Michigan Math. J. 52 (2004)

Correspondences between $K3$ Surfaces

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(with an Appendix by Igor Dolgachev)

1. Introduction

In this paper we study the existence of correspondences between $K3$ surfaces $X(k,m,n)$ with $k,m,n \in \mathbf{N}$, Picard number 17, and transcendental lattices $T(k,m,n) \cong U(k) \oplus U(m) \oplus \langle -2n \rangle$. In a fundamental paper, Mukai [Mu] showed that correspondences between $K3$ surfaces exist if the transcendental lattices are Hodge isometric over $\mathbf{Q}$. This construction holds if the Picard number of the surfaces is greater than or equal to 11. Nikulin [N2] later improved this result, obtaining the lower bound 5 for the Picard number.

The aim of our work is to realize examples of $K3$ surfaces with transcendental lattice that are not Hodge isometric but such that a correspondence between them already exists. This in particular implies the existence of an algebraic cycle on the middle cohomology of the product of two surfaces arbitrarily chosen in the constructed family.

In Sections 2 and 3 we recall some basic notions and results on lattices and correspondences. In Section 4 we consider a generic genus-2 curve and we show the existence of a correspondence between the Jacobian of the curve and a $K3$ surface with isomorphic transcendental lattice. Since this construction involves a second $K3$ surface whose transcendental lattice has quadratic form multiplied by 2, in Sections 5 and 6 we generalize this first example. First, we construct $K3$ surfaces "twisting" each direct summand of the transcendental lattice of the Jacobian by natural numbers. Then we find correspondences between them using both Mukai's theorem and Shioda–Inose structures that translate the problem into a problem of looking for isogenies between abelian varieties.

In this way we prove in Theorem 6.3 that all the $K3$ surfaces $X(k,m,n)$ are in correspondence to each other. Finally, in Theorem 6.5 we show the existence of a correspondence between a general $K3$ surface of Picard rank 17 and a Kummer surface of the same rank having transcendental lattices that are $\mathbf{Q}$-Hodge isomorphic.

In the Appendix, Igor Dolgachev realizes a geometric correspondence between the $K3$ surfaces of Section 4.

Received December 10, 2002. Revision received April 1, 2003.
The authors are supported by EAGER (European Algebraic Geometry Research) training network, Contract no. HPRN-CT-2000-00099, and by Progetto di Ricerca Nazionale COFIN 2000, "Geometria delle Varietà Algebriche".

2. Preliminary Notions

2.1. DEFINITIONS. A *lattice* is a free $\mathbf{Z}$-module $\mathcal{L}$ of finite rank with a $\mathbf{Z}$-valued symmetric bilinear form $b_{\mathcal{L}}(x,y)$. A lattice is called *even* if the quadratic form associated to the bilinear form has only even values, *odd* otherwise. A very useful invariant (under base change) of a lattice is its *discriminant* $d(\mathcal{L})$, defined as the determinant of the matrix of its bilinear form. A lattice is called *nondegenerate* if the discriminant is nonzero and *unimodular* if the discriminant is ± 1. If the lattice $\mathcal{L}$ is nondegenerate then the pair (s_+, s_-), where $s_\pm$ denotes the multiplicity of the eigenvalue ± 1 for the quadratic form associated to $\mathcal{L} \otimes \mathbf{R}$, is called *signature* of $\mathcal{L}$. Finally, we call $s_+ + s_-$ the *rank* of $\mathcal{L}$ and $s_+ - s_-$ its *index*; moreover, a lattice is *indefinite* if the associated quadratic form has both positive and negative values.

Given a lattice $\mathcal{L}$, we can construct the lattice $\mathcal{L}(m)$—that is, the $\mathbf{Z}$-module $\mathcal{L}$ with bilinear form $b_{\mathcal{L}(m)}(x,y) = m b_{\mathcal{L}}(x,y)$. An *isometry* of lattices is an isomorphism preserving the bilinear form. Given a sublattice $\mathcal{L} \hookrightarrow \mathcal{L}'$, the embedding is *primitive* if $\mathcal{L}'/\mathcal{L}$ is free.

2.2. EXAMPLES.

(i) The lattice $\langle n \rangle$ is a free $\mathbf{Z}$-module of rank 1, $\mathbf{Z}\langle e \rangle$, whose bilinear form is $b(e,e) = n$.

(ii) The *hyperbolic lattice* is the even, unimodular, indefinite lattice with $\mathbf{Z}$-module $\mathbf{Z}\langle e_1, e_2 \rangle$ and bilinear associated form whose matrix is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

(iii) The lattice E_8 has $\mathbf{Z}^8$ as $\mathbf{Z}$-module, and the matrix of the bilinear form is the Cartan matrix of the root system of E_8. It is an even, unimodular, and positive definite lattice.

2.3. $K3$ AND TORI LATTICES. If X is a $K3$ surface, one can show that $H^2(X,\mathbf{Z})$ is free of rank 22 and that there is an isometry $H^2(X,\mathbf{Z}) \cong U^3 \oplus (E_8(-1))^2$. From now on we use Λ to denote this $K3$ lattice. For X a complex torus, one has $H^2(X,\mathbf{Z}) \cong U^3$.

2.4. HODGE STRUCTURES. Let X be an abelian or $K3$ surface. If we consider the Hodge decomposition of $H^2(X,\mathbf{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$, inside $H^2(X,\mathbf{Z})$ there are two sublattices: the *Néron–Severi lattice,*

$$\mathrm{NS}(X) := H^2(X,\mathbf{Z}) \cap H^{1,1}(X);$$

and the orthogonal complement of $\mathrm{NS}(X)$, the *transcendental lattice* T_X, which has a natural Hodge structure induced by the one of $H^2(X,\mathbf{Z})$. The *Picard number* of X, denoted by $\rho(X)$, is the rank of $\mathrm{NS}(X)$.

A *Hodge isometry* between the transcendental lattices of two $K3$ (or abelian) surfaces is an isometry preserving the Hodge decomposition.

2.5. FOURIER–MUKAI PARTNERS. Two abelian (or $K3$) surfaces X and Y are called *Fourier–Mukai partners* if their derived categories of bounded complexes of coherent sheaves are equivalent. Mukai and Orlov have shown the following theorem.

THEOREM 2.6 [Mu; Or]. *X and Y as just described are Fourier–Mukai partners if and only if their transcendental lattices are Hodge-isometric.*

3. Known Results

In this section we recall some fundamental results that will play a key role in the sequel. First of all, that the period map is surjective for a $K3$ or abelian surface makes it possible to prove the following.

THEOREM 3.1 [Mo, Cor. 1.9]. *Suppose $T \hookrightarrow U^3$ (resp., $T \hookrightarrow \Lambda$) is a primitive sublattice of signature $(2, 4-\rho)$ (resp., $(2, 20-\rho)$). Then there exists an abelian surface (resp., algebraic $K3$ surface) X and an isometry $T_X \xrightarrow{\sim} T$.*

DEFINITION 3.2. A $K3$ surface X admits a *Shioda–Inose structure* if there is an involution ι on X such that $\iota^*(\omega) = \omega$ for every $\omega \in H^{2,0}(X)$ (ι is called a *Nikulin involution*), with rational quotient map $\pi : X \dashrightarrow Y$ where Y is a Kummer surface, and the map π_* induces a Hodge isometry $T_X(2) \cong T_Y$. This gives a diagram

$$\begin{array}{ccccc} X & & & & Z \\ & \searrow & & \swarrow & \\ & & Y & & \end{array}$$

of rational maps of degree 2, where Z is a complex torus and Y is the Kummer surface of Z.

One can detect the existence of a Shioda–Inose structure on a $K3$ surface by analyzing the transcendental lattice of that surface.

THEOREM 3.3 [Mo, Thm. 6.3]. *Let X be an algebraic $K3$ surface. Then X admits a Shioda–Inose structure if and only if there is a primitive embedding $T_X \hookrightarrow U^3$.*

REMARK 3.4. Obviously, the Shioda–Inose structure realizes a correspondence between the $K3$ surface X and the abelian variety Z. This correspondence will play a fundamental role in the development that follows.

4. Starting Problem

Let C be a generic genus-2 curve (i.e., such that its Jacobian surface has $\rho(JC) = 1$); JC is a principally polarized abelian variety and, if E is the principal polarization, we have $E^2 = 2$ so then $T := T_{JC} = U^2 \oplus \langle -2 \rangle$. Since we have obvious primitive embeddings of T in Λ, by Theorem 3.1 there exists an algebraic $K3$ surface X_1 such that $T_{X_1} \cong T_{JC}$. Moreover, by [Mu, Prop. 6.2], the number of Fourier–Mukai partners of X_1 is only one because $\rho(X_1) = 17$; hence such a X_1 is unique (up to isomorphisms).

We now analyze the relations between the Jacobian surface and this $K3$ surface. First, observe that we can construct an embedding $T_{X_1} \hookrightarrow U^3$ in the following way: we send the first two copies of $U \subset T_{X_1}$ to the corresponding ones of U^3 and send the element of square -2 to $e_1^3 - e_2^3$, where $\{e_j^i\}_{j=1,2}^{i=1,2,3}$ is a basis of U^3. Then, by Theorem 3.3, X_1 admits a Shioda–Inose structure

$$\begin{array}{ccc} X_1 & & Z \\ & \searrow \quad \swarrow & \\ & X_2 & \end{array}$$

and so one has $T_Z \cong T_{X_1}$ and thus $T_Z \cong T_{JC}$. More recently, the following theorem was proved by Hosono, Lian, Oguiso, and Yau.

THEOREM 4.1 [HLOY2, Main Theorem 1]. *Let A and B be abelian surfaces. They are Fourier–Mukai partners if and only if* $\mathrm{Kum}(A) \cong \mathrm{Kum}(B)$.

It follows that $\mathrm{Kum}(Z) \cong \mathrm{Kum}(JC)$ because Z and JC are Fourier–Mukai partners. In this way we obtain a correspondence between X_1, X_2, and JC:

$$\begin{array}{ccccccc} JC & & & & Z & & X_1 \\ & \searrow & & & & \searrow \quad \swarrow & \\ & & \mathrm{Kum}(JC) & \cong & & X_2 = \mathrm{Kum}(Z). & \end{array}$$

In the Appendix, Igor Dolgachev gives a geometric realization of the correspondence, between the $K3$ surface X_1 and the Kummer surface X_2, that is induced by the Shioda–Inose structure on X_1 itself.

REMARK 4.2. We observe that the Hodge isometry between T_{JC} and T_Z can be extended to a Hodge isometry between the second cohomology groups, since JC and Z are principally polarized abelian surfaces of rank 1. From [Sh] we obtain that $Z \cong JC$.

5. Embeddings of Twisted Lattices

We want to generalize the setting of the previous section. We consider the lattice $T(k,m,n) := U(k) \oplus U(m) \oplus \langle -2n \rangle$, with $k,m,n \in \mathbf{N}$ and obtained by twisting the summands of T, that is equipped with the Hodge structure induced by T itself. This is an indefinite even lattice of signature $(2,3)$. The aim is to prove the existence of correspondences between $K3$ surfaces having transcendental lattice Hodge-isometric to $T(k,m,n)$. In order to construct such surfaces, we start by proving the following lemma.

LEMMA 5.1. *The lattice $T(k,m,n)$ is a primitive sublattice of the $K3$-lattice Λ.*

Proof. First we observe that a lattice $\langle -2t \rangle$, $t \in \mathbf{N}$, can be primitively embedded in $E_8(-1)$ according to [N1, Thm. 1.12.4]. We denote by $\theta_t \in E_8(-1)$ its generator.

In order to construct the required primitive embedding of $T(k,m,n)$ in Λ, we consider a θ_k contained in the first copy of $E_8(-1)$ and a θ_n in the second one. Then, if $\{\delta_1,\dots,\delta_5\}$ is the standard basis of $T(k,m,n)$, we can obtain the desired embedding in the following way:

$$\begin{aligned} T(k,m,n) \hookrightarrow U^3 \oplus E_8(-1)^2; \\ \delta_1 &\mapsto (1,k)+(0,0)+(0,0)+\theta_k, \\ \delta_2 &\mapsto (1,0)+(0,0)+(0,0), \\ \delta_3 &\mapsto (0,0)+(1,0)+(0,m), \\ \delta_4 &\mapsto (0,0)+(0,0)+(1,0), \\ \delta_5 &\mapsto \theta_n. \end{aligned}$$

□

REMARK 5.2. Together with Theorem 3.1, this lemma implies the existence, for any $k,m,n \in \mathbf{N}$, of a $K3$ surface $X(k,m,n)$ with a Hodge isometry between the transcendental lattices $T_{X(k,m,n)}$ and $T(k,m,n)$. Such a surface is unique (up to isomorphisms) by [Mu, Prop. 6.2].

In order to generalize the example constructed in Section 4, we want to produce correspondences among these $K3$ surfaces. Observe that, with this notation, the surfaces of the previous section can be rewritten as $X_1 = X(1,1,1)$ and $X_2 = X(2,2,2)$.

The problem of finding correspondences among $K3$ surfaces was investigated by Mukai in [Mu]. His idea was to construct such correspondences starting from Hodge isometries (over $\mathbf{Q}$) between the transcendental lattices of the surfaces. This method works if the Picard number of the surfaces is sufficiently large. Mukai's result is as follows.

THEOREM 5.3 [Mu, Cor. 1.10]. *Let X,Y be $K3$ surfaces with $\rho(X),\rho(Y)\geq 11$. If $\varphi\colon T_X\otimes\mathbf{Q}\to T_Y\otimes\mathbf{Q}$ is a Hodge isometry, then φ is induced by an algebraic cycle.*

Our $K3$ surfaces, unfortunately, do not satisfy the condition of Mukai's theorem because an isometry between $T(k,m,n)$ and $T(k',m',n')$ doesn't exist, even over $\mathbf{Q}$. However, in the same article Mukai proved Oda's conjecture (as modified by Morrison in [Mo]), which realizes a correspondence between a $K3$ surface X and an abelian surface provided that the $\mathbf{Q}$-transcendental lattice $T_X\otimes\mathbf{Q}$ admits an embedding in $U^3\otimes\mathbf{Q}$. In order to construct the correspondences among the $X(k,m,n)$, we follow a similar approach, translating the problem into another one involving some (abelian) surfaces that are in correspondence with the given surfaces.

6. Abelian Surfaces and Correspondences

Motivated by the work of Morrison and Mukai, we start with the following lemma.

LEMMA 6.1. *There is an embedding of $\mathbf{Q}$-lattices $\phi\colon T(k,m,n)\otimes\mathbf{Q}\hookrightarrow U^3\otimes\mathbf{Q}$.*

Proof. Let $\{e_1^i, e_2^i\}$ $(i = 1, 2, 3)$ be the basis of the ith copy of the hyperbolic lattice U, and let $\{a_1, b_1, a_2, b_2, c\}$ be the basis of $T(k, m, n)$. We define ϕ as:

$$\begin{aligned} a_1 &\mapsto e_1^1 \otimes 1, \\ b_1 &\mapsto e_2^1 \otimes k, \\ a_2 &\mapsto e_1^2 \otimes 1, \\ b_2 &\mapsto e_2^2 \otimes m, \\ c &\mapsto e_1^3 \otimes 1 + e_2^3 \otimes (-n). \end{aligned}$$ □

The existence of such an embedding allows us to prove our next theorem.

THEOREM 6.2. *For any $k, m, n \in \mathbf{N}$ there exist abelian surfaces A_n and correspondences*

$$\begin{array}{ccc} Z(k,m,n) & \longrightarrow & X(k,m,n) \\ \big\downarrow & & \\ A_n. & & \end{array}$$

Proof. Let us consider the embedding $\phi\colon T(k, m, n) \otimes \mathbf{Q} \hookrightarrow U^3 \otimes \mathbf{Q}$ of Lemma 6.1 as well as the lattice $T_n := \phi(T(k, m, n) \otimes \mathbf{Q}) \cap U^3 = U^2 \oplus \langle -2n \rangle$ with Hodge structure induced by ϕ. Since T_n is a primitive sublattice of Λ, there exists a $K3$ surface Y_n and a Hodge isometry $T_{Y_n} \cong T_n$. Consider now the basis of $T_{X(k,m,n)}$ given in Lemma 6.1: the multiplication by $1/k$ on the sublattice $\mathbf{Z}\langle b_1 \rangle$ and by $1/m$ on $\mathbf{Z}\langle b_2 \rangle$ induces a Hodge isometry $T_{X(k,m,n)} \otimes \mathbf{Q} \cong T_{Y_n} \otimes \mathbf{Q}$, and Theorem 5.3 gives a correspondence $Z'(k, m, n)$ between $X(k, m, n)$ and Y_n.

We can define also a primitive embedding $T_n \cong T_{Y_n} \hookrightarrow U^3$ that sends $U^2 \subset T_n$ to the first two copies of U in U^3 via the identity and sends the element of square $-2n$ to $e_1^3 - n e_2^3$. By Theorem 3.3, the existence of such an embedding is equivalent to the existence of a Shioda–Inose structure on the $K3$ surface Y_n. Hence there exist an abelian surface A_n, a Hodge isometry $T_{A_n} \cong T_n$, and a correspondence Z_n'' between Y_n and A_n. The composition of the two correspondences

$$\begin{array}{ccccccc} & Z'(k,m,n) & & & & Z_n'' & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ X(k,m,n) & & & Y_n & & & A_n \end{array}$$

gives the desired one. □

Now, we are able to prove the following theorem.

THEOREM 6.3. *Let $X(k, m, n)$, $k, m, n \in \mathbf{N}$, be a $K3$ surface with transcendental lattice Hodge-isometric to $T(k, m, n)$. For any $k', m', n' \in \mathbf{N}$ there exist correspondences*

$$\begin{array}{ccc} Z^{k',m',n'}_{k,m,n} & \longrightarrow & X(k,m,n) \\ \downarrow & & \\ X(k',m',n'), & & \end{array}$$

and every $X(k,m,n)$ has a correspondence with the Jacobian surface JC of Section 4.

Proof. Let Z be a principally polarized abelian surface with $\rho(Z) = 1$, and let $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ be a symplectic basis for the lattice of Z. For any n, let $B_n := \mathbf{C}^2/\Lambda_n$ be the complex torus with lattice $\Lambda_n = \mathbf{Z}\langle\lambda_1, n\lambda_2, \lambda_3, \lambda_4\rangle$. Then the polarization E defines on each B_n a polarization of type $(1,n)$; moreover, the abelian surfaces B_n are all obviously isogenous to each other.

On the other hand, we know that $T_{B_n} \cong U^2 \oplus \langle -2n\rangle$, so every B_n is a Fourier–Mukai partner of the surface A_n constructed in Theorem 6.2. This means, by Theorem 4.1, that $\mathrm{Kum}(A_n) \cong \mathrm{Kum}(B_n)$, so there is a correspondence between A_n and B_n. In this way we have constructed a correspondence between A_n and A_m for any n, m:

$$\begin{array}{ccccccc} A_n & & B_n & \xrightarrow{\sim} & B_m & & A_m \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{Kum}(A_n) & \cong & \mathrm{Kum}(B_n) & & \mathrm{Kum}(B_m) & \cong & \mathrm{Kum}(A_m). \end{array}$$

But Theorem 6.2 gives a correspondence between $X(k,m,n)$ and A_n for any n, so the desired $Z^{k',m',n'}_{k,m,n}$ is the composition of these correspondences. Moreover, there is a correspondence between $X(1,1,1)$ and JC from Section 4, so the statement follows. □

REMARK 6.4. We observe that the existence of the correspondence is independent of the chosen Hodge structure on T. However, it is necessary *not* to change the structure when we "twist" the starting lattice T by (k,m,n). This allows us to obtain, for any chosen Hodge structure H on T, a family of $K3$ surfaces $X_H(k,m,n)$ in correspondence to each other.

Morrison [Mo, Cor. 4.4] has shown that a $K3$ surface K of Picard rank 17 is a Kummer surface if and only if there is an even lattice T' with $T_K \cong U(2)^2 \oplus T'(2)$. This allows us to prove the following theorem, which generalizes Theorem 6.3.

THEOREM 6.5. *Let X be a general $K3$ surface with $\rho(X) = 17$ and with $T_X \otimes \mathbf{Q}$ isomorphic as a Hodge structure to $T_K \otimes \mathbf{Q}$, where K is a Kummer surface. Then there is a correspondence between X and K.*

Proof. Let $\psi_K = U^2(2) \oplus \langle -4n\rangle$ be the polarization on T_K; we show that there is a $\mathbf{Q}$-basis of $T_X \otimes \mathbf{Q}$ in which the polarization is $\psi_X = a\psi_K$ with $a \in \mathbf{Q}$. Since $\psi_X \in \mathrm{Sym}^2(T_X \otimes \mathbf{Q})^{\mathrm{MT}(T_X \otimes \mathbf{Q})}$ (where MT denotes the Mumford–Tate group), it suffices to show that this space has dimension 1. This is a consequence of Schur's lemma, since $\mathrm{MT}(T_x \otimes \mathbf{Q})(\mathbf{C}) = \mathrm{SO}(5)(\mathbf{C})$ and the action on the Hodge structure is irreducible. Thus, we have the Hodge $\mathbf{Q}$-isometries $T_X \otimes \mathbf{Q} \cong a(U^2(2) \oplus \langle -4n \rangle) \cong U^2 \oplus \langle -4na_1a_2 \rangle$, where $a = a_1/a_2$. From the surjectivity of the period map, the last one is a transcendental lattice of a $K3$ surface $X(1, 1, -4na_1a_2)$, which by Mukai's work is in correspondence with X. The statement now follows from Theorem 6.3. □

ACKNOWLEDGMENTS. We would like to thank Prof. Bert van Geemen for having introduced us to this problem and for many useful discussions and valuable suggestions. We also thank Prof. Igor Dolgachev for his interest in this work.

References

[BiLa] C. Birkenhake and H. Lange, *Complex abelian varieties,* Grundlehren Math. Wiss., 302, Springer-Verlag, New York, 1992.

[BrMa] T. Bridgeland and A. Maciocia, *Complex surfaces with equivalent derived categories,* Math. Z. 236 (2001), 677–697.

[HLOY1] S. Hosono, B. H. Lian, K. Oguiso, and S. Yau, *Counting Fourier-Mukai partners and applications,* preprint (alg-geom 0202014).

[HLOY2] ———, *Kummer structures on a K3 surface—An old question of T. Shioda,* preprint (alg-geom 0202082).

[Mo] D. R. Morrison, *On K3 surfaces with large Picard number,* Invent. Math. 75 (1984), 105–121.

[Mu] S. Mukai, *On the moduli space of bundles on K3 surfaces I,* Vector bundles on algebraic varieties (Bombay, 1984), pp. 341–413, Tata Inst. Fund. Res., Bombay, 1987.

[N1] V. Nikulin, *Integer symmetric bilinear forms and some of their geometric applications,* Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), 111–177, 238; Math. USSR-Izv. 14 (1980), 103–167.

[N2] ———, *On correspondences between surfaces of K3 type,* Izv. Akad. Nauk SSSR Ser. Mat. 51 (1987), 402–411; Math USSR-Izv. 30 (1988), 375–383.

[Or] D. O. Orlov, *Equivalences of derived categories and K3 surfaces,* J. Math. Sci. (New York) 84 (1997), 1361–1381.

[Sh] T. Shioda, *The period map of abelian surfaces,* J. Fac. Sci. Univ. Tokyo Sect. 1A 25 (1978), 47–59.

[ShIn] T. Shioda and H. Inose, *On singular K3 surfaces,* Complex analysis and algebraic geometry: Papers dedicated to K. Kodaira, pp. 119–136, Iwanami Shoten, Tokyo, 1977.

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Appendix: A Geometric Correspondence between X_1 and a Kummer Surface

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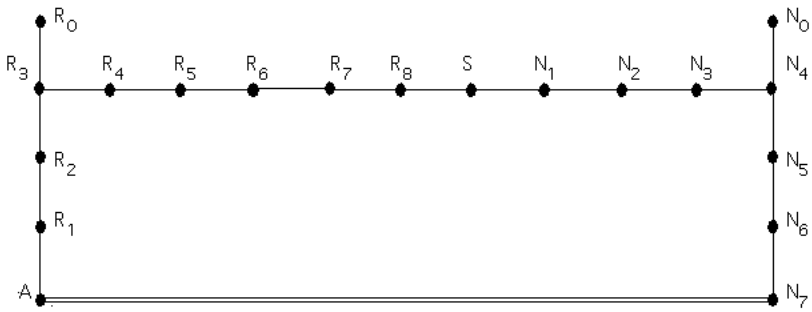
We use the following result of V. Nikulin [2].

THEOREM 1. *Let X be a K3-surface with $T_X \cong U^2 \oplus \langle -2 \rangle$. Then* $\mathrm{Pic}(X)$ *has only finitely many smooth rational curves which form the following graph.*



The automorphism group of X is generated by two commuting involutions σ and τ; the involution σ has eight isolated fixed points; the set of fixed points of the involution τ is the union of eight smooth rational curves and a smooth curve of genus 2.

It follows from observing the graph that X admits an elliptic fibration $|F|$ with two singular fibres

$$F_1 = 3R_0 + 2R_1 + 4R_2 + 6R_3 + 5R_4 + 4R_5 + 3R_6 + 2R_7 + R_8$$

and

$$F_2 = 2N_0 + N_1 + 2N_2 + 3N_3 + 4N_4 + 3N_5 + 2N_6 + N_7$$

of type $\tilde{E}_8$ and $\tilde{E}_7$, respectively. It also has a section S. The fixed locus of τ consists of smooth rational curves $R_1, R_3, R_5, R_7, N_2, N_4, N_6, S$ and a genus-2 curve W that intersects R_0, N_0, N_7 with multiplicity 1. Let $p = W \cap R_0$, $q = W \cap N_0$, and $a = W \cap N_7$. We have $3p \sim 2q + a$, and the fibration defines a g^1_3 on W spanned by the divisors $3p$ and $2q + a$.

We also observe that X contains another elliptic fibration $|F'|$ with two reducible fibres

$$F_1' = 3N_0 + R_8 + 2S + 3N_1 + 4N_2 + 5N_3 + 6N_4 + 4N_5 + 2N_6$$

and

$$F_2' = 2R_0 + A + 2R_1 + 3R_2 + 4R_3 + 3R_4 + 2R_5 + R_6$$

of type $\tilde{E}_8$ and $\tilde{E}_7$, respectively. The curve R_7 is a section. The involution σ switches the two fibres and induces the hyperelliptic involution on W. Its set of fixed points are two points on the curve R_8 and six points on W. Also note that σ maps the fibration $|F|$ to the fibration $|F'|$. It is easy to see that $q = \sigma(p)$. This gives $3p \sim 2K_W - 2p + a$, hence

$$|5p| = |2K_W + a|.$$

THEOREM 2. *The linear system $|F' + F|$ defines a map $f\colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ of degree 2. Its branch locus is a curve of bidegree $(4,4)$ that is equal to the union of a curve B of bidegree $(3,3)$ and two rulings E_1, E_2. The curve B has two cusps q_1, q_2. The cuspidal tangent at q_i is equal to E_i. The automorphism τ of X is the deck transformation of f, and the quotient $X/(\tau)$ admits a birational morphism to $\mathbb{P}^1 \times \mathbb{P}^1$ that resolves the singularities of the branch curve. The automorphism σ is induced by the automorphism $\bar{\sigma}$ of $\mathbb{P}^1 \times \mathbb{P}^1$, which switches the factors. It leaves the curve B invariant and switches E_1 and E_2.*

Proof. We have $(F + F')^2 = 4$, and the restriction of $|F' + F|$ to a nonsingular fibre of each fibration is a degree-2 map. This easily implies that the linear system defines a degree-2 map f to a nonsingular quadric in $\mathbb{P}^3$, and the pre-images of the rulings are the fibrations $|F|$ and $|F'|$. The map f blows down the curves $R_0, R_1, R_2, R_3, R_4, R_6, N_0, N_1, N_2, N_3, N_4, N_5, N_6$. Its restriction to W is a birational map defined by a 3-dimensional linear subsystem of $|3p + 3q| = |3K_W|$. The rest of the assertions are easy to verify. □

Consider the automorphism $\bar{\sigma}$ of $\mathbb{P}^1 \times \mathbb{P}^1$, and let

$$\pi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1/(\bar{\sigma}) \cong \mathbb{P}^1$$

be the natural projection to the orbit space. Its locus of fixed points is the diagonal Δ. The image of Δ on $\mathbb{P}^2$ is a conic Q, and the image of W is a cuspidal cubic G. The image of both E_1 and E_2 is the cuspidal tangent T of G, which is also a tangent of the conic Q. The curves Q and G intersect at six points, the ramification points of the hyperelliptic involution of W.

THEOREM 3. *Let $\bar{f}\colon Y \to \bar{Y} \to \mathbb{P}^2$ be a minimal resolution of the double cover $\bar{Y}$ of $\mathbb{P}^2$ branched along the union of the curves G, Q, and T. Then Y is a Kummer surface that is birationally isomorphic to the quotient of X by σ.*

Proof. We have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & \mathbb{P}^1 \times \mathbb{P}^1 \\ {\scriptstyle /\sigma}\downarrow & & \downarrow{\scriptstyle /\bar{\sigma}} \\ Y & \xrightarrow{\bar{f}} & \mathbb{P}^2. \end{array}$$

It follows from [1] that Y is a Kummer surface of $\mathrm{Jac}(C)$ for some curve C (not isomorphic to W). □

Remark. One can reverse the construction of Theorem 2. Starting from a genus-2 curve W together with a point $p \in W$ such that $|3K_W - p| \neq \emptyset$ and p is not a Weierstrass point, we construct a 2-cuspidal model W in $\mathbb{P}^1 \times \mathbb{P}^1$ as in the theorem. Then, taking the double cover, we obtain a $K3$ surface X with Picard lattice containing $U \oplus E_8 \oplus E_7$ and $U^2 \oplus \langle -2 \rangle \subset T_X$. Replacing p by the conjugate point q under the hyperelliptic involution, we get the same surface X. As explained to me by J. Harris, the number of pairs (p, q) as above is equal to 16. Thus we obtain that the moduli space of $K3$ surfaces marked with the lattice $U \oplus E_8 \oplus E_7$ is isomorphic to a (16 : 1)-cover $\mathcal{M}_2'$ of the moduli space $\mathcal{M}_2$ of genus-2 curves; on the other hand, via periods this space is isomorphic to the moduli space $\mathcal{A}_2$ of principally polarized abelian surfaces. This defines a birational isomorphism between $\mathcal{M}_2'$ and $\mathcal{M}_2$.

References

[1] I. Naruki, *On metamorphosis of Kummer surfaces,* Hokkaido Math. J. 20 (1991), 407–415.

[2] V. Nikulin, *Quotient-groups of groups of automorphisms of hyperbolic forms by subgroups generated by 2-reflections. Algebro-geometric applications* [Russian], Current problems in mathematics, v. 18, pp. 3–114, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1981; J. Soviet Math. 22 (1983), 1401–1476.

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