4\,4',6\,6'}. \quad (51)$$

Since B_4 has degree 12, the total degree of the braid monodromy factorization Δ^2_{12} should be $12 \cdot 11 = 132$ (see [21]). By the foregoing regeneration, B_4 has eight branch points, 24 cusps, and 25 nodes. Their related braids give a total degree of 130. The missing braids correspond to two extra branch points. We explain how to find them.

Look at the preimage in X_4 of a vertical line in $\mathbb{CP}^2$ (a fiber of the projection); this is an elliptic curve (a 6-fold cover of $\mathbb{CP}^1$ branched in twelve points). Considering the entire family of vertical lines in $\mathbb{CP}^2$, we get that X_4 admits a projection to $\mathbb{CP}^1$ whose generic fiber is an elliptic curve. The preimage of a vertical line in $\mathbb{CP}^2$ is singular if and only if that vertical line is tangent to the branch curve or if it passes through the intersection of the lines 6 and $6'$.

There is a "lifting homomorphism" from the braid group B_{12} to the mapping class group $\mathrm{SL}(2, \mathbb{Z})$ obtained by considering the aforementioned 6-fold cover of $\mathbb{CP}^1$: if the twelve branch points are moved by a braid, this induces a homeomorphism of the covering [12, Sec. 5.2]. Now, since the abelianization of $\mathrm{SL}(2, Z)$ is $\mathbb{Z}/12$

and since the quotient homomorphism $\mathrm{SL}(2, Z) \to \mathbb{Z}/12$ takes Dehn twists to the integer 1, the number of Dehn twists is a multiple of 12. However, we get two from $Z^2_{6\,6'}$ and one from each of the eight branch points.

In order to check which braids are missing, we consider a homomorphism from the pure braid group on twelve strings to the pure braid group on two strings, which is defined by deleting all the strands except i and i'; it should map Δ^2_{12} to $\Delta^2_2 = Z^2_{i\,i'}$. By [25, Lemma 2.I], $Z^3_{i\,i',j} = Z^2_{i'\,j} Z^2_{i\,j} Z^2_{i'\,j} Z^2_{i\,j} Z_{i\,i'}$. Therefore, by Theorem 21, $\Delta^2_2 = Z^2_{i\,i'}$ for $i = 1, 2, 4, 6$. Now, forgetting all indices and remembering 3 and $3'$ (resp. 5 and $5'$) yields the half-twist $Z_{3\,3'}$ (resp. $Z_{5\,5'}$) counted three times. But by [25, Lemma 8.IV], $\varphi_{m_4} = \Delta^2_8 Z^{-2}_{2\,2'} Z^{-2}_{6\,6'} Z^{-2}_{3\,3'} Z^{-2}_{5\,5'}$. In his thesis [28], Robb discusses the existence of extra branch points. According to our results, there is an extra branch point that contributes the half-twist $Z_{3\,3'}$ (resp. $Z_{5\,5'}$). By [28, Prop. 3.3.1], the relation in $\pi_1(\mathbb{CP}^2 \setminus B_4)$ should be $\Gamma_3 = \Gamma_{3'}$ (resp. $\Gamma_5 = \Gamma_{5'}$).

REMARK 22. There is also a group-theoretic justification for the missing braids. Because Moishezon–Teicher's formulas for arrangements of lines [25] deal only with what happens before each line regenerates to a pair i, i', their global formula ($\Delta^2 = \prod C_i \varphi_i$ with C_i the parasitic braids) is correct only up to half-twists of the form $Z_{i\,i'}$, which are not seen at all by configurations at the level of the double lines (before regeneration). In our case this product is not Δ^2_{12} but rather $\Delta^2_{12} Z^{-1}_{3\,3'} Z^{-1}_{5\,5'}$; the implication is that there are two extra half-twists, which must be $Z_{3\,3'}$ and $Z_{5\,5'}$.

COROLLARY 23. *The braid monodromy factorization Δ^2_{12} is a product of the braids from Theorem 21 and the extra branch points braids $Z_{3\,3'}$ and $Z_{5\,5'}$.*

Now we are ready to compute the group $\pi_1(\mathbb{CP}^2 \setminus B_4)$.

THEOREM 24. *Let $\tilde{\mathcal{B}}_6$ be the quotient of the braid group $\mathcal{B}_6$ by $\langle [X, Y] \rangle$, where X and Y are transversal. The fundamental group $\pi_1(\mathbb{CP}^2 \setminus B_4)$ is isomorphic to a quotient of $\tilde{\mathcal{B}}_6$ by* (92). *The group $\Pi_{(B_4)}$ is isomorphic to S_6.*

Proof. Applying the van Kampen theorem [31] to the factorization Δ^2_{12} gives a presentation of $\pi_1(\mathbb{CP}^2 \setminus B_4)$ with the generators $\{\Gamma_i, \Gamma_{i'}\}^6_{i=1}$.

The monodromy φ_{m_1} contributes the following relations:

$$\langle \Gamma_{22'}, \Gamma_4 \rangle = \langle \Gamma_{2'} \Gamma_2 \Gamma^{-1}_{2'}, \Gamma_4 \rangle = e, \tag{52}$$

$$\Gamma_4^{\Gamma_2^{-1} \Gamma_{2'}^{-1} \Gamma_4^{-1}} = \Gamma_{4'}; \tag{53}$$

$$[\Gamma_{11'}, \Gamma_4^{\Gamma_2^{-1} \Gamma_{2'}^{-1} \Gamma_4^{-1}}] = [\Gamma_{11'}, \Gamma_{4'}] = e, \tag{54}$$

$$\langle \Gamma_{1'}, \Gamma_{22'} \rangle = \langle \Gamma_{1'}, \Gamma_{2'} \Gamma_2 \Gamma^{-1}_{2'} \rangle = e, \tag{55}$$

$$\Gamma_{2'} \Gamma_2 \Gamma_{1'} \Gamma^{-1}_2 \Gamma^{-1}_{2'} = \Gamma_1. \tag{56}$$

From the monodromies φ_{m_3} and φ_{m_5} we have:

$$\langle \Gamma_{1'}, \Gamma_{33'} \rangle = \langle \Gamma_{1'}, \Gamma_{3'}\Gamma_3\Gamma_{3'}^{-1} \rangle = e, \tag{57}$$

$$\Gamma_{3'}\Gamma_3\Gamma_{1'}\Gamma_3^{-1}\Gamma_{3'}^{-1} = \Gamma_1; \tag{58}$$

$$\langle \Gamma_{4'}, \Gamma_{55'} \rangle = \langle \Gamma_{4'}, \Gamma_{5'}\Gamma_5\Gamma_{5'}^{-1} \rangle = e, \tag{59}$$

$$\Gamma_{5'}\Gamma_5\Gamma_{4'}\Gamma_5^{-1}\Gamma_{5'}^{-1} = \Gamma_4. \tag{60}$$

By φ_{m_4} we have

$$\langle \Gamma_{22'}, \Gamma_3 \rangle = \langle \Gamma_{22'}, \Gamma_{3'} \rangle = \langle \Gamma_{22'}, \Gamma_{3'}\Gamma_3\Gamma_{3'}^{-1} \rangle = e, \tag{61}$$

$$\langle \Gamma_{55'}, \Gamma_6 \rangle = \langle \Gamma_{5'}\Gamma_5\Gamma_{5'}^{-1}, \Gamma_6 \rangle = e, \tag{62}$$

$$\langle \Gamma_{55'}, \Gamma_6^{-1}\Gamma_{6'}\Gamma_6 \rangle = \langle \Gamma_{5'}\Gamma_5\Gamma_{5'}^{-1}, \Gamma_6^{-1}\Gamma_{6'}\Gamma_6 \rangle = e, \tag{63}$$

$$[\Gamma_2, \Gamma_6] = [\Gamma_{2'}^{\Gamma_2}, \Gamma_{6'}^{\Gamma_6}] = e, \tag{64}$$

$$[\Gamma_{2'}, \Gamma_6^{\Gamma_{3'}\Gamma_3}] = [\Gamma_2, \Gamma_{6'}^{\Gamma_6\Gamma_{3'}\Gamma_3}] = e; \tag{65}$$

$$\Gamma_3^{\Gamma_{2'}^{-1}\Gamma_3^{-1}\Gamma_{3'}^{-1}} = \Gamma_{5'}^{\Gamma_6^{-1}}, \tag{66}$$

$$\Gamma_{3'}^{\Gamma_{2'}^{-1}\Gamma_3^{-1}\Gamma_{3'}^{-1}} = \Gamma_5^{\Gamma_{5'}^{-1}\Gamma_6^{-1}}, \tag{67}$$

$$\Gamma_3^{\Gamma_2^{-1}\Gamma_3^{-1}\Gamma_{3'}^{-1}} = \Gamma_{5'}^{\Gamma_6^{-1}\Gamma_{6'}^{-1}\Gamma_6}, \tag{68}$$

$$\Gamma_{3'}^{\Gamma_2^{-1}\Gamma_3^{-1}\Gamma_{3'}^{-1}} = \Gamma_5^{\Gamma_{5'}^{-1}\Gamma_6^{-1}\Gamma_{6'}^{-1}\Gamma_6}; \tag{69}$$

and φ_{m_6} contributes

$$[\Gamma_6, \Gamma_{6'}] = e. \tag{70}$$

The parasitic intersections braids yield

$$[\Gamma_{11'}, \Gamma_{55'}^{\Gamma_{4'}\Gamma_4}] = e, \tag{71}$$

$$[\Gamma_{11'}, \Gamma_{66'}] = e, \quad \text{and} \tag{72}$$

$$[\Gamma_{44'}, \Gamma_{ii'}] = e \quad \text{for } i = 3, 6; \tag{73}$$

the extra branch points contribute

$$\Gamma_3 = \Gamma_{3'}, \tag{74}$$

$$\Gamma_5 = \Gamma_{5'}. \tag{75}$$

The projective relation is

$$\Gamma_{6'}\Gamma_6\Gamma_{5'}\Gamma_5\Gamma_{4'}\Gamma_4\Gamma_{3'}\Gamma_3\Gamma_{2'}\Gamma_2\Gamma_{1'}\Gamma_1 = e. \tag{76}$$

LEMMA 25. *The presentation just described is a complete one.*

Proof. Considering the complex conjugations (details in [20; 25]) of the braids, we obtain a complete set of relations. Simplifying them gives the same list as before. □

Continuing with the proof of Theorem 24, we outline now our simplification of the foregoing presentation. We will express the relations in terms of Γ_1, Γ_2, Γ_3,

Γ_4, Γ_5, and $\Gamma_{6'}$. First we use relations (74) and (75) to omit the generators $\Gamma_{3'}$ and $\Gamma_{5'}$ from all the given relations.

The branch points relations (53), (56), (58), (60), and (66)–(69) are rewritten as

$$\Gamma_{4'} = \Gamma_2^{-1}\Gamma_{2'}^{-1}\Gamma_4\Gamma_{2'}\Gamma_2 \quad \text{by (52)}, \tag{77}$$

$$\Gamma_{1'} = \Gamma_2^{-1}\Gamma_{2'}^{-1}\Gamma_1\Gamma_{2'}\Gamma_2, \tag{78}$$

$$\Gamma_{1'} = \Gamma_3^{-2}\Gamma_1\Gamma_3^2, \tag{79}$$

$$\Gamma_{4'} = \Gamma_5^{-2}\Gamma_4\Gamma_5^2, \tag{80}$$

$$\Gamma_6 = \Gamma_5\Gamma_3\Gamma_{2'}\Gamma_3^{-1}\Gamma_5^{-1}, \tag{81}$$

$$\Gamma_{6'} = \Gamma_5\Gamma_3\Gamma_2\Gamma_3^{-1}\Gamma_5^{-1}. \tag{82}$$

Now we rewrite the commutations. Using (82), relation (62) gives the form $\langle\Gamma_3, \Gamma_6\rangle = e$, and this enables us to prove that (65) is

$$\begin{aligned} e &= [\Gamma_{2'}, \Gamma_3^{-2}\Gamma_6\Gamma_3^2] = [\Gamma_3^{-1}\Gamma_6\Gamma_5\Gamma_6^{-1}\Gamma_3, \Gamma_3^{-2}\Gamma_6\Gamma_3^2] \\ &= [\Gamma_6\Gamma_5\Gamma_6^{-1}, \Gamma_3^{-1}\Gamma_6\Gamma_3] = [\Gamma_6\Gamma_5\Gamma_6^{-1}, \Gamma_6\Gamma_3\Gamma_6^{-1}] = [\Gamma_3, \Gamma_5]. \end{aligned} \tag{83}$$

Relation (71) is rewritten as $[\Gamma_1, \Gamma_5] = e$ by using (59), (80), (79), and $[\Gamma_3, \Gamma_5] = e$. This enables us to prove from (54) that $[\Gamma_1, \Gamma_4] = e$. Using these two resulting relations together with (81), (77), and (73), we can rewrite the relation $[\Gamma_1, \Gamma_6] = e$ as follows:

$$\begin{aligned} e &= [\Gamma_1, \Gamma_6] = [\Gamma_1, \Gamma_5\Gamma_3\Gamma_{2'}\Gamma_3^{-1}\Gamma_5^{-1}] = [\Gamma_1, \Gamma_3\Gamma_{2'}\Gamma_3^{-1}] = [\Gamma_3^{-1}\Gamma_1\Gamma_3, \Gamma_{2'}] \\ &= [\Gamma_3^{-1}\Gamma_1\Gamma_3, \Gamma_4^{-1}\Gamma_2\Gamma_{4'}\Gamma_2^{-1}\Gamma_4] = [\Gamma_3^{-1}\Gamma_1\Gamma_3, \Gamma_{4'}^{-1}\Gamma_2\Gamma_{4'}] = [\Gamma_3^{-1}\Gamma_1\Gamma_3, \Gamma_2] \\ &= [\Gamma_1, \Gamma_3\Gamma_2\Gamma_3^{-1}]. \end{aligned}$$

In a similar way, $[\Gamma_{1'}, \Gamma_6] = e$ can be rewritten as $[\Gamma_1, \Gamma_3^{-1}\Gamma_2\Gamma_3] = e$. Using (80) and $[\Gamma_3, \Gamma_5] = e$, the relation $[\Gamma_3, \Gamma_{4'}] = e$ gets the form $[\Gamma_3, \Gamma_4] = e$. Relation (62) is rewritten as

$$\begin{aligned} e &= \langle\Gamma_5, \Gamma_6\rangle = \langle\Gamma_5, \Gamma_5\Gamma_3\Gamma_{2'}\Gamma_3^{-1}\Gamma_5^{-1}\rangle = \langle\Gamma_5, \Gamma_{2'}\rangle \\ &= \langle\Gamma_5, \Gamma_1^{-1}\Gamma_2\Gamma_{1'}\Gamma_2^{-1}\Gamma_1\rangle = \langle\Gamma_5, \Gamma_{1'}^{-1}\Gamma_2\Gamma_{1'}\rangle = \langle\Gamma_5, \Gamma_2\rangle. \end{aligned} \tag{84}$$

Thus $[\Gamma_{44'}, \Gamma_6] = e$ is rewritten as

$$[\Gamma_4, \Gamma_5\Gamma_2\Gamma_5^{-1}] = [\Gamma_4, \Gamma_5^{-1}\Gamma_2\Gamma_5] = e. \tag{85}$$

Now, by (72), (55), and (78), the relation $[\Gamma_{2'}, \Gamma_{6'}] = e$ gets the form $[\Gamma_2, \Gamma_{6'}] = e$. This relation, together with (77) and (78), enables us to prove that $[\Gamma_{1'}, \Gamma_{6'}] = e$ and $[\Gamma_{4'}, \Gamma_{6'}] = e$ get the forms $[\Gamma_1, \Gamma_{6'}] = e$ and $[\Gamma_4, \Gamma_{6'}] = e$, respectively. Since (63) can be rewritten as $\langle\Gamma_3, \Gamma_{6'}\rangle = e$, it follows by (65) that the relation $[\Gamma_2, \Gamma_3^{-2}\Gamma_{6'}\Gamma_3^2] = e$ gets the form $[\Gamma_3, \Gamma_5] = e$. The relation $[\Gamma_2, \Gamma_6] = e$ from (64) now gets the following form:

$$e = [\Gamma_2, \Gamma_6] = [\Gamma_2, \Gamma_5\Gamma_3\Gamma_{2'}\Gamma_3^{-1}\Gamma_5^{-1}] \quad \text{by (81)}$$

$$e = [\Gamma_5^{-1}\Gamma_2\Gamma_5, \Gamma_3\Gamma_{2'}\Gamma_3^{-1}] = [\Gamma_5^{-1}\Gamma_2\Gamma_5, \Gamma_3\Gamma_4^{-1}\Gamma_2\Gamma_{4'}\Gamma_2^{-1}\Gamma_4\Gamma_3^{-1}] \quad \text{by (77)}$$

$$e = [\Gamma_2^{-1}\Gamma_3^{-1}\Gamma_2\Gamma_5\Gamma_2^{-1}\Gamma_3\Gamma_2, \Gamma_{4'}] \quad \text{by (73), (85), and (84)}$$

$$e = [\Gamma_3\Gamma_2^{-1}\Gamma_3^{-1}\Gamma_5\Gamma_3\Gamma_2\Gamma_3^{-1}, \Gamma_{4'}] \quad \text{by (61)}$$

$$e = [\Gamma_2^{-1}\Gamma_5\Gamma_2, \Gamma_{4'}] \quad \text{by (73) and (83)}$$

$$e = [\Gamma_2^{-1}\Gamma_5\Gamma_2, \Gamma_2^{-1}\Gamma_{2'}^{-1}\Gamma_4\Gamma_{2'}\Gamma_2] \quad \text{by (77)}$$

$$e = [\Gamma_5, \Gamma_4\Gamma_{2'}\Gamma_4^{-1}] \quad \text{by (52)}$$

$$e = [\Gamma_5, \Gamma_4\Gamma_1^{-1}\Gamma_2\Gamma_{1'}\Gamma_2^{-1}\Gamma_1\Gamma_4^{-1}] \quad \text{by (78)}$$

$$e = [\Gamma_5, \Gamma_4\Gamma_{1'}^{-1}\Gamma_2\Gamma_{1'}\Gamma_4^{-1}] \quad \text{by (54), (71), and (55)}$$

$$e = [\Gamma_5, \Gamma_4\Gamma_2\Gamma_4^{-1}] = [\Gamma_4, \Gamma_5^{-1}\Gamma_2\Gamma_5] \quad \text{by (52) and (84).}$$

The only relation which is left for now in its original form is (70). We shall prove that $\Gamma_6 = \Gamma_{6'}$, and this equality will eliminate it.

The triple relations are rewritten as follows: (57) and (59), respectively, get the forms $\langle \Gamma_1, \Gamma_3 \rangle = e$ and $\langle \Gamma_4, \Gamma_5 \rangle = e$ by (79) and (80). It is also easy to prove that (52), (55), and (61) yield $\langle \Gamma_2, \Gamma_4 \rangle = e$, $\langle \Gamma_1, \Gamma_2 \rangle = e$, and $\langle \Gamma_2, \Gamma_3 \rangle = e$, respectively.

Relation (76) is now

$$\Gamma_{6'}\Gamma_5\Gamma_3\Gamma_4^{-1}\Gamma_2\Gamma_5^{-2}\Gamma_4\Gamma_5^2\Gamma_2^{-1}\Gamma_4\Gamma_5^{-1}\Gamma_4\Gamma_5^2\Gamma_3\Gamma_2\Gamma_5^{-2}\Gamma_4\Gamma_5^2\Gamma_2^{-1}\Gamma_4\Gamma_2\Gamma_3^{-2}\Gamma_1\Gamma_3^2\Gamma_1 = e. \tag{86}$$

Equating the two expressions of $\Gamma_{2'}$ given by (77) and (78), we obtain

$$\Gamma_1^{-1}\Gamma_2\Gamma_3^{-2}\Gamma_1\Gamma_3^2\Gamma_2^{-1}\Gamma_1 = \Gamma_4^{-1}\Gamma_2\Gamma_5^{-2}\Gamma_4\Gamma_5^2\Gamma_2^{-1}\Gamma_4, \tag{87}$$

which will be redundant later on.

The relations we now have are (70), (82), (86), (87), and:

$$\langle \Gamma_i, \Gamma_j \rangle = e \quad (\Gamma_i \text{ and } \Gamma_j \text{ share a common triangle}), \tag{88}$$

$$[\Gamma_i, \Gamma_j] = e \quad (\Gamma_i \text{ and } \Gamma_j \text{ share no common triangle}); \tag{89}$$

$$[\Gamma_1, \Gamma_3^{-1}\Gamma_2\Gamma_3] = [\Gamma_1, \Gamma_3\Gamma_2\Gamma_3^{-1}] = e, \tag{90}$$

$$[\Gamma_4, \Gamma_5^{-1}\Gamma_2\Gamma_5] = [\Gamma_4, \Gamma_5\Gamma_2\Gamma_5^{-1}] = e. \tag{91}$$

Using (82), we omit $\Gamma_{6'}$ and hence the group $\pi_1(\mathbb{CP}^2 \setminus B_4)$ has the generators $\{\Gamma_i\}_{i=1}^5$ and admits the relations (70), (87), (88)–(91) for $i, j \neq 6'$, and the new form of (86):

$$\Gamma_5\Gamma_3\Gamma_2\Gamma_4^{-1}\Gamma_2\Gamma_5^{-2}\Gamma_4\Gamma_5^2\Gamma_2^{-1}\Gamma_4\Gamma_5^{-1}\Gamma_4\Gamma_5^2\Gamma_3\Gamma_2\Gamma_5^{-2}\Gamma_4\Gamma_5^2\Gamma_2^{-1}\Gamma_4\Gamma_2\Gamma_3^{-2}\Gamma_1\Gamma_3^2\Gamma_1 = e. \tag{92}$$

Now we show that $\pi_1(\mathbb{CP}^2 \setminus B_4)$ is isomorphic to a quotient of $\mathcal{B}_6/\langle [X, Y] \rangle$, where X and Y are transversal half-twists. We choose a point in each triangle in

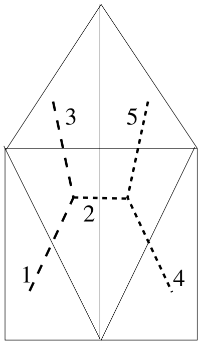

Figure 13 The tree with five generators

Figure 8. Then we choose a path h_i that connects two points in neighboring triangles, skipping the one that crosses the edge 6. This yields a tree; see Figure 13. The paths represent generators $\{H_i\}_{i=1}^5$ of the braid group $\mathcal{B}_6$ with the following complete list of relations:

$$\langle H_i, H_j\rangle = e \quad (H_i \text{ and } H_j \text{ are consecutive}), \tag{93}$$

$$[H_i, H_j] = e \quad (H_i \text{ and } H_j \text{ are disjoint}); \tag{94}$$

$$[H_4, H_5H_2H_5^{-1}] = e, \tag{95}$$

$$[H_1, H_3^{-1}H_2H_3] = e. \tag{96}$$

Let $H_{6'} = H_5H_3H_2H_3^{-1}H_5^{-1}$, where $H_{6'}$—being transversal to H_1 and H_2 and disjoint from H_4—corresponds to the missing path h_6. Recall the definition in [26, Sec. IV] of the group $\tilde{\mathcal{B}}_6$ as $\mathcal{B}_6/\langle [X,Y]\rangle$ for X and Y transversal. Denote the images of H_i as $\tilde{H}_i$ in $\tilde{\mathcal{B}}_6$. Then the group $\tilde{\mathcal{B}}_6$ is generated by $\tilde{H}_i$ $(i = 1,\dots,5,6')$, and the only relations are:

$$\langle \tilde{H}_i, \tilde{H}_j\rangle = e \quad (\tilde{H}_i \text{ and } \tilde{H}_j \text{ are consecutive}, i,j \neq 6'), \tag{97}$$

$$[\tilde{H}_i, \tilde{H}_j] = e \quad (\tilde{H}_i \text{ and } \tilde{H}_j \text{ are disjoint}, i,j \neq 6'); \tag{98}$$

$$[\tilde{H}_4, \tilde{H}_5\tilde{H}_2\tilde{H}_5^{-1}] = [\tilde{H}_4, \tilde{H}_5^{-1}\tilde{H}_2\tilde{H}_5] = e, \tag{99}$$

$$[\tilde{H}_1, \tilde{H}_3^{-1}\tilde{H}_2\tilde{H}_3] = [\tilde{H}_1, \tilde{H}_3\tilde{H}_2\tilde{H}_3^{-1}] = e, \tag{100}$$

$$\tilde{H}_5\tilde{H}_3\tilde{H}_2\tilde{H}_3^{-1}\tilde{H}_5^{-1} = \tilde{H}_{6'}. \tag{101}$$

Here $\tilde{H}_4$ and $\tilde{H}_5^{-1}\tilde{H}_2\tilde{H}_5$ ($\tilde{H}_1$ and $\tilde{H}_3\tilde{H}_2\tilde{H}_3^{-1}$, respectively) are transversal. We note that (101) can be used to remove $\tilde{H}_{6'}$ from the list of generators in the same way that $\Gamma_{6'}$ was eliminated from the presentation of $\pi_1(\mathbb{CP}^2 \setminus B_4)$.

According to our result, $\pi_1(\mathbb{CP}^2 \setminus B_4)$ is a quotient of $\tilde{\mathcal{B}}_6$. Now we eliminate (87). Since $\Gamma_3^{-2}\Gamma_1\Gamma_3^2$ and $\Gamma_3^{-1}\Gamma_2\Gamma_3$ are transversal, the relations in $\tilde{\mathcal{B}}_6$ imply that they commute; hence the left-hand side of (87) is equal to

$$\Gamma_1^{-1}\Gamma_2(\Gamma_3^{-1}\Gamma_2^{-1}\Gamma_3)\Gamma_3^{-2}\Gamma_1\Gamma_3^2(\Gamma_3^{-1}\Gamma_2\Gamma_3)\Gamma_2^{-1}\Gamma_1 = \Gamma_1^{-1}\Gamma_3^{-1}\Gamma_2^{-1}\Gamma_1\Gamma_2\Gamma_3\Gamma_1 = \Gamma_2.$$

Similarly, the right-hand side of (87) is also equal to Γ_2. This allows us to eliminate (87). Since both sides of (87) are equal to Γ_2, we have shown that $\Gamma_2 = \Gamma_{2'}$; therefore, $\Gamma_6 = \Gamma_{6'}$ (see (81) and (82)). That means that (70) is redundant, too. Thus $\pi_1(\mathbb{CP}^2 \setminus B_4)$ is isomorphic to $\tilde{\mathcal{B}}_6/\langle(92)\rangle$.

In order to get the group $\Pi_{(B_4)}$, we take $\Gamma_j^2 = e$ for each j. Relation (92) is then redundant. By [29], the rest of the relations in $\pi_1(\mathbb{CP}^2 \setminus B_4)$, together with the ones $\Gamma_j^2 = e$, are the only ones necessary to make $\Pi_{(B_4)}$ isomorphic to S_6. □

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