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On the Local Behavior of the Carmichael λ-Function

Nicolas Doyon & Florian Luca

1. Introduction

Let ϕ denote the *Euler function,* which, for an integer $n \geq 1$, is defined as usual by

$$\phi(n) = \#(\mathbb{Z}/n\mathbb{Z})^\times = \prod_{p^\nu \| n} p^{\nu-1}(p-1).$$

The *Carmichael function* λ is defined for each integer $n \geq 1$ as the largest order of any element in the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^\times$. More explicitly, for any prime power p^ν we have:

$$\lambda(p^\nu) = \begin{cases} p^{\nu-1}(p-1) & \text{if } p \geq 3 \text{ or } \nu \leq 2, \\ 2^{\nu-2} & \text{if } p = 2 \text{ and } \nu \geq 3; \end{cases}$$

and, for an arbitrary integer $n \geq 2$,

$$\lambda(n) = \mathrm{lcm}[\lambda(p_1^{\nu_1}), \dots, \lambda(p_k^{\nu_k})],$$

where $n = p_1^{\nu_1} \cdots p_k^{\nu_k}$ is the prime factorization of n. Note that $\lambda(1) = 1$.

For a positive integer n, let $\Omega(n)$, $\omega(n)$, $\tau(n)$, and $\sigma(n)$ denote (respectively) the number of prime divisors of n with and without repetitions, the total number of divisors of n, and their sum. Let f be any one of the functions Ω, ω, τ, ϕ, or σ. It is well known that, if t is any positive integer and a is any permutation of $\{1, \dots, t\}$, then there exist infinitely many positive integers n such that all inequalities $f(n + a(i)) > f(n + a(i + 1))$ hold for $i = 1, \dots, t - 1$. In fact, in [3] it is shown that, if a, b are any two permutations of $\{1, \dots, t\}$, then there exist infinitely many positive integers n such that all inequalities $\omega(n + a(i)) > \omega(n + a(i + 1))$ and $\tau(n + b(i)) > \tau(n + b(i + 1))$ hold for $i = 1, \dots, t - 1$.

In this note, we prove some effective versions of this result from [3] with the pair of functions $\{\omega, \tau\}$ replaced by the pair $\{\lambda, \phi\}$.

We use the Vinogradov symbols $\gg$, $\ll$, and $\asymp$ as well as the Landau symbols O and o with their usual meaning. We use the letters p and q for prime numbers. For a positive real number x we write $\log_1 x = \max\{1, \log x\}$, where log is the natural logarithm, and for a positive integer $k \geq 2$ we define $\log_k x = \log_1(\log_{k-1} x)$. When $k = 1$, we omit the subscript and thus understand that all the logarithms that will appear are ≥ 1. We write $\pi(x)$ for the number of primes $p \leq x$ and

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write $\pi(x; a, b)$ for the number of primes $p \le x$ in the arithmetical progression $a \pmod{b}$.

We derive two results as follows.

THEOREM 1.1. *Let t be a positive integer and let a be any permutation of the integers $\{1, 2, \ldots, t\}$. Then there exist infinitely many positive integers n such that the inequality $\lambda(n + a(i)) > \lambda(n + a(i + 1))$ holds for all $i = 1, \ldots, t - 1$. Furthermore, if $n := n(t)$ denotes the minimal value of n such that the preceding inequality holds, then the estimate $t \gg \log_2 n(t)$ holds as t tends to infinity.*

THEOREM 1.2. *Let t be a positive integer and let a and b be any permutations of the integers $\{1, 2, \ldots, t\}$. Then there exist infinitely many positive integers n such that both the inequalities $\lambda(n + a(i)) > \lambda(n + a(i + 1))$ and $\phi(n + b(i)) > \phi(n + b(i + 1))$ hold for all $i = 1, \ldots, t - 1$. Furthermore, if $n := n(t)$ denotes the minimal value of n such that the preceding inequalities hold, then the estimate $t \gg \sqrt{(\log_3 n(t))/(\log_5 n(t))}$ holds as t tends to infinity.*

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2. Preliminary Results

In this section, we prove some lower bounds on the value of the Carmichael λ-function of n when n runs in some arithmetic progression with the first term coprime to the difference. Our estimates are uniform as long as the difference of the progression does not exceed $n^{1/20}$, and as such our bounds might be of some independent interest.

We will use the following well-known facts.

LEMMA 2.1. (i) *The estimate*

$$\sum_{p>y} \frac{1}{p^2} = O\left(\frac{1}{y \log y}\right)$$

holds as y tends to infinity.

(ii) *The estimate*

$$\sum_{\substack{p \equiv A \pmod{B} \\ p \le x}} \frac{1}{p} = O\left(\frac{\log_2 x}{\phi(B)}\right)$$

holds uniformly when $1 \le A \le B \le x$ and A and B are coprime.

Proof. The first estimate follows immediately from the Prime Number Theorem by partial summation. The second estimate follows easily from the Brun–Titchmarsh theorem after some simple calculations (see also the bound (3.1) in [2] or Lemma 1 in [1]). □

LEMMA 2.2. *Let $\alpha > e$ be any constant, and put $\beta = \alpha \log(\alpha/e)$. Then the estimate*

$$\sum_{\substack{\omega(m)\ge \lfloor \alpha \log_2 x\rfloor \\ m<x}} \frac{1}{m} \ll \frac{x}{(\log x)^{\beta}}$$

holds as x tends to infinity.

Proof. This follows easily by partial summation from Theorem 4 in Section III.6 of [7] (see also [6, Sec. 2] for an elementary approach). □

The following is a well-known result due to Montgomery and Vaughan [5].

LEMMA 2.3. *Let x be a large positive real number. Assume that A and B are coprime positive integers with $B < x$. Then*

$$\pi(x; A, B) \le \frac{2x}{\phi(B)\log(x/B)}.$$

We now start our analysis by noting that $\lambda(n) = \phi(n)/S(n)$, where

$$S(n) = \prod_{p\mid\phi(n)} p^{\beta_p(n)}$$

with $\beta_p(n)$ given by

$$\beta_p(n) = \sum_{\substack{q^{\gamma}\,\|\, n \\ p^{\alpha}\,\|\,\phi(q^{\gamma})}}{}' \alpha;$$

by $\sum'$ we mean that the maximal term of the sum is not considered.

We now write $S(n)$ as $S(n) = S_1(n)\cdot S_2(n)$, where

$$S_1(n) = \prod_{\substack{p^a\,\|\,S(n) \\ p>(\log_2 n)^3}} p^a \quad\text{and}\quad S_2(n) = \prod_{\substack{p^b\,\|\,S(n) \\ p\le(\log_2 n)^3}} p^b.$$

Of course, an empty product is taken to be 1. We are now ready to prove the following lemma.

LEMMA 2.4. *Let x be a large positive real number. Assume that A and B are coprime positive integers with $B < x^{1/20}$. Then*

$$\#\{n < x : n \equiv A \pmod B \text{ and } S_1(n) > 1\} \ll \frac{x}{B\log_2 x}.$$

Proof. Let $\mathcal{A}_{A,B}(x)$ be the set in question. To simplify notation, we omit reference to the pair (A, B) and write the set simply as $\mathcal{A}(x)$.

We put $\mathcal{A}_1(x) = \{n \in \mathcal{A}(x) : p^2 | n \text{ for some prime } p > x^{1/3}\} \cup \{n < x^{1/2}\}$. We bound the cardinality of $\mathcal{A}_1(x)$. Let $p > x^{1/3}$ be some fixed prime. Then the number of positive integers $n < x$ such that $p^2|n$ is $\le x/p^2$ (note that the prime p must satisfy $p < x^{1/2}$). Therefore,

$$\#\mathcal{A}_1(x) \le x^{1/2} + \sum_{p > x^{1/3}} \frac{x}{p^2} \ll \frac{x^{2/3}}{\log x} = o\left(\frac{x}{B \log_2 x}\right). \tag{1}$$

Thus, from now on we work only with the positive integers $n \in \mathcal{A}(x) \backslash \mathcal{A}_1(x)$.

Let $n \in (x^{1/2}, x)$ be such that $S_1(n) > 1$. Since $n > x^{1/2}$, it follows that $\log_2 n > (\log_2 x)/2^{1/3}$ holds for large values of x. We may then replace $S_1(n)$ by the (possibly larger) number

$$S_1(n, x) = \prod_{\substack{p^a \,\|\, S(n) \\ p > (\log_2 x)^3/2}} p^a,$$

and this is still > 1. From now on, we look at such positive integers n.

If $p^a \,|\, S_1(n, x)$, we then distinguish two possibilities:

(i) $p^2|n$ and there exists a prime $q|n$ such that $p|q-1$;

(ii) there exist two distinct prime factors q_1 and q_2 of n such that $p|q_i - 1$ for $i = 1, 2$.

We consider these two possibilities separately.

Let $\mathcal{A}_2(x)$ be the subset of $\mathcal{A}(x) \backslash \mathcal{A}_1(x)$ consisting of those positive integers n such that there exist a prime $p > (\log_2 x)^3/2$ with $p^2|n$ and another prime factor q of n such that $q \equiv 1 \pmod{p}$. Since $n \equiv A \pmod{B}$ and $\gcd(A, B) = 1$, we must also have $\gcd(p, B) = 1$. Hence, by the Chinese Remainder Theorem, it follows that $n \equiv C \pmod{Bp^2}$ with some positive integer C that depends on A, B, and p. We now fix the prime number p. The number of such positive integers $n < x$ is at most $x/Bp^2 + 1 \le 2x/Bp^2$ if x is large enough, where we have used that $B < x^{1/20}$ and $p \le x^{1/3}$. Hence,

$$\#\mathcal{A}_2(x) \le \sum_{p > (\log_2 x)^3/2} \frac{x}{Bp^2} = o\left(\frac{x}{B(\log_2 x)^3}\right). \tag{2}$$

Next we let $\mathcal{A}_3(x)$ be the subset of $\mathcal{A}(x) \backslash \mathcal{A}_1(x)$ consisting of those positive integers n such that there exist a prime $p > (\log_2 x)^3/2$ and two prime factors q_1 and q_2 of n such that $q_i \equiv 1 \pmod{p}$ for $i = 1, 2$. We then have that $n \equiv A \pmod{B}$ and $n \equiv 0 \pmod{q_1 q_2}$. Since $\gcd(A, B) = 1$, it follows that $\gcd(q_1 q_2, B) = 1$. According to the Chinese Remainder Theorem, these two congruences are equivalent to a congruence of the form $n \equiv C \pmod{Bq_1q_2}$ with some positive integer C. The number of positive integers $n < x$ satisfying this last congruence certainly cannot exceed $x/Bq_1q_2 + 1$.

We now distinguish two cases.

Case 1: $Bq_1q_2 \le x$. We write $\mathcal{A}_{3,1}(x)$ for the subset of $\mathcal{A}_3(x)$ consisting of those positive integers n that satisfy the hypothesis of Case 1 for some triple of primes p, q_1, and q_2.

In this case, the number of such positive integers $n < x$ that are $\equiv C \pmod{Bq_1q_2}$ is $\le 2x/Bq_1q_2$. Hence, the number of such positive integers does not exceed

$$\begin{aligned}
\#\mathcal{A}_{3,1}(x) &\ll \sum_{p>(\log_2 x)^3/2} \sum_{\substack{q_1\equiv 1 \pmod p\\ q_1<x}} \sum_{\substack{q_2\equiv 1 \pmod p\\ q_2<x/Bq_1}} \frac{x}{Bq_1q_2}\\
&\ll \frac{x}{B} \sum_{p>(\log_2 x)^3/2} \sum_{\substack{q_1\equiv 1 \pmod p\\ q_1<x}} \frac{\log_2 x}{q_1 p}\\
&\ll \frac{x(\log_2 x)^2}{B} \sum_{p>(\log_2 x)^3/2} \frac{1}{p^2}\\
&= o\left(\frac{x}{B\log_2 x}\right). \qquad (3)
\end{aligned}$$

Case 2: $Bq_1q_2 > x$. In this case, we write n both as $n = m_1B + A$ and as $n = m_2q_1q_2$. We are looking for the number of solutions $n < x$ to the equation

$$n = m_1B + A = m_2q_1q_2. \qquad (4)$$

Suppose $q_1 < q_2$. We fix m_2 and q_1. Observe that $\gcd(m_2q_1, B) = 1$ because $\gcd(m_2q_1, B)$ divides both A and B, which are coprime by hypothesis. We also fix the prime number p. We first assume that $p \nmid B$, and we write $\mathcal{A}_{3,2}(x)$ for the set of such positive integers n.

Let $(m_2q_1)^{-1}$ stand for the multiplicative inverse of m_2q_1 modulo B. We then have that $q_2 \equiv A(m_2q_1)^{-1} \pmod B$ and also that $q_2 \equiv 1 \pmod p$. So, according to the Chinese Remainder Theorem, we must have $q_2 \equiv C \pmod{pB}$ with some positive integer C that is determined in terms of A, B, m_2, q_1, and p. Since we also know that $q_2 < x/m_2q_1$, by Lemma 2.3 it follows that the number of such primes q_2 does not exceed

$$\frac{2x}{m_2q_1\phi(Bp)\log(x/(Bpm_2q_1))}.$$

Because $q_1 < q_2$, q_1q_2 divides n, and $n < x$, we immediately obtain $q_1 < x^{1/2}$. Since $n = m_2q_1q_2 < x$ and $Bq_1q_2 > x$, we get $m_2 < B < x^{1/20}$. Finally, since $p < x^{1/3}$ (because $n \notin \mathcal{A}_1(x)$), we have

$$Bpm_2q_1 < x^{1/20+1/3+1/20+1/2} = x^{14/15};$$

therefore,

$$\log(x/(Bpm_2q_1)) \gg \log x.$$

As a result, the number of primes q_2 is $\ll x/(m_2q_1p\phi(B)\log x)$. It now follows that the total number of positive integers n in $\mathcal{A}_{3,2}(x)$ does not exceed

$$\begin{aligned}
\#\mathcal{A}_{3,2}(x) &\ll \sum_{p>(\log_2 x)^3/2}\ \sum_{\substack{q_1\equiv 1 \pmod p\\ q_1<x}}\ \sum_{m_2<B} \frac{x}{m_2 q_1 \phi(B) p \log x}\\
&\ll \frac{x\log B}{\phi(B)\log x} \sum_{p>(\log_2 x)^3/2}\ \sum_{\substack{q_1\equiv 1 \pmod p\\ q_1<x}} \frac{1}{q_1 p}\\
&\ll \frac{x\log_2 x}{\phi(B)} \sum_{p>(\log_2 x)^3/2} \frac{1}{p^2}\\
&= o\left(\frac{x}{\phi(B)(\log_2 x)^2}\right) = o\left(\frac{x}{B\log_2 x}\right). \qquad (5)
\end{aligned}$$

In these inequalities we used both that $\log B < \log x$ and that $\phi(B) \gg B/\log_2 B \gg B/\log_2 x$.

We finally look at the possibility when $p|B$. We write $\mathcal{A}_{3,3}(x)$ for the subset of $\mathcal{A}_3(x)$ formed by these last numbers n. Fixing m_2, p, and q_1, we find that the number of possible primes $q_2 \le x/m_2 q_1$ is (again by Lemma 2.3) at most

$$O\left(\frac{x}{q_1 m_2 \phi(B)\log(x/(Bq_1 m_2))}\right) = O\left(\frac{x\log_2 x}{q_1 B m_2 \log x}\right), \qquad (6)$$

provided that $m_2 \equiv A \pmod p$ and that there is no such prime otherwise. Summing up inequalities (6) over all possible values of $m_2 < x$ such that $m_2 \equiv A \pmod p$, and then over all primes $q_1 < x$ such that $q_1 \equiv 1 \pmod p$, we get that the number of such possibilities is

$$\ll \frac{x\log_2 x}{B\log x} \sum_{\substack{m_2\equiv A \pmod p\\ m_2<x}}\ \sum_{\substack{q_1\equiv 1 \pmod p\\ q_1<x}} \frac{1}{m_2 q_1} \ll \frac{x(\log_2 x)^2}{Bp^2}.$$

Summing up the last of these bounds over all those prime factors $p > (\log_2 x)^3/2$ of B, we find that the number of such numbers $n < x$ is

$$\begin{aligned}
\#\mathcal{A}_{3,3}(x) &\ll \frac{x(\log_2 x)^2}{B} \sum_{\substack{p>(\log_2 x)^3/2\\ p|B}} \frac{1}{p^2} \le \frac{x(\log_2 x)^2}{B} \sum_{p>(\log_2 x)^3/2} \frac{1}{p^2}\\
&= o\left(\frac{x}{B\log_2 x}\right). \qquad (7)
\end{aligned}$$

From inequalities (3), (5), and (7) it follows that

$$\#\mathcal{A}_3(x) \le \#\mathcal{A}_{3,1}(x) + \#\mathcal{A}_{3,2}(x) + \#\mathcal{A}_{3,3}(x) \ll \frac{x}{B\log_2 x},$$

which together with estimates (1) and (2) completes the proof of Lemma 2.4. □

We will also need the following result, which gives us information about the behavior of the function $S_2(n)$ when n runs in arithmetical progressions with the first term coprime to the difference (which is again uniform in the difference of the progression).

LEMMA 2.5. *Let again x, A, B be as in the statement of Lemma 2.4. Assume further that B is cubefree. Let*

$$\mathcal{B}(x) = \left\{ n < x : n \equiv A \pmod{B} \text{ and } S_2(n) > \exp\left(\frac{(\log_2 x)^5}{2}\right) \right\}.$$

Then

$$\#\mathcal{B}(x) \ll \frac{x}{B \log_2 x}.$$

Proof. If $S_2(n) > 1$, it follows that we may replace $S_2(n)$ by the (presumably larger) number

$$S_2(n, x) = \prod_{\substack{p^b \| S(n) \\ p \le (\log_2 x)^3}} p^b$$

and look at the set of positive integers n such that $S_2(n, x) > \exp((\log_2 x)^5/2)$.

We now write $\mathcal{B}_1(x)$ for the subset of $\mathcal{B}(x)$ consisting of those positive integers n such that $\omega(n) < 10 \log_2 x$. Since the only prime factors of $S_2(n)$ are smaller than $(\log_2 x)^3$, we trivially have that $\omega(S_2(n)) < (\log_2 x)^3$. Thus, in order to deduce Lemma 2.5 at least for those $n \in \mathcal{B}_1(x)$, it suffices to prove that the inequality

$$\#\Big\{ n < x : n \equiv A \pmod{B},\ \omega(n) < 10 \log_2 x, \\ \text{and } \max\{p^\alpha : p^\alpha \| S_2(n)\} > \exp\left(\frac{(\log_2 x)^2}{2}\right) \Big\} \ll \frac{x}{B \log_2 x} \tag{8}$$

holds. But all our primes p under scrutiny are $< (\log_2 x)^3$ and so, if we put $y = y(x) = (\log_2 x)^2/(6 \log_3 x)$, it then suffices to show that the inequality

$$\#\{ n < x : n \equiv A \pmod{B},\ \omega(n) < 10 \log_2 x, \\ \text{and } p^\alpha \| S_2(n) \text{ for some prime } p \text{ and } \alpha > y \} \ll \frac{x}{B \log_2 x} \tag{9}$$

holds.

We write $\mathcal{B}_2(x)$ for the set appearing in the left-hand side of inequality (9). Let $n \in \mathcal{B}_2(x)$. Since $\omega(n) < 10 \log_2 x$, there exists a prime q such that $q^a \| n$ as well as a prime p and an integer b such that $p^b | \phi(q^a)$, where $b \ge z = z(x) = \lfloor (\log_2 x)/(60 \log_3 x) \rfloor$.

From here on, we distinguish several cases. Let $\mathcal{B}_3(x)$ be the subset of $\mathcal{B}_2(x)$ for which $p \ne q$. We then have that $p^z | (q - 1)$. We also have that $(q, B) = 1$ because $q | n$, $n \equiv A \pmod{B}$, and A and B are coprime. The Chinese Remainder Theorem now implies that $n \equiv C \pmod{Bq}$ holds with some positive integer C depending on A, B, and q. The number of such positive integers $n < x$ is at most $x/Bq + 1$. We write $\mathcal{B}_{3,1}(x)$ for the subset of $\mathcal{B}_3(x)$ such that $Bq \le x$ and write $\mathcal{B}_{3,2}(x)$ for the complement of $\mathcal{B}_{3,1}(x)$ in $\mathcal{B}_3(x)$.

We find an upper bound on $\#\mathcal{B}_{3,1}(x)$. Since for such n we have $Bq \le x$, it follows that $x/Bq + 1 \le 2x/Bq$. We allow p, q to vary and conclude that

$$\begin{aligned}\#\mathcal{B}_{3,1}(x) &\ll \sum_{p<(\log_2 x)^3} \sum_{\substack{q\equiv 1 \pmod{p^z}\\ q<x}} \frac{x}{Bq} \\ &\ll \frac{x\log_2 x}{B} \sum_{p<(\log_2 x)^3} \frac{1}{p^z} \\ &\ll \frac{x\log_2 x}{B2^{z-1}} = o\left(\frac{x}{B\log_2 x}\right). \end{aligned} \tag{10}$$

We now find an upper bound on $\#\mathcal{B}_{3,2}(x)$. Let B and q be such that $Bq > x$. Since B is cubefree by hypothesis, we have that $\gcd(B, p^z)|p^2$. In this situation, we write $n = m_1B + A = m_2q < x$. Fixing m_2, we see that q is in a certain arithmetical progression modulo B. Since $q \equiv 1 \pmod{p^z}$, the Chinese Remainder Theorem tells us that $q \equiv C \pmod{Bp^{z-2}}$, where C is some positive integer depending on A, B, m_2, p, and z. Note that $B < x^{1/20}$ and $p^z < \exp(O(\log_2 x)) = x^{o(1)}$; therefore, $Bp^{z-2} < x^{1/3}$ if x is sufficiently large. By applying Lemma 2.3 once again, we obtain that the number of eligible primes q is

$$\leq \frac{2x}{m_2\phi(B)p^{z-3}(p-1)\log(x/Bp^{z-2}m_2)} \ll \frac{x\log_2 x}{m_2Bp^{z-2}\log x}. \tag{11}$$

Here we have used the fact that $m_2 < B < x^{1/20}$ to conclude that $Bp^{z-2}m_2 < x^{1/2}$ and hence that $\log(x/Bp^{z-2}m_2) \gg \log x$ if x is sufficiently large.

Summing up inequality (11) over all the possible values of $m_2 < x^{1/20}$ and $p < (\log_2 x)^3$, the result is

$$\begin{aligned}\#\mathcal{B}_{3,2}(x) &\ll \frac{x\log_2 x}{B\log x} \sum_{m_2<x^{1/20}} \sum_{p<(\log_2 x)^3} \frac{1}{m_2p^{z-2}} \ll \frac{x\log_2 x}{B2^{z-3}} \\ &= o\left(\frac{x}{B\log_2 x}\right). \end{aligned} \tag{12}$$

From (10) and (12) we may then deduce

$$\#\mathcal{B}_3(x) \leq \#\mathcal{B}_{3,1}(x) + \#\mathcal{B}_{3,2}(x) \ll \frac{x}{B\log_2 x}. \tag{13}$$

We now look at the set $\mathcal{B}_4(x)$, which is the subset of $\mathcal{B}_2(x)$ consisting of those positive integers n for which $p = q$. In this case, $p^z|n$ and it is clear that p and B are coprime. It is also clear that $p^zB < x$ when x is sufficiently large (because $B < x^{1/20}$ and $p^z = x^{o(1)}$), so by the Chinese Remainder Theorem we must again have $n \equiv C \pmod{Bp^z}$ for some positive integer C depending on A, B, p, and z. The number of positive integers $n < x$ satisfying this congruence is smaller than $2x/Bp^z$. Summing this inequality over all possible values for p yields

$$\#\mathcal{B}_4(x) \ll \sum_{p<(\log_2 x)^3} \frac{x}{Bp^z} \ll \frac{x}{B2^{z-1}} = o\left(\frac{x}{B\log_2 x}\right). \tag{14}$$

From inequalities (13) and (14) we obtain

$$\#\mathcal{B}_2(x) \le \#\mathcal{B}_{3,1}(x) + \#\mathcal{B}_{3,2}(x) + \#\mathcal{B}_4(x) \ll \frac{x}{B \log_2 x},$$

which proves inequality (9) and thus inequality (8) as well.

We now let

$$\mathcal{B}_5(x) = \{n < x : n \equiv A \pmod{B} \text{ and } \omega(n) > 10 \log_2 x\}.$$

To complete the proof of Lemma 2.5, it suffices to show that

$$\#\mathcal{B}_5(x) \ll \frac{x}{B \log_2 x}. \tag{15}$$

Put $\lambda = \lambda(x) = \lfloor 5 \log_2 x \rfloor - 1$.

Each integer n in $\mathcal{B}_5(x)$ can be written as $u \cdot v$, with

$$u = \prod_{i=1}^{\lambda} p_i(n) \quad \text{and} \quad v = \frac{n}{u};$$

here $p_i(n)$ stands for the ith distinct prime factor of n when arranged in increasing order. Since n satisfies both $n < x$ and $\omega(n) > 10 \log_2 x$, we have $\omega(u) \le \omega(n)/2$ and therefore $u < x^{1/2}$. Since $u | n$, it follows that u and B are coprime. If we fix u then $n \equiv C \pmod{uB}$ for some positive integer C depending on A, B, and u. The number of such $n < x$ is clearly at most $x/Bu + 1 \le 2x/Bu$, since $Bu < x^{1/20+1/2} < x$. It now follows that

$$\#\mathcal{B}_5(x) \le \sum_{\substack{u < x^{1/2} \\ \omega(u) = \lambda}} \frac{x}{Bu}. \tag{16}$$

Since

$$\lambda = \lfloor 5 \log_2 x \rfloor - 1 = \lfloor 5 \log_2(x^{1/2}) + 5 \log 2 \rfloor - 1 > 5 \log_2(x^{1/2})$$

(because $5 \log 2 > 3$), Lemma 2.2 with $\alpha = 5$ and $\beta = 5 \log(5/e) > 1$ gives us immediately that

$$\sum_{\substack{m < x^{1/2} \\ \omega(u) = \lambda}} \frac{x}{Bu} = O\left(\frac{x}{B(\log x)^{\beta}}\right) = o\left(\frac{x}{B \log_2 x}\right),$$

which together with estimate (16) implies estimate (15) and so completes the proof of Lemma 2.5. □

LEMMA 2.6. *Let x be a large positive real number and let A and B be coprime positive integers such that $B < x^{1/20}$ and B is cubefree. Then there exists an x_0 such that the estimate*

$$\#\left\{n < x : n \equiv A \pmod{B} \text{ and } \lambda(n) < \frac{n}{\exp(2(\log_2 x)^5/3)}\right\} \ll \frac{x}{B \log_2 x}$$

holds for all $x > x_0$.

Proof. Lemma 2.6 follows immediately from Lemma 2.4, Lemma 2.5, and the fact that the estimate

$$\phi(n) \gg \frac{n}{\log_2 n} \gg \frac{n}{\log_2 x}$$

holds for all positive integers $n < x$; hence, the estimate

$$\phi(n) > \frac{n}{\exp((\log_2 x)^5/6)}$$

holds for all positive integers $n < x$ and all $x > x_0$. □

3. Proofs of the Main Results

We are now ready to tackle the proofs of Theorems 1 and 2.

Proof of Theorem 1.1

We begin with the following result due to Heath-Brown [4].

LEMMA 3.1. *There exists a positive constant B_0 such that, if $B > B_0$ is an integer and A is coprime to B, then there exists a prime $P < B^6$ in the arithmetical progression A* (mod B).

For the purpose of the proof of Theorem 1.1, we let $M_1 < M_2 < \cdots < M_t$ be t distinct integers with $M_1 > t$ and let $P_{i,j}$ be distinct primes that are less than M_i^{36} for $j = 1, 2$ such that $P_{i,j} \equiv 1 \pmod{M_i}$ for all $i = 1, \ldots, t$. The existence of such numbers $P_{i,j}$ is guaranteed by twice applying Lemma 3.1 to each M_i for $i = 1, \ldots, t$. Further, we will also see that the numbers M_i can be chosen to be primes and that the inequality $M_{i+1} > M_i^{36}$ holds for all $i \in \{1, \ldots, t-1\}$. Thus, the primes $P_{i,j}$ are all distinct for $i = 1, \ldots, t$ and $j = 1, 2$.

We now set B as

$$B = \prod_{i \le t} (P_{i,1} P_{i,2})^2.$$

Let x be a large positive real number (to be determined later) that depends on t and consider those positive integers $n < x$ such that

$$n \equiv -a(i) + P_{i,1} P_{i,2} \pmod{(P_{i,1} P_{i,2})^2} \quad \text{for all } i = 1, \ldots, t.$$

The Chinese Remainder Theorem allows us to conclude that these congruences are equivalent to $n \equiv A \pmod{B}$ for some positive integer A coprime to B, because $P_{i,j} > t$ and $a(i) \le t$ for all $i = 1, \ldots, t$ and $j = 1, 2$. For such a positive integer n, we write $n + a(i) = m_i P_{i,1} P_{i,2}$, where m_i and $P_{i,1} P_{i,2}$ are coprime. Then we have

$$\lambda(n + a(i)) = \frac{\lambda(m_i)(P_{i,1} - 1)(P_{i,2} - 1)}{\gcd(\lambda(m_i), (P_{i,1} - 1)(P_{i,2} - 1)) \gcd((P_{i,1} - 1), (P_{i,2} - 1))}. \tag{17}$$

For the time being, we suppose that

$$\lambda(m_i) \ge \frac{m_i}{\exp(2(\log_2 x)^5/3)} \quad \text{for all } i = 1, \ldots, t. \tag{18}$$

Therefore,

$$\lambda(n+a(i)) \ge \frac{\lambda(m_i)}{P_{i,1}-1} > \frac{m_i}{P_{i,1}\exp(2(\log_2 x)^5/3)} = \frac{n+a(i)}{P_{i,1}^2 P_{i,2}\exp(2(\log_2 x)^5/3)}.$$

We then obtain

$$\frac{n+a(i)}{P_{i,1}^2 P_{i,2}\exp(2(\log_2 x)^5/3)} < \lambda(n+a(i)) < \frac{n+a(i)}{M_i}. \tag{19}$$

We want to make sure that the inequality $\lambda(n+a(i)) > \lambda(n+a(i+1))$ holds. According to (19), this inequality will certainly hold provided that

$$P_{i,1}^2 P_{i,2}\exp\left(\frac{2(\log_2 x)^5}{3}\right) < M_{i+1}\left(\frac{n+a(i)}{n+a(i+1)}\right). \tag{20}$$

We will choose n such that $n > A$; hence, $n > B > t^t$. It is then clear that the inequality

$$\frac{n+a(i)}{n+a(i+1)} \ge \frac{n}{n+t} > \frac{1}{2} > \exp\left(-\frac{(\log x)^5}{3}\right)$$

holds once t (and hence x) is sufficiently large. Thus, inequality (20) will certainly be satisfied if

$$P_{i,1}^2 P_{i,2}\exp((\log_2 x)^5) \le M_{i+1},$$

when x is sufficiently large. Since $P_{i,1}^2 P_{i,2} < (M_i^{36})^3 = M_i^{108}$, it suffices that the inequality

$$108\log M_i + (\log_2 x)^5 \le \log M_{i+1} \quad \text{holds for } i = 1,\ldots,t-1.$$

Since the interval $(y, 2y)$ contains a prime number for all $y > 1$, it follows that we may further assume that M_i is prime for all $i = 1,\ldots,t$ and that

$$\log M_{i+1} \le 108\log M_i + (\log_2 x)^5 + \log 2$$

holds for all $i = 1,\ldots,t-1$, where $\log M_1 \le \log(2t)$. By induction on i, one shows that the inequality

$$\log M_i \le 108^{i-1}(\log(2t) + (\log_2 x)^5 + \log 2)$$

holds for all $i = 1,\ldots,t$; therefore,

$$\begin{aligned}\log B &= 2\sum_{i=1}^{t}\sum_{j=1}^{2}\log P_{i,j} \le 4\cdot 36\sum_{i=1}^{t}\log M_i \\ &\le 144\left(\frac{108^t-1}{107}\right)(\log(2t) + (\log_2 x)^5 + \log 2).\end{aligned} \tag{21}$$

We will apply Lemma 2.6 and so will need the inequality $B < x^{1/20}$ to hold. Since $144/107 < 2$, by (21) it suffices that the inequality

$$2\cdot 108^t(\log(2t) + (\log_2 x)^5 + \log 2) < \frac{\log x}{20}$$

holds. Since $e^5 > 108$, it follows easily that this last inequality is satisfied provided that we choose

$$t \le \tfrac{1}{5} \log_2 x, \tag{22}$$

and that x is large.

Finally, we must show that we can choose $n \equiv A \pmod{B}$ such that all inequalities (18) hold. In order to do so, we shall apply Lemma 2.6. Let n be such that, for some $i = 1, \dots, t$,

$$\lambda(m_i) < \frac{m_i}{\exp(2(\log_2 x)^5/3)}. \tag{23}$$

Note that $m_i < x_i = x/P_{i,1}P_{i,2}$ and that $m_i \equiv A_i \pmod{B_i}$ with

$$B_i = \frac{B}{P_{i,1}P_{i,2}} < x_i^{1/20},$$

where the above inequality holds because it is implied by the inequality $B < x^{1/20}$. Here, A_i is some positive integer coprime to B_i that depends on A, B, $P_{i,1}$, and $P_{i,2}$. Note also that B is cubefree (and hence B_i is cubefree as well). Inequality (23) shows that

$$\lambda(m_i) < \frac{m_i}{\exp(2(\log_2 x_i)^5/3)},$$

and now Lemma 2.6 guarantees that the number of such positive integers is

$$\ll \frac{x_i}{B_i \log_2 x_i} \ll \frac{x}{B \log_2 x}. \tag{24}$$

Let γ be the absolute constant implied by (24). It follows that the cardinality of the set of positive integers $n < x$ in the arithmetical progression $n \equiv A \pmod{B}$ such that one of the inequalities (18) fails for some $i = 1, \dots, t$ is at most $\le \gamma t x/(B \log_2 x)$. Putting $\kappa = \min\{\gamma/2, 1/5\}$ (see (22)) and choosing

$$t = \lfloor \kappa \log_2 x \rfloor, \tag{25}$$

we see that the number of such positive integers is $\le x/(2B)$. Since there are at least $x/B - 2$ such positive integers, it follows that there exist at least $x/(2B) - 2$ positive integers $n \equiv A \pmod{B}$ that satisfy all the inequalities asserted at (18). Theorem 1.1 is therefore completely proved.

Proof of Theorem 1.2

The method is somewhat similar to the proof of Theorem 1.1, although a bit more complicated.

We let x be a large positive real number and put $t = t(x)$ for some integer to be determined later. Let $y_0 = y_0(x) = t$. For $i = 1, \dots, t$, let $y_i = y_i(x)$ be defined inductively as the smallest positive integer such that the inequality

$$\prod_{y_{i-1} < p < y_i} \left(1 - \frac{1}{p}\right) < (5 \log t)^{-i} \tag{26}$$

holds. It is clear that, if t is large, then

$$\prod_{y_{i-1}<p<y_i}\left(1-\frac{1}{p}\right) > \frac{1}{2(5\log t)^i} > \frac{1}{10(\log t)}\prod_{y_{i-2}<p<y_{i-1}}\left(1-\frac{1}{p}\right) \tag{27}$$

for all $i = 2, \ldots, t$. Note also that all numbers y_i are primes. Inductively, it follows that

$$\prod_{p<y_i}(1-p^{-1}) > \frac{1}{2^i(5\log t)^{i(i+1)/2}}\prod_{p<t}\left(1-\frac{1}{p}\right) \gg \frac{1}{2^i(5\log t)^{i(i+1)/2+1}}.$$

Since

$$\prod_{p<y_i}(1-p^{-1}) \asymp \frac{1}{\log y_i},$$

we deduce that

$$\log y_i \ll 2^i(5\log t)^{i(i+1)/2+1} \ll (\log t)^{2(i+1)^2}$$

holds for all $i = 1, \ldots, t$. Therefore,

$$y_t < \exp(\exp((1+o(1))2t^2\log_2 t)). \tag{28}$$

A similar argument shows that

$$\log y_i \gg (5\log t)^{i(i+1)/2}$$

and hence

$$y_t > \exp\left(\exp\left(\tfrac{1}{2}t^2\log_2 t\right)\right) \tag{29}$$

for large t. We now set

$$Y_i = \prod_{y_{i-1}<p<y_i} p \quad \text{for all } i = 1, \ldots, t.$$

To define the number B, we put $T = y_t$ and let $M_1 < \cdots < M_t$ be distinct integers, with $M_1 > T$ and primes $P_{i,j}$ that are $< M_i^{36}$ in the arithmetical progression $1 \pmod{M_i}$ for $j = 1, 2$ and all $i = 1, \ldots, t$. The existence of such primes is guaranteed by applying Lemma 3.1 twice for each M_i for $i = 1, \ldots, t$; if we further assume that $M_{i+1} > M_i^{36}$ holds for all $i = 1, \ldots, t-1$, then all these primes $P_{i,j}$ are distinct.

We now set

$$B = \prod_{i=1}^{t}(P_{i,1}P_{i,2}Y_i)^2.$$

Choose $n < x$ such that

$$n + a(i) + P_{i,1}P_{i,2} \equiv 0 \pmod{(P_{i,1}P_{i,2})^2} \quad \text{and}$$

$$n + b(i) + Y_i \equiv 0 \pmod{Y_i^2}.$$

The Chinese Remainder Theorem now shows that $n < x$ is in an arithmetical progression $n \equiv A \pmod{B}$.

Next we examine the values of the Carmichael λ-functions. For a positive integer $n < x$ in the arithmetical progression $A \pmod{B}$, we write it as $n + a(i) = m_i Y_{j(i)} P_{i,1} P_{i,2}$. Here, $j(i)$ is the only index such that $a(i) = b(j(i))$. We note that

$$\gcd(m_i, Y_{j(i)}P_{i,1}P_{i,2}) = \gcd(Y_{j(i)}, P_{i,1}P_{i,2}) = 1.$$

Now we have

$$\lambda(n+a(i)) \\ = \frac{\lambda(m_iY_{j(i)})(P_{i,1}-1)(P_{i,2}-1)}{\gcd(\lambda(m_iY_{j(i)}), (P_{i,1}-1)(P_{i,2}-1))\gcd((P_{i,1}-1),(P_{i,2}-1))}. \quad (30)$$

For the time being, suppose that

$$\lambda(m_i) > \frac{m_i}{\exp(2(\log_2 x)^5/3)} \quad \text{for all } i = 1, \ldots, t. \quad (31)$$

Since $\lambda(m_i)|\lambda(m_iY_{j(i)})$, it follows that

$$\lambda(n+a(i)) \geq \frac{\lambda(m_i)}{P_{i,1}-1} > \frac{m_i}{P_{i,1}\exp(2(\log_2 x)^5/3)} \\ = \frac{n+a(i)}{Y_{j(i)}P_{i,1}^2P_{i,2}\exp(2(\log_2 x)^5/3)}.$$

Therefore,

$$\frac{n+a(i)}{Y_{j(i)}P_{i,1}^2P_{i,2}\exp(2(\log_2 x)^5/3)} < \lambda(n+a(i)) < \frac{n+a(i)}{M_i}. \quad (32)$$

We want to make sure that the inequality $\lambda(n+a(i)) > \lambda(n+a(i+1))$ holds. According to (32), this inequality will certainly hold provided that

$$P_{i,1}^2P_{i,2}Y_{j(i)}\exp\left(\frac{2(\log_2 x)^5}{3}\right) < M_{i+1}\left(\frac{n+a(i)}{n+a(i+1)}\right). \quad (33)$$

We will choose n such that $n > A$; hence, $n > B > y_t^t$. It is then clear that

$$\frac{n+a(i)}{n+a(i+1)} \geq \frac{n}{n+t} > \frac{1}{2} > \exp\left(-\frac{(\log_2 x)^5}{3}\right)$$

holds once t (and hence x) is sufficiently large. Thus, (33) will certainly be satisfied if

$$P_{i,1}^2P_{i,2}Y_{j(i)}\exp((\log_2 x)^5) \leq M_{i+1},$$

provided that x is sufficiently large. Since $P_{i,1}^2P_{i,2} < M_i^{108}$ and since

$$\log Y_{j(i)} \leq 2\sum_{p<y_t}\log p < 3y_t$$

(provided that t is sufficiently large), we need only show

$$108\log M_i + 3y_t + (\log_2 x)^5 \leq \log M_{i+1} \quad \text{for } i = 1, \ldots, t-1.$$

The interval $(y, 2y)$ contains a prime number for all $y > 1$, so we may further assume that M_i is prime for all $i = 1, \ldots, t$ and that

$$\log M_{i+1} \leq 108\log M_i + 3y_t + (\log_2 x)^5 + \log 2$$

for all $i = 1, \dots, t-1$, where $\log M_1 \le \log(2y_t)$. By induction on i, one shows that the inequality

$$\log M_i \le 108^{i-1}(\log(2y_t) + 3y_t + (\log_2 x)^5 + \log 2)$$

holds for all $i = 1, \dots, t$; therefore,

$$\begin{aligned}\log B = 2\sum_{i=1}^{t}\sum_{j=1}^{2}\log P_{i,j} &\le 4\cdot 36\sum_{i=1}^{t}\log M_i \\ &\le 144\left(\frac{108^t-1}{107}\right)(\log(2t) + 3y_t + (\log_2 x)^5 + \log 2).\end{aligned}$$

We will apply Lemma 2.6, so we will need the inequality $B < x^{1/20}$ to hold. Since $144/107 < 2$ and $3y_t > \log(2y_t) + \log 2$ once t is large, it suffices that

$$4\cdot 108^t(3y_t + (\log_2 x)^5) < \frac{\log x}{20}.$$

Using estimate (28), it follows easily that this last inequality is satisfied provided we choose x large and t such that

$$3t^2\log_2 t < \log_3 x. \tag{34}$$

This means that we need only show that

$$t < \frac{1}{2}\sqrt{\frac{\log_3 x}{\log_5 x}} \tag{35}$$

and that x is large.

The concluding argument from the proof of Theorem 1.1 demonstrates that, in the arithmetical progression $A \pmod B$, the number of positive integers $n < x$ such that at least one of the inequalities (31) fails is

$$O\left(\frac{xt}{B\log_2 x}\right).$$

Observe, moreover, that for such integers we have $n + b(i) = Y_i l_i$, where Y_i and l_i are coprime. Also, l_i is in a certain arithmetical progression $A_i \pmod{B_i}$, where $B_i = B/Y_i$ and A_i is an integer that is coprime to B_i. The concluding argument from the proof of Lemma 2.5 shows that the number of such positive integers $l_i < x/Y_i$ such that $\omega(l_i) > 10\log_2 x$ is $O(x/B\log_2 x)$. This argument—and given that t is small (see (35))—shows that, except for a set of cardinality

$$O\left(\frac{xt}{B\log_2 x}\right) = o\left(\frac{x}{B}\right),$$

all numbers $n < x$ in the arithmetical progression $A \pmod B$ fulfill all inequalities (31) for $i = 1, \dots, t$ as well as the inequalities $\omega(l_i) < 10\log_2 x$. In particular, we know that $\lambda(n + a(i)) > \lambda(n + a(i+1))$ for all $i = 1, \dots, t-1$.

We are now finally ready to look at the values of the Euler function. On the one hand, we have

$$\phi(n+b(i)) \le (n+b(i)) \prod_{y_i<p<y_{i+1}} \left(1-\frac{1}{p}\right). \tag{36}$$

On the other hand, we also have

$$\phi(n+b(i)) \ge (n+b(i)) \prod_{y_i<p<y_{i+1}} \left(1-\frac{1}{p}\right) \prod_{\substack{p\mid n+b(i)\\ p<t}} \left(1-\frac{1}{p}\right) \prod_{\substack{p\mid n+b_i\\ p>y_t}} \left(1-\frac{1}{p}\right).$$

Now

$$\prod_{\substack{p\mid n+b(i)\\ p<t}} \left(1-\frac{1}{p}\right) > \prod_{p<t}\left(1-\frac{1}{p}\right) > \frac{1}{2(\log t)},$$

where the last inequality holds for large t by Mertens's estimate. Using that

- $1-y > \exp(-y/2)$ if y is a sufficiently small positive number,
- $\omega(l_i) \le 10\log_2 x$, and
- every prime factor dividing $n+b(i)$ and larger than y_t divides also l_i,

it follows that

$$1 > \prod_{\substack{p\mid n+b(i)\\ p>y_t}} \left(1-\frac{1}{p}\right) > \exp\left(-\frac{1}{2}\sum_{\substack{p\mid n+b(i)\\ p>y_t}} \frac{1}{p}\right) > \exp\left(-\frac{5\log_2 x}{y_t}\right).$$

From estimate (29) we conclude easily that if

$$\exp\big(\exp\big(\tfrac{1}{2}t^2\log_2 t\big)\big) > (\log_2 x)^2 \tag{37}$$

then

$$\prod_{\substack{p\mid n+b(i)\\ p>y_t}} \left(1-\frac{1}{p}\right) = 1+o(1).$$

Inequality (37) is satisfied for large x if

$$t > 2\sqrt{\frac{\log_4 x}{\log_6 x}},$$

which is consistent with (35). Using now the fact that

$$\frac{n+b(i)}{n+b(i-1)} = 1+o(1) > \frac{1}{2}$$

(because $n > B > y_t^t$) as well as inequality (27), we obtain

$$\begin{aligned}
\phi(n+b(i)) &\ge (n+b(i))\left(\frac{1+o(1)}{2\log t}\right) \prod_{y_i<p<y_{i+1}} \left(1-\frac{1}{p}\right)\\
&> (n+b(i-1))\left(\frac{1+o(1)}{4\log t}\right) \prod_{y_i<p<y_{i+1}} \left(1-\frac{1}{p}\right)\\
&> (n+b(i-1))\left(\frac{1}{10\log t}\right) \prod_{y_{i+1}<p<y_{i+2}} \left(1-\frac{1}{p}\right)\\
&> \phi(n+b(i-1))
\end{aligned}$$

for all $i = 2, \ldots, t$ when x is large. Since b was an arbitrary permutation of $\{1, \ldots, t\}$, we may replace b by its inverse and obtain the desired inequalities. This completes the proof of Theorem 1.2.

4. Final Remarks

Theorems 1.1 and 1.2 show (respectively) that the estimates

$$t \gg \log_2 n(t) \quad \text{and} \quad t \gg \sqrt{\frac{\log_3 n(t)}{\log_5 n(t)}}$$

hold. It would be interesting to estimate the true value of $n(t)$. In what follows, we give some nontrivial lower bounds on these functions.

Let us consider first the case when the permutations a and b are taken to be identical. Since for large values of x there is always a prime number between x and $x + x^{7/12}$, we obviously have that $n(t) \gg t^{12/7}$ in the case of Theorem 1.1. In the case of Theorem 1.2, we can easily prove a slightly better lower bound. Namely, for large t, the interval $[n, n+t]$ contains a positive integer (let's call it n_0) such that $Y(t)|n_0$, where $Y = \prod_{p<(\log t)/2} p$. We then have

$$\phi(n_0) < n_0 \prod_{p<(\log t)/2} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma} n_0}{\log_2 t}.$$

On the other hand,

$$\begin{aligned}
\phi(n_0 \pm 1) &= (n_0 \pm 1) \prod_{p|n_0 \pm 1} \left(1 - \frac{1}{p}\right) \\
&> (n_0 \pm 1) \prod_{(\log t)/2 < p < 2\log(n+t)} \left(1 - \frac{1}{p}\right) \\
&\sim \frac{(n_0 \pm 1) \log_2 t}{\log_2(n+t)}.
\end{aligned}$$

Since $\phi(n_0 - 1) > \phi(n_0) > \phi(n_0 + 1)$, we deduce that necessarily

$$n(t) > \exp(\exp((e^{-\gamma} + o(1))(\log_2 t)^2)).$$

Furthermore, there are $t!$ permutations of the integers $\{1, 2, \ldots, t\}$ in the case of Theorem 1.1, so there exists a permutation a such that $n(t) > t!$. However, this trivial inequality leaves a huge open gap for investigation.

Given a positive integer t and two fixed permutations a and b of the integers $1, 2, \ldots, t$, one could also investigate the quantities

$$S_1(x) = \#\{n < x : \lambda(n + a(i)) < \lambda(n + a(i+1)) \text{ for } i = 1, \ldots, t-1\}$$

and

$$\begin{aligned}
S_2(x) = \#\{n < x : \ &\lambda(n + a(i)) < \lambda(n + a(i+1)) \text{ and} \\
&\phi(n + b(i)) < \phi(n + b(i+1)) \text{ for } i = 1, \ldots, t-1\}.
\end{aligned}$$

Since the value of $\lambda(n)$ does not depend much on the small prime factors of n, we conjecture that $S_1(x) \sim x/t!$. That is to say, the asymptotic value of $S_1(x)$ should not depend on the chosen permutation a. However, since the value of $\phi(n)$ depends heavily on the small prime factors of n, it is likely that a similar result might not hold for $S_2(x)$.

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N. Doyon
Départment de Mathématiques
et de Statistique
Université Laval
Québec G1K7P4
Canada

doyon@dms.umontreal.ca

F. Luca
Instituto de Matemáticas
Universidad Nacional Autónoma
de México
C.P. 58089
Morelia, Michoacán
Mexico

fluca@matmor.unam.mx