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Intersective Sets and Diophantine Approximation

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1. Introduction

In 1939, Koksma [16] introduced a classification of the real transcendental numbers ξ in terms of the quality of their algebraic approximations. For any positive integer n, denote by $w_n^*(\xi)$ the supremum of the real numbers w for which there exist infinitely many real algebraic numbers α of degree at most n satisfying

$$0 < |\xi - \alpha| \le \mathrm{H}(\alpha)^{-w-1},$$

where $\mathrm{H}(\alpha)$ is the naïve height of α, that is, the maximum of the absolute values of the coefficients of its minimal defining polynomial over the integers. Following Koksma, set

$$w^*(\xi) = \limsup_{n \to +\infty} \frac{w_n^*(\xi)}{n}$$

and call ξ an

S^*-number if $w^*(\xi) < +\infty$;

T^*-number if $w^*(\xi) = +\infty$ and $w_n^*(\xi) < +\infty$ for any $n \ge 1$;

U^*-number if $w^*(\xi) = +\infty$ and $w_n^*(\xi) = +\infty$ from some n onward.

It turns out (see e.g. Schneider [20]) that this classification coincides with that of Mahler introduced in 1932 [17], which depends on the accuracy with which nonzero integer polynomials evaluated at ξ approach zero. Sprindžuk [21] proved that almost all real numbers (in the sense of Lebesgue measure) are S^*-numbers and, moreover, satisfy $w_n^*(\xi) = n$ for any positive integer n. Using this result and the theory of Hausdorff dimension, Baker and Schmidt [1] established that, for any $n \ge 1$, the function w_n^* takes any value in the range $[n, +\infty[$ and even that, for any $\tau \ge 1$,

$$\dim\{\xi \in \mathbf{R} : w_n^*(\xi) \ge \tau(n+1) - 1\} = 1/\tau, \tag{1}$$

$$\dim\{\xi \in \mathbf{R} : w_n^*(\xi) = \tau(n+1) - 1\} = 1/\tau, \tag{2}$$

and

$$\dim \bigcap_{n \ge 1} \{\xi \in \mathbf{R} : w_n^*(\xi) \ge \tau(n+1) - 1\} = \frac{1}{\tau}, \tag{3}$$

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where dim denotes the Hausdorff dimension. We would like to point out that until now this has been the only known method of ensuring that, for any integer $n \geq 2$, the set of values taken by w_n^* includes the interval $[n, 2n]$. Further results on the functions w_n^* are given in [10].

In [7] we see an extension of Baker and Schmidt's results that involves general dimension functions rather than the family of power functions $x \mapsto x^s$. Basically, under some natural assumptions on the functions f and Ψ (observe that the technical condition (1) in [7, Thm. 1] can be removed; see [9; 6]), the Hausdorff $\mathcal{H}^f$-measure of the set

$$\mathcal{K}_n^*(\Psi) := \{\xi \in \mathbf{R} : |\xi - \alpha| < \Psi(\mathrm{H}(\alpha)) \text{ for infinitely many real algebraic numbers } \alpha \text{ of degree at most } n\}$$

is equal to 0 or $+\infty$ according as the sum $\sum_{x \geq 1} x^n f(\Psi(x))$ converges or diverges. However, the approach followed in [7] does not seem to yield such a precise statement for countable intersections of sets of this form.

The purpose of the present work is to consider these questions from another point of view. Our main tool is the notion of intersective sets, introduced and systematically studied by Falconer [12; 14]. These are classes of sets of Hausdorff dimension at least s with the property that countable intersections of the sets also have dimension at least s. Examples include the "regular sets" introduced by Baker and Schmidt [1] (which allowed them to obtain (3)), the $\mathcal{M}_\infty^s$-sequences of Rynne [19], and constructions using the "ubiquitous systems" of Dodson, Rynne, and Vickers [11]. Falconer [12; 13; 14] pointed out various applications of the notion of intersective sets to Diophantine approximation. Thanks to an extension of [14] to classes of sets defined in terms of general dimension functions (see Section 4, at the end of which we correct a slight mistake in [14]), we refine an auxiliary result of [1], allowing us to obtain sharp, new results in Diophantine approximation (stated in Section 3 and proved in Section 6). These complement our previous work in [7].

2. Background on Hausdorff Measure Theory

The notion of intersective sets that we consider was introduced by Falconer [14], and we refer to that paper for some background and notation. In [14], Falconer dealt with the scale of functions $x \mapsto x^s$; however, we need to work in a more general setting.

DEFINITION 1. A *dimension function* f is a strictly increasing continuous function defined on $\mathbf{R}_{\geq 0}$ and satisfying $f(0) = 0$.

Let E be some set in $\mathbf{R}^n$. Let f be a dimension function and, for any positive real number δ, set

$$\mathcal{H}_\delta^f(E) := \inf_J \sum_{j \in J} f(|U_j|),$$

where the infimum is taken over all countable coverings $(U_j)_{j\in J}$ of E by cubes of diameter at most δ. Clearly, the function $\delta \mapsto \mathcal{H}^f_\delta(E)$ is nonincreasing. Consequently,

$$\mathcal{H}^f(E) := \sup_{\delta>0} \mathcal{H}^f_\delta(E)$$

is well-defined and lies in $[0, +\infty]$; this is the $\mathcal{H}^f$-measure of E.

If f and g are two dimension functions, we say that g corresponds to a "smaller" generalized dimension than f (and write $g \prec f$) if

$$x \mapsto \frac{g(x)}{f(x)} \text{ tends monotonically to infinity as } x \text{ tends to zero.}$$

Observe that if $g \prec f$ then g increases faster than f in a neighborhood of the origin. Usually, the monotonicity is omitted in the definition of the ordering $\prec$, but in our present context this assumption cannot be dropped. Clearly, $\prec$ does not define a total ordering.

Definition 2 generalizes the definition in [14].

Definition 2. Let f be a dimension function. We define $\mathcal{G}^f(\mathbf{R}^n)$ to be the class of G_δ-subsets F of $\mathbf{R}^n$ such that

$$\mathcal{H}^g\left(\bigcap_{i=1}^{+\infty} f_i(F)\right) = +\infty$$

for any dimension function g with $g \prec f$ and any sequence of similarity transformations $\{f_i\}_{i=1}^{+\infty}$. If E is an open cube in $\mathbf{R}^n$, we define $\mathcal{G}^f(E)$ to be the class of G_δ-subsets F of E such that the set $\bigcup_j \sigma_j(F)$ is in $\mathcal{G}^f(\mathbf{R}^n)$. Here, the σ_j are translations such that $\bigcup_j \sigma_j(E)$ is a disjoint union of cubes and covers $\mathbf{R}^n$ up to a set of Lebesgue n-dimensional measure 0.

Observe that a subset F of $\mathbf{R}^n$ is in $\mathcal{G}^f(\mathbf{R}^n)$ if $F \cap E$ is in $\mathcal{G}^f(E)$ for any bounded open cube E.

Theorem 1 extends [14, Thm. A] to the case of general dimension functions.

Theorem 1. *The class $\mathcal{G}^f(\mathbf{R}^n)$ is closed under countable intersections and under bi-Lipschitz transformations on $\mathbf{R}^n$. Furthermore, if $f(x) = x^s$ for some real number s with $0 < s \le n$, then any set in $\mathcal{G}^f(\mathbf{R}^n)$ has Hausdorff dimension at least equal to s.*

We outline the proof of Theorem 1 in Section 4. Except for some minor changes, it follows the same lines as the proofs of Theorems B and C in [14].

3. Diophantine Approximation

In order to study sets of real numbers close to infinitely many algebraic numbers of bounded degree, Baker and Schmidt [1] introduced the notion of a "regular system". Roughly speaking: an infinite sequence of points form a *regular* system if they are well distributed; they form an *optimal* regular system (or, in the terminology of [4], a *best possible* regular system) if they are as well distributed as they could be, in the following sense.

DEFINITION 3. Let E be a bounded open real interval. Let $\mathcal{S} = (\alpha_j)_{j\geq 1}$ be a sequence of real numbers. Then $\mathcal{S}$ is called an *optimal regular system* of points in E if there exist positive constants c_1, c_2, c_3 depending only on $\mathcal{S}$ and, for any interval I in E, a number $K_0 = K_0(\mathcal{S}, I)$ such that, for any $K \geq K_0$, there exist integers

$$c_1 K \leq i_1 < \cdots < i_t \leq K$$

with α_{i_h} in I for $h = 1, \ldots, t$,

$$|\alpha_{i_h} - \alpha_{i_\ell}| \geq \frac{c_2}{K} \quad (1 \leq h \neq \ell \leq t),$$

and

$$c_3 |I| K \leq t \leq |I| K.$$

We emphasize that we do not assume that every point in $\mathcal{S}$ belongs to E. In the original work of Baker and Schmidt [1], the set $\mathcal{S}$ is not indexed. However, as in [9], we choose to number its elements; an alternative presentation can be found in [2; 4; 7] and in the impressive work [6]. Furthermore, we have supposed that E is bounded, although this was not assumed in [1]. This does not involve any loss of generality because any unbounded set can be covered by a countable collection of bounded, open sets to which the results may be applied.

It is an easy exercise to show that the rational numbers, ordered by increasing height and increasing numerical order, form an optimal regular system in any bounded interval. The importance of this notion has been pointed out in a series of papers [2; 4; 6; 7; 8; 9]. In particular, Beresnevich [3] proved a Khintchine-type statement for sets of real numbers close to infinitely many points in an optimal regular system.

Examples of optimal regular systems in any bounded interval include real algebraic numbers of fixed degree ([2]; see Proposition 2 to follow), real algebraic integers of fixed degree ≥ 2 (see [8]), and real algebraic units of fixed degree ≥ 3 [8].

Theorem 2 asserts that sets of real numbers close to infinitely many points in an optimal regular system turn out to be intersective sets.

THEOREM 2. *Let E be a bounded, open real interval. Let $\mathcal{S} = (\alpha_j)_{j\geq 1}$ be a sequence of real numbers that is an optimal regular system in E. Let $\Psi \colon \mathbf{R}_{\geq 1} \to \mathbf{R}_{>0}$ be a nonincreasing function such that $\sum_{j\geq 1} \Psi(j)$ converges. Set*

$$E(\alpha_j) := \{\xi \in E : |\xi - \alpha_j| < \Psi(j)\}$$

for any $j \geq 1$, and let

$$E(\Psi) = \limsup_{j\to+\infty} E(\alpha_j).$$

Let f be a dimension function with $f \prec \mathrm{Id}$ such that $x \mapsto x f(2\Psi(x))$ tends to zero as x goes to infinity. If the sum $\sum_{j\geq 1} f(2\Psi(j))$ diverges, then the set $E(\Psi)$ is in the class $\mathcal{G}^f(E)$.

Theorem 2 neither follows from nor implies Theorem 3 of [9], which asserts that $\mathcal{H}^f(E(\Psi)) = +\infty$ if the sum $\sum_{j\geq 1} f(2\Psi(j))$ diverges. However, it may be seen as a refinement of Lemma 1 in [1], where the assumption "$x \mapsto xf(\Psi(x)/2)$ tends to infinity" is demanded instead of the divergence of the sum $\sum_{j\geq 1} f(2\Psi(j))$. Here, f can increase more slowly.

Thanks to Theorem 1 and to the observations following Definition 3, Theorem 2 allows us to prove the existence of real numbers with various approximation properties by real algebraic numbers or/and by real algebraic integers or/and by real algebraic units.

We first give an application to Koksma's classification of real numbers. A well-known refinement of this classification consists of dividing the class of S^*-numbers into uncountably many subclasses according to the value of $w^*(\xi)$, which is called the *type* of ξ. Actually, Koksma [16] called "Index der S^*-Zahl ξ" the quantity $\sup_{n\geq 1} w_n^*(\xi)/n$, but—in view of the results from [1] quoted in the Introduction—it is much more natural to consider

$$\limsup_{n\to+\infty} \frac{w_n^*(\xi)+1}{n+1} \quad \text{or} \quad \sup_{n\geq 1} \frac{w_n^*(\xi)+1}{n+1}. \tag{4}$$

In this author's opinion, the limsup is much more relevant than the supremum; hence we define the type $t^*(\xi)$ of an S^*-number ξ by

$$t^*(\xi) = \limsup_{n\to+\infty} \frac{w_n^*(\xi)+1}{n+1} \quad (= w^*(\xi)).$$

THEOREM 3. *For any real number $\tau \geq 1$,*

$$\dim\{\xi \in \mathbf{R} : \xi \textit{ is an } S^*\textit{-number of type } \tau\} = 1/\tau.$$

Notice that Theorem 3 does not follow from (1), (2), and (3). Although the tools developed in [1] are sufficient to obtain Theorem 3 (see [10, Ch. V]), this statement has not previously appeared in print.

For $\tau = 1$, Theorem 3 follows from the result of Sprindžuk quoted in the Introduction. For $\tau > 1$, Theorem 3 is an easy consequence of Theorem 4, which deals with the more general sets introduced in [7]. In the sequel of this paper we denote by $\log_i r$ the i-fold iterated logarithm

$$\underbrace{\log \circ \cdots \circ \log r}_{i \text{ times}}.$$

For positive integers n and t and for real numbers $\nu_0 \geq 1$ and $\nu_1, \dots, \nu_{t-1}$, set $\tilde{\nu} := (\nu_0, \dots, \nu_{t-1})$ and, for any real number τ, consider the set

$$\mathcal{K}_n^*(\tilde{\nu}, \tau) := \mathcal{K}_n^*(\nu_0, \dots, \nu_{t-1}, \tau) = \mathcal{K}_n^*(x \mapsto x^{-(n+1)\nu_0} (\log x)^{-\nu_1} \cdots (\log_t x)^{-\tau})$$

of real numbers ξ for which the inequality

$$|\xi - \alpha| < (H(\alpha))^{-(n+1)\nu_0} \big(\log(H(\alpha))\big)^{-\nu_1} \cdots \big(\log_{t-1}(H(\alpha))\big)^{-\nu_{t-1}} (\log_t H(\alpha))^{-\tau}$$

is satisfied by infinitely many algebraic numbers α of degree at most n. The $\tilde{\nu}$-exact logarithmic order (a terminology introduced by Beresnevich, Dickinson, and Velani [5]) of ξ is, by definition,

$$\tau_{n,\tilde{\nu}}(\xi) := \sup\{\tau : \xi \in \mathcal{K}_n^*(\tilde{\nu}, \tau)\}.$$

We adopt the convention $\tilde{\nu} = (0)$ for $t = 0$. Then,

$$\mathcal{K}_n^*((0), \tau) = \mathcal{K}_n^*(x \mapsto x^{-\tau}) \quad \text{and} \quad \tau_{n,(0)}(\xi) = \frac{w_n^*(\xi) + 1}{n+1}.$$

For any $\underline{\nu} = (\nu_0, \ldots, \nu_{t-1}, \nu_t)$, put $\tilde{\nu} = (\nu_0, \ldots, \nu_{t-1})$ and consider the set

$$\mathcal{E}_n(\underline{\nu}) := \{\xi \in \mathbf{R} : \tau_{n,\tilde{\nu}}(\xi) = \tau\}$$

of real numbers whose $\tilde{\nu}$-exact logarithmic order is equal to τ. In particular, $\mathcal{E}_n((0), \tau) = \mathcal{E}_n(\tau)$ is the set of real S^*-numbers ξ such that $(w_n^*(\xi)+1)/(n+1) = \tau$. For $\underline{\nu} = (\nu_0, \ldots, \nu_t)$, define the dimension function $f_{\underline{\nu}}$ by

$$f_{\underline{\nu}}(u) := u^{1/\nu_0} \prod_{i=1}^{t} \left(\log_i \frac{1}{u}\right)^{-1+\nu_i/\nu_0},$$

where an empty product is taken to be 1. With the foregoing notation, Theorem 4 is an easy consequence of Theorem 1 and provides an extension of some results in [5].

THEOREM 4. *Let* $\underline{\nu} = (\nu_0, \ldots, \nu_t)$ *with* $\nu_0 > 1$. *For any integer* $n \geq 1$, *the set* $\mathcal{K}_n^*(\underline{\nu}) := \mathcal{K}_n^*(\nu_0, \ldots, \nu_t)$ *is in the class* $\mathcal{G}^{f_{\underline{\nu}}}(\mathbf{R})$ *and we have*

$$\mathcal{H}^g\left(\bigcap_{n\geq 1} \mathcal{E}_n(\underline{\nu})\right) = +\infty$$

for any dimension function g *with* $g \prec f_{\underline{\nu}}$.

The tools developed in [1] are not precise enough to get Theorem 4 for two reasons: (i) we make considerable use of the fact that real algebraic numbers of bounded degree form an optimal regular system (a weaker result is sufficient to get (1), (2), and (3)); and (ii) we also need a refinement of [1, Lemma 1].

Using the properties of intersective sets yields the following statement, which seems to be out of reach via the methods of [9] or [7].

THEOREM 5. *Let* $(\varphi_k)_{k\geq 1}$ *be a sequence of real numbers. For any real number* $\tau \geq 1$,

$$\dim\{\xi \in \mathbf{R} : \textit{for all } k \geq 1,\ \xi^k + \varphi_k \textit{ is an } S^*\textit{-number of type } \tau\} = 1/\tau.$$

Observe that there exist real numbers ξ for which $w_n^*(\xi) \neq w_n^*(\xi^2)$ (see e.g. [15]) for some integers n, although it is still unknown whether real numbers ξ with $t^*(\xi) \neq t^*(\xi^2)$ do exist.

Theorem 5 is one among many examples of results in Diophantine approximation that we can obtain thanks to the properties of intersective sets. We may apply it, for example, to a sequence $(\varphi_k)_{k\geq 1}$ that is composed of Liouville numbers (i.e., real numbers ξ with $w_1^*(\xi) = +\infty$) or of other real numbers having various Diophantine approximation properties.

Notice that Theorem 5 (and hence Theorem 3 also) holds for either definition of the type of an S^*-number chosen in (4).

4. Proof of Theorem 1

As pointed out by Falconer [12; 14], proving Theorem 1 is much more convenient with the net-premeasures $\mathcal{M}^f_\infty$. According to [12] (but not to [14]), a dyadic cube in $\mathbf{R}^n$ is a set of the form

$$[2^{-k}m_1, 2^{-k}(m_1+1)[\times\cdots\times[2^{-k}m_n, 2^{-k}(m_n+1)[,$$

where k is a nonnegative integer and $m_1,\dots,m_n$ are integers.

Definition 4. Let f be a dimension function. Let $\varepsilon(f)$ be the supremum of the real numbers x in $[0,1]$ such that f is increasing and concave on $[0,x]$. Then, for any subset F of $\mathbf{R}^n$, we set

$$\mathcal{M}^f_\infty(F) = \inf \sum_{j\ge 1} f(|I_j|),$$

where the infimum is taken over all countable coverings $(I_j)_{j\ge1}$ of F by dyadic cubes of diameter $|I_j|$ that are less than or equal to $\varepsilon(f)$.

Note that $\mathcal{M}^f_\infty(I) = f(|I|)$ for any dyadic cube I of diameter at most $\varepsilon(f)$. Furthermore, $\varepsilon(f)$ is positive if f satisfies $f \prec \mathrm{Id}$, which is assumed in Theorem 2.

Since the proof of Theorem 1 requires only slight modifications of the proofs in [14], we direct the reader to [14] for the notation and content ourselves here with stating the main lines. However, for sake of simplicity, we assume until Lemma 5 that the dimension functions f, g, and h satisfy $\varepsilon(f)=\varepsilon(g)=\varepsilon(h)=1$.

The next statement is a generalization of [14, Thm. B]—which, however, contains a (slight) mistake; see the end of this section for a correction.

Theorem 6. *Let f be a dimension function and let F be a subset of $\mathbf{R}^n$. Then the following implications between the statements hereunder are valid*:

$$\text{(a)} \implies \text{(b)} \implies \text{(c)} \iff \text{(d)} \implies \text{(e)} \iff \text{(f)}.$$

If F is a G_δ-set, then (a)–(f) *are equivalent.*

(a) *For every nonempty open subset V of $\mathbf{R}^n$ and every sequence of bi-Lipschitz transformations $f_i\colon V \to \mathbf{R}^n$,*

$$\mathcal{H}^g\left(\bigcap_{i=1}^{+\infty} f_i^{-1}(F)\right) = +\infty$$

for any dimension function g with $g \prec f$.

(b) *For every sequence of similarity transformations $f_i\colon \mathbf{R}^n \to \mathbf{R}^n$,*

$$\mathcal{H}^g\left(\bigcap_{i=1}^{+\infty} f_i(F)\right) = +\infty$$

for any dimension function g with $g \prec f$.

(c) *For all dyadic cubes I,*

$$\mathcal{M}^g_\infty(F \cap I) = \mathcal{M}^g_\infty(I)$$

for any dimension function g with $g \prec f$.

(d) *For all open sets U,*

$$\mathcal{M}^g_\infty(F \cap U) = \mathcal{M}^g_\infty(U)$$

for any dimension function g with $g \prec f$.

(e) *There exists a c with $0 < c \leq 1$ such that, for all dyadic cubes I,*

$$\mathcal{M}^g_\infty(F \cap I) \geq c\mathcal{M}^g_\infty(I)$$

for any dimension function g with $g \prec f$.

(f) *There exists a c with $0 < c \leq 1$ such that, for all open sets U,*

$$\mathcal{M}^g_\infty(F \cap U) \geq c\mathcal{M}^g_\infty(U)$$

for any dimension function g with $g \prec f$.

The proof of Theorem B in [14] depends on four lemmas. Instead of giving complete proofs of their extensions to the case of general dimension functions, we merely point out which changes have to be made.

LEMMA 1. *Let f be a dimension function, let $0 < c \leq 1$, and let $F \subset \mathbf{R}^n$. If U is an open subset of $\mathbf{R}^n$ such that*

$$\mathcal{M}^f_\infty(F \cap I) \geq c\mathcal{M}^f_\infty(I)$$

for all dyadic cubes I contained in U, then

$$\mathcal{M}^f_\infty(F \cap U) \geq c\mathcal{M}^f_\infty(U).$$

Proof. This is a straightforward adaptation of [14, Lemma 1]. □

LEMMA 2. *Let f be a dimension function, and let $F \subset \mathbf{R}^n$ and $c > 0$ be such that*

$$\mathcal{M}^f_\infty(F \cap I) \geq c\mathcal{M}^f_\infty(I)$$

for all dyadic cubes I of diameter at most 1. Then

$$\mathcal{M}^g_\infty(F \cap I) = \mathcal{M}^g_\infty(I)$$

for all dimension functions g with $g \prec f$ and for all dyadic cubes I of diameter at most 1.

Proof. We follow the same lines as [14] and set $h = g/f$. There are, however, some minor changes. Let I be a dyadic cube of side 2^m for some integer $m \leq 0$. Let m' be an integer with $m' \leq m$ and $h(2^{m'}) \geq h(2^m)c^{-1}$. We replace inequality (7) of [14] by

$$\sum_{i \in Q(j)} g(|I_i|) = g(|J_j|) \geq h(|I|)f(|J_j|)$$

as well as the next two displayed inequalities of [14] by

$$g(|I_i|) \geq h(|I|)c^{-1}f(|I_i|)$$

and

$$\sum_{i \in Q(j)} g(|I_i|) \geq h(|I|)f(|J_j|).$$

Summing over all j we then have

$$\sum_{i=1}^{\infty} g(|I_i|) \geq h(|I|) \sum_{j=1}^{k} f(|J_j|) \geq h(|I|)\mathcal{M}_{\infty}^{f}(I) = g(|I|),$$

as expected. □

LEMMA 3. *Let V be a nonempty subset of $\mathbf{R}^n$ and let $f \colon V \to \mathbf{R}^n$ be a bi-Lipschitz mapping that satisfies*

$$c_1|x - y| \leq |f(x) - f(y)| \leq c_2|x - y| \quad (x, y \in V),$$

where $0 < c_1 < c_2 < \infty$. Let h be a dimension function and assume that

$$\mathcal{M}_{\infty}^{h}(F \cap U) \geq c\mathcal{M}_{\infty}^{h}(U)$$

for some $0 < c \leq 1$, for $F \subset \mathbf{R}^n$, and for all open sets U. Then, for all open $U \subset V$,

$$\mathcal{M}_{\infty}^{h}(f^{-1}(F) \cap U) \geq c_0\mathcal{M}_{\infty}^{h}(U)$$

for some positive real number c_0 and also

$$\mathcal{M}_{\infty}^{g}(f^{-1}(F) \cap U) = \mathcal{M}_{\infty}^{g}(U)$$

for any dimension function g with $g \prec h$.

Proof. This is a straightforward adaptation of [14, Lemma 3]. □

We write $\mathcal{C}^f(V)$ for the class of sets F such that

$$\mathcal{M}_{\infty}^{f}(F \cap U) = \mathcal{M}_{\infty}^{f}(U)$$

for all open $U \subset V$.

LEMMA 4. *Let f be a dimension function, and let $\{F_k\}_{k=1}^{\infty}$ be a sequence of G_δ-sets in $\mathcal{C}^f(V)$. Then there exists a positive constant c such that*

$$\mathcal{M}_{\infty}^{f}\left(\bigcap_{k=1}^{\infty} F_k \cap U\right) \geq c\mathcal{M}_{\infty}^{f}(U)$$

for all open $U \subset V$.

Proof. This goes exactly along the same lines as in [14]. Notice that we need a version of the increasing sets lemma in this general context (see e.g. [18, Thm. 52] for a suitable candidate). □

We have now all the tools necessary for proving Theorems 1 and 6.

Proof of Theorem 6. As remarked in [14], the implications $(a) \Rightarrow (b)$ and $(c) \Leftrightarrow (d) \Rightarrow (e) \Leftrightarrow (f)$ are immediate. To prove that $(b) \Rightarrow (c)$ we argue by contradiction. We assume that there exists a dimension function g with $g \prec f$ and $\mathcal{M}_\infty^g(F \cap I) < \alpha \mathcal{M}_\infty^g(I) = \alpha g(|I|)$ for some dyadic cube I and some $\alpha < 1$. Then there is a sequence of dyadic cubes $\{I_i\}_{i=1}^{+\infty}$ such that $\sum_{i=1}^{+\infty} g(|I_i|) < \alpha g(|I|)$. We obtain the analogue of [14, (16)] with $\mathcal{M}_\infty^t$ replaced by $\mathcal{M}_\infty^g$, and we end up with a doubly infinite sequence of similarity transformations $\{h_m \circ g_j\}$ such that

$$\mathcal{M}_\infty^g\left(\bigcap_{j=1}^{\infty}\bigcap_{m=1}^{\infty}(h_m \circ g_j)(F)\right) = 0,$$

which is the desired contradiction to (b).

Assume now that F is a G_δ-set satisfying (f). Let g be a dimension function with $g \prec f$. Then there exists a dimension function h with $g \prec h \prec f$. Let $f_i : V \to \mathbf{R}^n$ be bi-Lipschitz transformations $(i = 1, 2, \dots)$. Lemma 3 yields that $\mathcal{M}_\infty^h(f_i^{-1}(F) \cap U) = \mathcal{M}_\infty^h(U)$ holds for all open subsets U of V. Since the sets $f_i^{-1}(F)$ are G_δ, we infer from Lemma 4 that

$$\mathcal{M}_\infty^h\left(\bigcap_{i=1}^{\infty} f_i^{-1}(F) \cap V\right) > 0;$$

thus we have

$$\mathcal{H}^g\left(\bigcap_{i=1}^{\infty} f_i^{-1}(F) \cap V\right) = +\infty$$

as expected. □

Proof of Theorem 1. This follows the same lines as the proof of assertions (a) and (e) of [14, Thm. C]. Indeed, let $F_1, F_2, \dots$ be in $\mathcal{G}^f(\mathbf{R}^n)$. Let g be a dimension function with $g \prec f$. Then there exists a dimension function h with $g \prec h \prec f$. By Theorem 6(d),

$$\mathcal{M}_\infty^h(f_i(F_k) \cap U) = \mathcal{M}_\infty^h(U)$$

for all open sets U and for all integers k and similarity transformations $f_i : \mathbf{R}^n \to \mathbf{R}^n$. Applying Lemma 4, we then get

$$\mathcal{M}_\infty^h\left(\bigcap_{i=1}^{\infty}\bigcap_{k=1}^{\infty} f_i(F_k) \cap U\right) > 0$$

for all open sets U. Consequently, we have

$$\mathcal{H}^g\left(\bigcap_{i=1}^{\infty} f_i\left(\bigcap_{k=1}^{\infty} F_k\right)\right) = +\infty.$$

We conclude that $\bigcap_{k=1}^{+\infty} F_k$ is in $\mathcal{G}^f(\mathbf{R}^n)$ by Theorem 6(b). □

As pointed out in [14], the following lemma provides a useful test for $\mathcal{G}^f$-sets. Since the condition $\varepsilon(f) = 1$ is not always satisfied in the applications we have in mind, this restriction does not appear in Lemma 5.

LEMMA 5. *Let $(F_k)_{k\geq 1}$ be a sequence of open subsets of $\mathbf{R}^n$. Assume that there exist a dimension function f with $\varepsilon(f) > 0$ and positive real numbers ε and c such that $\varepsilon < \varepsilon(f)$ and*

$$\lim_{k\to\infty} \mathcal{M}^g_\infty(F_k \cap I) \geq c\mathcal{M}^g_\infty(I)$$

for every dyadic cube I of diameter less than ε and any dimension function g with $g \prec f$. Then

$$\limsup_{k\to\infty} F_k \in \mathcal{G}^f(\mathbf{R}^n).$$

Proof. This is a straightforward adaptation of [14, Lemma 7]. If $\varepsilon(f)$ is strictly less than 1, we adapt Theorem 6 with obvious modifications. □

In the applications, we are not always able to work directly in $\mathbf{R}^n$ and we deal only with bounded sets. Hence, Lemma 6 turns out to be very useful.

LEMMA 6. *Let E be an open cube in $\mathbf{R}^n$. Let $(F_k)_{k\geq 1}$ be a sequence of open subsets of E, and assume that there exist a dimension function f with $\varepsilon(f) > 0$ and positive real numbers ε and c such that $\varepsilon < \varepsilon(f)$ and*

$$\lim_{k\to\infty} \mathcal{M}^g_\infty(F_k \cap I) \geq c\mathcal{M}^g_\infty(I)$$

for every dyadic cube I in E of diameter less than ε and any dimension function g with $g \prec f$. Then

$$\limsup_{k\to\infty} F_k \in \mathcal{G}^f(E).$$

Proof. This follows immediately from Lemma 5 and the definition of $\mathcal{G}^f(E)$. □

We end this section by pointing out a (slight) mistake in [14].

In the proof of the implication $(b) \Rightarrow (c)$ of Theorem B [14, p. 273], it is asserted that "we may choose $t < s$ such that $\sum_{i=1}^\infty |I_i|^t < \alpha|I|^t$". This statement does not automatically follow from the assumption $\sum_{i=1}^\infty |I_i|^s < \alpha|I|^s$ unless the sum is finite (which will not occur in the cases of interest). This slight mistake can be easily corrected by changing statement (c) (and likewise statements (d), (e), and (f)) of Theorem B as follows:

(c′) *For all dyadic cubes I,*

$$\mathcal{M}^t_\infty(F \cap I) = \mathcal{M}^t_\infty(I)$$

for any positive real number $t < s$.

Another consequence is that assertion (a) in [14, Thm. D] does not hold true.

Furthermore, Example 3 of [14] seems to be incorrect because there is no reason for infinitely many rational approximants of the x_i to have the same denominator.

5. Proof of Theorem 2

Proposition 1 is the key tool in proving Theorem 2.

PROPOSITION 1. *Let $\mathcal{S} = (\alpha_j)_{j\geq 1}$ be an optimal regular system in a bounded, open real interval E. Let I be an interval in E. Let F be a positive, nonincreasing function such that the sum $\sum_{j\geq 1} F(j)$ diverges and $x \mapsto xF(x)$ is nonincreasing and tends to zero as x goes to infinity. For any real number m, there exist a positive constant $c(\mathcal{S}) \leq 1$, depending only on $\mathcal{S}$, and integers $m \leq i_1 < \cdots < i_t$ such that the intervals*

$$[\alpha_{i_h} + F(i_h), \alpha_{i_h} - F(i_h)]$$

are included in I and are pairwise disjoint and

$$\sum_{h=1}^{t} F(i_h) \geq c(\mathcal{S})|I|.$$

Proof. This is [9, Prop. 1] in the case $s = 1$. $\square$

We now show how Proposition 1 implies Theorem 2.

Proof of Theorem 2. In order to simplify the exposition, we assume that the length of E is 1. We construct inductively open real subsets $E_0, E_1, \ldots$ such that

$$E(\Psi) \supset \limsup_{k\to+\infty} E_k,$$

and we aim to conclude by Lemma 5. We first apply Proposition 1 to the interval E, the function $F = f \circ \Psi$, and a real number $H_0 \geq 2$ such that

$$f \circ \Psi(x) > \Psi(x) \quad \text{for any } \; x \geq H_0. \tag{5}$$

This is possible since $f \prec \mathrm{Id}$ and since the function Ψ tends to zero at infinity. We then have a set of distinct integers $\mathcal{A}(0) := \{i_1^{(0)}, \ldots, i_{t_1}^{(0)}\}$ with

$$\sum_{j\in\mathcal{A}(0)} F(j) \geq \kappa,$$

where $\kappa = c(\mathcal{S}) \cdot |E| = c(\mathcal{S})$. We define the set E_0 to be the intersection of E with the union of the intervals

$$]\alpha_j - \Psi(j), \alpha_j + \Psi(j)[, \quad j \in \mathcal{A}(0),$$

which are pairwise disjoint by (5).

Let k be a nonnegative integer and assume that (i) the sets of integers $\mathcal{A}(0), \ldots, \mathcal{A}(k)$ have been constructed and (ii) the sets $E_0, \ldots, E_k$ are finite unions of open intervals centered at real numbers α_j with j in $\mathcal{A}(0) \cup \cdots \cup \mathcal{A}(k)$. Denote by H_k an upper bound for the integers contained in $\mathcal{A}(0) \cup \cdots \cup \mathcal{A}(k)$, and apply Proposition 1 to each dyadic closed interval I in E of length 2^{-k-1}, to the real number H_k, and to the function F. We get a set of distinct integers $\mathcal{A}(k+1, I) := \{i_1^{(k+1)}, \ldots, i_{t_I}^{(k+1)}\}$ such that

$$\sum_{j\in\mathcal{A}(k+1,I)} F(j) \geq \kappa 2^{-k-1},$$

and we define the set E_{k+1} as the intersection of E with the union of the pairwise disjoint intervals

$$]\alpha_j - \Psi(j), \alpha_j + \Psi(j)[, \quad j \in \mathcal{A}(k+1, I),$$

where I runs through the dyadic closed intervals of length 2^{-k-1} in E. Thanks to this inductive process, we have constructed the sets E_k, which by (5) clearly satisfy

$$E(\Psi) \supset \bigcap_{i=1}^{\infty}\bigcup_{k=i}^{\infty} E_k.$$

Let k_0 be such that $2^{-k_0} \leq \varepsilon(f)$. Observe that, if $I \subset E$ is a dyadic interval of length $|I| = 2^{-k_0}$, then for any $k \geq k_0$ we have

$$\sum_j F(j) \geq \kappa |I|, \tag{6}$$

where the summation is taken over all indices j in $\mathcal{A}(k)$ for which α_j belongs to I.

Let I be a dyadic interval contained in E of length less than $\varepsilon(f)$. Since $f \prec \mathrm{Id}$, we may further assume that $f(x) \geq x$ for any $x \leq |I|$. Let $k \geq k_0$ be an integer. We want to prove that $\mathcal{M}^f_\infty(I \cap E_k) \geq \kappa f(|I|)$ for any integer k sufficiently large. Consider a finite covering $U_1 \cup \cdots \cup U_m$ of $I \cap E_k$, where the U_i are pairwise disjoint dyadic intervals such that their endpoints coincide with those of intervals composing E_k. Without any restriction we can take only finite coverings, as observed by Falconer [12, Proof of Lemma 6.1]. By definition, we have

$$\mathcal{M}^f_\infty(I \cap E_k) \geq \sum_{j=1}^{m} f(|U_j|). \tag{7}$$

For any integer j with $1 \leq j \leq m$, either (i) U_j is one of the intervals composing E_k, say $U_j =]\alpha_h - \Psi(h), \alpha_h + \Psi(h)[$, or (ii) there exist $h_1, \ldots, h_v$ with $v \geq 2$ and $\alpha_{h_1} < \cdots < \alpha_{h_v}$ such that

$$[\alpha_{h_1}, \alpha_{h_1} + F(h_1)[\cup \bigcup_{\ell=2}^{v-1}]\alpha_{h_\ell} - F(h_\ell), \alpha_{h_\ell} + F(h_\ell)[\cup]\alpha_{h_v} - F(h_v), \alpha_{h_v}] \subset U_j$$

and

$$U_j \subset]\alpha_{h_1} - \Psi(h_1), \alpha_{h_v} + \Psi(h_v)[.$$

In case (i) we have $f(|U_j|) = f(2(\Psi(h)) \geq F(h)$; in case (ii),

$$f(|U_j|) \geq f(F(h_1) + \cdots + F(h_v)) \geq F(h_1) + \cdots + F(h_v),$$

since $f \prec \mathrm{Id}$. Consequently, we have by (6) and (7) that

$$\mathcal{M}_\infty^f(I \cap E_k) \geq \sum_{j \in \mathcal{A}(k)} F(j) \geq \kappa|I| \geq \kappa f(|I|).$$

Thus, the assumptions of Lemma 6 are satisfied, and the desired result follows. □

6. Proofs of Theorem 4 and 5

Proof of Theorem 4. Before applying Theorem 2, we recall a deep result of Beresnevich [2] on the distribution of real algebraic numbers of bounded degree.

PROPOSITION 2. *Let $n \geq 1$ and $M \geq 2$ be integers and let I be an interval contained in $(-M+1, M-1)$. There exist positive constants c_4, c_5 depending only on n, and $K_0 = K_0(n, I)$. For any $K \geq K_0$, there are $\alpha_1, \ldots, \alpha_t$ in $\mathbf{A}_n \cap I$ such that*

$$c_4 M^n K \leq \mathrm{H}(\alpha_h) \leq M^n K \quad (1 \leq h \leq t),$$

$$|\alpha_h - \alpha_\ell| \geq K^{-n-1} \quad (1 \leq h < \ell \leq t),$$

$$t \geq c_5 |I| K^{n+1}.$$

Proof. This is Theorem 3 of Beresnevich [2]. Actually, the existence of c_4 is not proved in [2], but we may easily deduce it by following his proof (see e.g. [7, Thm. G]). □

To prove that the set $\mathbf{A}_n$ of real algebraic numbers of bounded degree n forms an optimal regular system in any bounded, open real interval E, it remains for us (in view of Proposition 2) to order $\mathbf{A}_n$ in a suitable manner, as follows.

LEMMA 7. *Let $n \geq 1$ be an integer. We number the elements of $\mathbf{A}_n := (\alpha_j)_{j \geq 1}$ by increasing order of their height and, when the heights are equal, by increasing numerical order. Then, there exist two positive constants c_1 and c_2 depending only on n and such that, for any $j \geq 1$,*

$$c_1(n) j^{1/(n+1)} \leq \mathrm{H}(\alpha_j) \leq c_2(n) j^{1/(n+1)}. \tag{8}$$

Proof. The left-hand inequality in (8) is clear, since an easy counting argument shows that, for any positive integer H, there are at most $n(2H+1)^{n+1}$ algebraic numbers of height at most H and degree at most n. As for the right-hand side, let $h \geq 5$ be an odd integer. Consider an integer polynomial

$$P(X) := hX^n - a_{n-1}X^{n-1} - \cdots - a_1 X - a_0,$$

where a_0 is congruent to 2 modulo 4 and, for $0 \leq j \leq n-1$, the integer a_j is even and belongs to $\{0, 2, \ldots, 2[h/2]\}$. By Eisenstein's criterion, the polynomial $P(X)$ is irreducible. Furthermore, it clearly has (at least) one real root. Consequently, there are at least $c_3(n) h^n$ real algebraic numbers of height h and degree n. Hence, for any positive integer H, there are at least $c_4(n) H^{n+1}$ real algebraic numbers of height at most H and degree at most n. This proves the right-hand inequality of (8). □

Let E be a bounded, real closed interval. By Proposition 2 and Lemma 7, the set $\mathbf{A}_n$ is an optimal regular system in E. In order to apply Theorem 2 with the function $\tilde{\Psi}$ defined by $\tilde{\Psi}(j) := \Psi(\mathrm{H}(\alpha_j))$ for $j \geq 1$, we need only check that the sums $\sum_{j\geq 1} \tilde{\Psi}(j)$ and $\sum_{j\geq 1} j^n \Psi(j)$ have the same behavior, which holds true. Indeed, the functions $\tilde{\Psi}$ and $j \mapsto j^n\Psi(j)$ are both nonincreasing and so we may, for example, use comparisons between sums and integrals to derive from (8) that the sum $\sum_{j\geq 1} \tilde{\Psi}(j)$ converges if and only if the sum $\sum_{j\geq 1} j^n \Psi(j)$ converges.

Let $n \geq 1$ be an integer. For any real number $x \geq 1$, set

$$\Psi_{n,\underline{\nu}}(x) = x^{-(n+1)\nu_0}(\log x)^{-\nu_1}\cdots(\log_t x)^{-\nu_t}.$$

Because the sum $\sum_{j\geq 1} f_{\underline{\nu}}(2\tilde{\Psi}_{n,\underline{\nu}}(j))$ diverges, Theorem 2 implies that the set $\mathcal{K}_n^*(\underline{\nu}) \cap E = \mathcal{K}_n^*(\Psi_{n,\underline{\nu}}) \cap E$ is in the class $\mathcal{G}^{f_{\underline{\nu}}}(E)$.

This holds for any bounded open interval E, so the set $\mathcal{K}_n^*(\underline{\nu})$ is in the class $\mathcal{G}^{f_{\underline{\nu}}}(\mathbf{R}^n)$ and, by Theorem 1, the intersection $\bigcap_{n\geq 1}\mathcal{K}_n^*(\underline{\nu})$ also belongs to that class. Setting $g(u) = f_{\underline{\nu}}(u) \times \log_{t+1}(1/u)$ yields

$$\mathcal{H}^g\Bigg(\bigcap_{n\geq 1}\mathcal{K}_n^*(\underline{\nu})\Bigg) = +\infty. \tag{9}$$

For positive integers n and k, define the function $\Psi_{n,\underline{\nu},k}$ on $\mathbf{R}_{\geq 1}$ by $\Psi_{n,\underline{\nu},k}(x) = \Psi_{n,\underline{\nu}}(x) \times (\log_t x)^{-1/k}$, and set

$$\mathcal{E} := \bigcap_{n\geq 1}\mathcal{E}_n(\underline{\nu}) = \bigcap_{n\geq 1}\mathcal{K}_n^*(\Psi_{n,\underline{\nu}}) \setminus \bigcup_{n_0\geq 1}\bigcup_{k\geq 1}\Bigg(\bigcap_{n\neq n_0}\mathcal{K}_n^*(\Psi_{n,\underline{\nu}}) \cap \mathcal{K}_{n_0}^*(\Psi_{n_0,\underline{\nu},k})\Bigg).$$

For any integers $k \geq 1$ and $n_0 \geq 1$, we have $\mathcal{H}^g(\mathcal{K}_{n_0}^*(\Psi_{n_0,\underline{\nu},k})) = 0$; hence it follows from (9) that $\mathcal{H}^g(\mathcal{E}) = +\infty$, as claimed.

Theorem 3 for $\tau > 1$ follows by simply taking $t = 0$ and $\tilde{\nu} = (\tau)$: we obtain that the Hausdorff dimension of the set of real numbers of type τ is greater than or equal to $1/\tau$. Actually, we have equality by (3). □

Proof of Theorem 5. It is sufficient to observe that, by Theorem 1, the image of an intersective set by a translation is an intersective set. Then we argue as at the end of the proof of Theorem 4, noticing that, for any positive integer k, $1/\tau$ is the dimension of the set of real numbers ξ such that $\xi^k + \varphi_k$ is an S^*-number of type τ. □

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