Michigan Math. J. 52 (2004)

Lattice Points inside Random Ellipsoids

S. Hofmann, A. Iosevich, & D. Weidinger

1. Introduction

Let

$$N_a(t) = \#\{t\Omega_a \cap \mathbb{Z}^d\}, \tag{0.1}$$

where

$$\Omega_a = \left\{(a_1^{-\frac{1}{2}}x_1, a_2^{-\frac{1}{2}}x_2, \dots, a_d^{-\frac{1}{2}}x_d) : x \in \Omega\right\} \tag{0.2}$$

with $\frac{1}{2} \le a_j \le 2$ and where Ω is the unit ball.

Let

$$N_a(t) = t^d|\Omega_a| + E_a(t). \tag{0.3}$$

A classical result due to Landau states that

$$|E_a(t)| \lesssim t^{d-2+\frac{2}{d+1}}; \tag{0.4}$$

here and throughout the paper, $A \lesssim B$ means that there exists a positive constant C such that $A \le CB$. Similarly, $A \lessapprox B$, with a parameter t, means that given $\delta > 0$ there exists a $C_\delta > 0$ such that $A \le C_\delta t^\delta B$.

A number of improvements over (0.4) have been obtained over the years in two and three dimensions. The best-known result in three dimensions (to the best of our knowledge) is $|E_a(t)| \lessapprox t^{\frac{21}{16}}$ proved by Heath-Brown [HB], improving on an earlier breakthrough due to Vinogradov [V]. It is proved by Szegö that

$$\left|E_{1,1,1}(t) - \frac{4\pi}{3}t^3\right| \gtrsim t\log(t). \tag{0.5}$$

In two dimensions, the best-known result is $|E_a(t)| \lessapprox t^{\frac{46}{73}}$ due to Huxley [Hu]. A classical result due to Hardy says that

$$|E_{1,1}(t) - \pi t^2| \gtrsim t^{\frac{1}{2}} \log^{\frac{1}{2}}(t). \tag{0.6}$$

Thus it is reasonable to conjecture that the estimate

$$|E_a(t)| \lessapprox t^{\frac{d-1}{2}} \tag{0.7}$$

holds in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Received July 22, 2002. Revision received July 21, 2003.
Research supported in part by NSF Grant no. DMS00-87339.

In higher dimensions, the problem of a pointwise estimate of $E_a(t)$ is completely solved. It is a result of Walfisch that if $d \geq 4$ then $|E_a(t)| \lessapprox t^{d-2}$, and a logarithm may be removed in dimension 5 and greater. It is also known that if the eccentricities $(a_1, \ldots, a_d)$ are rational, then this estimate is essentially sharp.

It is not known if there exists a single $a = (a_1, a_2, \ldots, a_d)$ such that $|E_a(t)| \lessapprox t^{\frac{d-1}{2}}$ in any dimension. The question of finding such an a was posed by Sarnak in a two-dimensional setting a number of years ago. Sarnak's question would be answered by the following estimate.

CONJECTURE. *Given any* $\delta > 0$,

$$\sup_{t \geq 1} t^{-\frac{d-1}{2}-\delta} |E_{(\cdot)}(t)| \in L^p\left(\left[\tfrac{1}{2}, 2\right] \times \left[\tfrac{1}{2}, 2\right] \times \cdots \times \left[\tfrac{1}{2}, 2\right]\right) \tag{0.8}$$

for some $p \geq 1$ *with a constant depending on* δ.

Of course, (0.8) would imply that the estimate $|E_a(t)| \lessapprox t^{\frac{d-1}{2}}$ holds for almost every $a \in \left(\left[\frac{1}{2}, 2\right] \times \left[\frac{1}{2}, 2\right] \times \cdots \times \left[\frac{1}{2}, 2\right]\right)$. We hope to address this issue in a subsequent paper.

Other types of square averages of lattice point discrepancy functions have been studied in the past and in recent years. For example, a classical result due to Kendall is that

$$\int_{\mathbb{T}^2} |\#\{(t\Omega + \tau) \cap \mathbb{Z}^d\} - t^d |\Omega||^2 \, d\tau \lesssim t^{\frac{d-1}{2}} \tag{0.9}$$

for every convex domain whose boundary has everywhere nonvanishing Gaussian curvature.

This result was recently sharpened by Magyar and Seeger, who proved that the estimate (0.9) still holds in $\mathbb{R}^d$ if the exponent 2 is replaced by $p \leq \frac{2d}{d-1}$.

Another type of average is studied in [ISS]. The authors prove that

$$\left(\frac{1}{h} \int_R^{R+h} |\#\{t\Omega \cap \mathbb{Z}^d\} - t^d |\Omega||^2 \, dt\right)^{\frac{1}{2}} \lesssim R^{\alpha_d}, \tag{0.10}$$

where

$$\alpha_2 = \tfrac{1}{2} \quad \text{with } h \geq \log(R) \tag{0.11}$$

and

$$\alpha_d = d - 2 \quad \text{with } h \approx R \tag{0.12}$$

for $d \geq 4$. If $d = 3$ then $\alpha_d = 1$ and an additional factor $\log(R)$ is present. These results improve upon those previously obtained by Muller [M]. See also [Hu] and [ISS] and the references contained therein.

Using (0.10), (0.11), (0.12), and their proofs, one can deduce the following result.

THEOREM 0.1. *Let* $E_a(t)$ *be as before. Then*

$$\int_{\frac{1}{2}}^{2} \int_{\frac{1}{2}}^{2} \cdots \int_{\frac{1}{2}}^{2} |E_a(t)|^2 \, da \lesssim R^{\alpha_d}, \tag{0.13}$$

where α_d is exactly as described previously and where the additional $\log(t)$ *factor is still present in three dimensions.*

The purpose of this paper is to give a simple and transparent proof of Theorem 0.1 in two and three dimensions. Similar two-dimensional results have recently been obtained by Toth and Petridis [TP] using different methods. We believe it is likely that our approach will lead to a better estimate in higher dimensions, where we conjecture that (0.13) holds with $\alpha_d = \frac{d-1}{2}$. We hope to address this issue in a subsequent paper.

We shall give the proof in three dimensions. We shall then indicate how a two-dimensional proof follows from a simpler version of the same argument.

1. Basic Setup

We start with the following standard reduction. Let $\rho_0 \in C_0^\infty\left(\frac{1}{4}, 4\right)$ with $\rho_0 \equiv 1$ on [1, 2], and let ρ be the radial extension of ρ_0 such that $\int \rho(x)\,dx = 1$.

Let $\rho_\varepsilon(x) = \varepsilon^{-3}\rho\left(\frac{x}{\varepsilon}\right)$, and let

$$\begin{aligned} N_a^\varepsilon(t) &= \sum_{k\in\mathbb{Z}^3} \chi_{t\Omega_a} * \rho_\varepsilon(k) = t^3|\Omega_a| + t^3 \sum_{k\neq(0,0,0)} \hat{\chi}_{\Omega_a}(tk)\hat{\rho}(\varepsilon k) \\ &= t^3|\Omega_a| + E_a^\varepsilon(t). \end{aligned} \tag{1.1}$$

It is not hard to see that there exists a $C > 0$ such that

$$N_a^\varepsilon(t - C\varepsilon) \leq N_a(t) \leq N_a^\varepsilon(t + C\varepsilon). \tag{1.2}$$

It follows that

$$\int_{\left[\frac{1}{2},2\right]\times\left[\frac{1}{2},2\right]\times\left[\frac{1}{2},2\right]} |E_a(t)|^2\,da \lesssim \int_{\left[\frac{1}{2},2\right]\times\left[\frac{1}{2},2\right]\times\left[\frac{1}{2},2\right]} |E_a^\varepsilon(t)|^2\,da + t^4\varepsilon^2. \tag{1.3}$$

We conclude that it suffices to establish estimates for $E_a^\varepsilon(t)$ with $\varepsilon = t^{-1}$.

Using the standard asymptotic formula for the Fourier transform of the characteristic function of a bounded smooth convex domain where the Gaussian curvature of the boundary is nonvanishing (see e.g. [H]), we see that $\hat{\chi}_{\Omega_a}(tk)$ is a sum of two terms of the form

$$e^{2\pi it|k|_a} t^{-2}|k|_a^{-2} + O((t|k|)^{-3}), \tag{1.4}$$

where

$$|k|_a = \sqrt{a_1k_1^2 + a_2k_2^2 + a_3k_3^2}. \tag{1.5}$$

It follows that

$$\begin{aligned} E_a^\varepsilon(t) &= t \sum_{k\neq(0,0,0)} e^{2\pi it|k|_a}|k|_a^{-2}\hat{\rho}(\varepsilon k) + t^3 \sum_{k\neq(0,0,0)} O((t|k|)^{-3})\hat{\rho}(\varepsilon k) \\ &= I + II. \end{aligned} \tag{1.6}$$

Since we can easily handle II pointwise, we turn our attention to I. Squaring, integrating in a, and replacing the limits of integration in a by a smooth cutoff function, we obtain

$$t^2 \sum_{k,l \neq (0,0,0)} |k|^{-2}|l|^{-2}\hat{\rho}(\varepsilon k)\hat{\rho}(\varepsilon l) \int e^{2\pi i t(|k|_a - |l|_a)} \psi_{k,l}(a)\, da$$

$$= t^2 \sum_{k,l \neq (0,0,0)} |k|^{-2}|l|^{-2}\hat{\rho}(\varepsilon k)\hat{\rho}(\varepsilon l) I_{k,l}(t); \quad (1.7)$$

here

$$\psi_{k,l}(a) = \left(\frac{|k|}{|k|_a}\right)^2 \left(\frac{|l|}{|l|_a}\right)^2 \psi(a), \quad (1.8)$$

where ψ is a positive smooth cutoff function that is supported in $\left[\frac{1}{4}, 4\right]$ and identically equal to 1 on $\left[\frac{1}{2}, 2\right]$. Observe that, if $k \neq (0,0,0)$ and $l \neq (0,0,0)$, then $\psi_{k,l} \in C_0^\infty$ with constants uniform in k and l. It suffices to show that (1.7) is bounded above by $C_\delta t^{2+\delta}$ for any $\delta > 0$.

2. Preliminary Reductions

This section contains some simple observations that we shall make use of in Section 3, where the main result of the paper is proved.

Lemma 2.1. *Let $\delta > 0$, and let $N > \frac{1}{\delta} + 1$. Then*

$$\sum_{|k| > \varepsilon^{-1-\delta}} |k|^{-2}|\varepsilon k|^{-N} \lesssim 1. \quad (2.1)$$

Proof. We have

$$\sum_{|k| > \varepsilon^{-1-\delta}} |k|^{-2}|\varepsilon k|^{-N} \lesssim \varepsilon^{-N} \int_{|x| > \varepsilon^{-1-\delta}} |x|^{-2-N}\, dx$$

$$\lesssim \varepsilon^{-N} \varepsilon^{-1-\delta} \varepsilon^{N} \varepsilon^{\delta N} \frac{1}{N-1} \lesssim 1 \quad (2.2)$$

if $N > \frac{1}{\delta} + 1$. □

Since $|\hat{\rho}(\varepsilon k)| \lesssim (1 + |\varepsilon k|)^{-N}$ for any $N > 0$ and since $|I_{k,l}(t)| \lesssim 1$, Lemma 2.1 shows that in estimating (1.7) we may sum over $|k|, |l| \lesssim \varepsilon^{-1-\delta}$ ($\delta > 0$). In particular, this means that we may sum over $|k_j|, |l_j| \lesssim \varepsilon^{-1-\delta}$.

Lemma 2.2. *Let S, S' be subsets of $\{1, 2, 3\}$ of cardinality at most 2. Then*

$$t^2 \sum_{1 \le |k_i|, |l_j| \lesssim \varepsilon^{-1-\delta}; i \in S, j \in S'} |k|^{-2}|l|^{-2} \lessapprox t^2. \quad (2.3)$$

Proof. The proof is immediate since we are down to at most two variables in k and l, so the power -2 suffices (up to logarithms). □

Lemma 2.3. *Let $U = \{k, l \in \mathbb{Z}^3 \times \mathbb{Z}^3 : |k_j|, |l_j| \lesssim \varepsilon^{-1-\delta};\ k_1 = 0, l_1 \neq 0\}$. Then*

$$t^2 \sum_U |k|^{-2}|l|^{-2} I_{k,l}(t) \lessapprox t^2. \quad (2.4)$$

Proof. Let $\Phi_{k,l}(a) = |k|_a - |l|_a$. We have

$$\nabla\Phi_{k,l}(a) = \frac{1}{2}\left(\frac{k_1^2}{|k|_a} - \frac{l_1^2}{|l|_a}, \frac{k_2^2}{|k|_a} - \frac{l_2^2}{|l|_a}, \frac{k_3^2}{|k|_a} - \frac{l_3^2}{|l|_a}\right). \tag{2.5}$$

Since $k_1 = 0$, it follows that $|\nabla\Phi_{k,l}(a)| \gtrsim l_1^2/|l|$. Integrating by parts once (see Section 5) shows that

$$|I_{k,l}(t)| \lesssim t^{-1}\frac{|l|}{l_1^2}. \tag{2.6}$$

We then have

$$t^2 t^{-1} \sum_{1\le|k_j|,|l_j|\lesssim\varepsilon^{-1-\delta};\, k_1=0} |k|^{-2}|l|^{-2}|l|l_1^{-2} \lessapprox t \sum_{1\le|l_j|\lesssim\varepsilon^{-1-\delta}} (|l_2|+|l_3|)^{-1}l_1^{-2} \lessapprox t\varepsilon^{-1} \lesssim t^2. \tag{2.7}$$

□

The same argument works if $k_2 = 0$ and $l_2 \neq 0$, or if $k_3 = 0$ and $l_3 \neq 0$.

The basic idea of these reductions is that we need only sum up to $|k|, |l| \lesssim \varepsilon^{-1-\delta}$ and that it suffices to consider the case where $k_j, l_j \neq 0$ for $j = 1, 2, 3$.

3. $\left|\left|\frac{k_1}{k_2}\right| - \left|\frac{l_1}{l_2}\right|\right| + \left|\left|\frac{k_1}{k_3}\right| - \left|\frac{l_1}{l_3}\right|\right| + \left|\left|\frac{k_2}{k_3}\right| - \left|\frac{l_2}{l_3}\right|\right| \neq 0$

The determinant of the Hessian matrix of $\Phi_{k,l}$ with respect to (a_1, a_2) equals

$$-\frac{1}{16}\frac{(k_1^2l_2^2 - k_2^2l_1^2)^2}{|k|_a^3|l|_a^3}, \tag{3.1}$$

and its absolute value is bounded from below by a constant multiple of

$$\frac{(k_1^2l_2^2 - k_2^2l_1^2)^2}{|k|^3|l|^3}. \tag{3.2}$$

It follows that

$$\begin{aligned} t^2 &\sum_{1\le|k_j|,|l_j|\lesssim\varepsilon^{-1-\delta};\, \left|\left|\frac{k_1}{k_2}\right|-\left|\frac{l_1}{l_2}\right|\right|\neq0} |k|^{-2}|l|^{-2}I_{k,l}(t) \\ &\lesssim t \sum_{1\le|k_j|,|l_j|\lesssim\varepsilon^{-1-\delta};\, \left|\left|\frac{k_1}{k_2}\right|-\left|\frac{l_1}{l_2}\right|\right|\neq0} |k|^{-\frac12}|l|^{-\frac12}|k_1^2l_2^2 - k_2^2l_1^2|^{-1} \\ &\lesssim t \sum_{1\le|k_j|,|l_j|\lesssim\varepsilon^{-1-\delta};\, \left|\left|\frac{k_1}{k_2}\right|-\left|\frac{l_1}{l_2}\right|\right|\neq0} |k_3|^{-\frac12}|l_3|^{-\frac12}|k_1^2l_2^2 - k_2^2l_1^2|^{-1} \\ &\lessapprox t\varepsilon^{-1} \sum_{1\le|k_j|,|l_j|\lesssim\varepsilon^{-1-\delta};\, j=1,2;\, \left|\left|\frac{k_1}{k_2}\right|-\left|\frac{l_1}{l_2}\right|\right|\neq0} |k_1^2l_2^2 - k_2^2l_1^2|^{-1}. \end{aligned} \tag{3.3}$$

Either $\operatorname{sgn}(k_1l_2) = \operatorname{sgn}(l_1k_2)$ or $\operatorname{sgn}(k_1l_2) = -\operatorname{sgn}(l_1k_2)$. Without loss of generality, suppose that $k_j, l_j > 0$. It follows that (3.3) is bounded by the expression of the form

$$t\varepsilon^{-1}\sum_{m=0}^{\approx\log(\varepsilon^{-2})}2^{-m}\left|\sum_{\substack{1\le k_j,l_j\lesssim\varepsilon^{-1-\delta},\,j=1,2;\\ 2^m\le|k_1l_2-k_2l_1|\le 2^{m+1}}}k_1^{-1}l_2^{-1}\right|$$

$$\lessapprox t\varepsilon^{-1}\sum_{m=0}^{\approx\log(\varepsilon^{-2})}2^{-m}\left|\int_{\substack{1\le x_j,\,y_j\le\varepsilon^{-1};\\ 2^m\le|x_1x_2-y_1y_2|\le 2^{m+1}}}x_1^{-1}x_2^{-1}\,dx\,dy\right|. \quad (3.4)$$

Let

$$u_1=x_1x_2,\quad u_2=x_2;\quad v_1=y_1y_2,\quad v_2=y_2. \quad (3.5)$$

It follows that

$$\begin{aligned} du_1&=x_2dx_1+x_1dx_2,\quad du_2=dx_2;\\ dv_1&=y_2dy_1+y_1dy_2,\quad dv_2=dy_2. \end{aligned} \quad (3.6)$$

Also, $x_1=u_1/u_2$ and so $x_1x_2=u_1$. Combining this with (3.5) and (3.6), we see that (3.4) is bounded by

$$t\varepsilon^{-1}\sum_{m=0}^{\approx\log(\varepsilon^{-2})}2^{-m}\left|\int_{\substack{1\le u_1,\,v_1\le\varepsilon^{-2},1\le u_2,\,v_2\le\varepsilon^{-1};\\ 2^m\le|u_1-v_1|\le 2^{m+1}}}u_1^{-1}u_2^{-1}v_2^{-1}\,du\,dv\right|$$

$$\lessapprox t\varepsilon^{-1}\sum_{m=0}^{\approx\log(\varepsilon^{-2})}2^{-m}\left|\int_{\substack{1\le u_1,\,v_1\le\varepsilon^{-2};\\ 2^m\le|u_1-v_1|\le 2^{m+1}}}u_1^{-1}\,du_1\,dv_1\right|\lessapprox t\varepsilon^{-1}\le t^2. \quad (3.7)$$

Clearly, the same argument works if

$$\left|\left|\frac{k_1}{k_3}\right|-\left|\frac{l_1}{l_3}\right|\right|\ne 0\quad\text{or}\quad\left|\left|\frac{k_2}{k_3}\right|-\left|\frac{l_2}{l_3}\right|\right|\ne 0.$$

4. $\left|\left|\frac{k_1}{k_2}\right|-\left|\frac{l_1}{l_2}\right|\right|+\left|\left|\frac{k_1}{k_3}\right|-\left|\frac{l_1}{l_3}\right|\right|+\left|\left|\frac{k_2}{k_3}\right|-\left|\frac{l_2}{l_3}\right|\right|=0$

In this case,

$$\left|\frac{k_1}{l_1}\right|=\left|\frac{k_2}{l_2}\right|=\left|\frac{k_3}{l_3}\right|. \quad (4.1)$$

It follows that $k=\alpha l$. Dominating $|I_{k,l}(t)|$ by 1, we have

$$t^2\sum_{1\le|k_j|,|l_j|\lesssim\varepsilon^{-1-\delta};\left|\frac{k_1}{l_1}\right|=\left|\frac{k_2}{l_2}\right|=\left|\frac{k_3}{l_3}\right|}|k|^{-2}|l|^{-2}I_{k,l}(t). \quad (4.2)$$

We are summing over the set where $l=\alpha k$. Observe that α must be of the form $m/\gcd(k_1,k_2,k_3)$. It follows that the expression in (4.2) is bounded by a constant multiple of

$$\lesssim t^2 \sum_{1\le|k|\lesssim\varepsilon^{-1-\delta}} \sum_{\alpha=m/\gcd(k_1,k_2,k_3)\lesssim\varepsilon^{-1-\delta}} \alpha^{-2}|k|^{-4}$$

$$= t^2 \sum_{1\le|k|\lesssim\varepsilon^{-1-\delta}} \sum_{m=1}^{\approx\varepsilon^{-1-\delta}/\gcd(k_1,k_2,k_3)} \frac{(\gcd(k_1,k_2,k_3))^2}{m^2}|k|^{-4}$$

$$\lesssim t^2 \sum_{1\le|k|\lesssim\varepsilon^{-1-\delta}} (\gcd(k_1,k_2,k_3))^2|k|^{-4}$$

$$= t^2 \sum_{n=1}^{\approx\log(\varepsilon^{-1-\delta})} 2^{-4n} \sum_{|k|\approx 2^n} \sum_{j=1}^{\approx\varepsilon^{-1-\delta}} \sum_{\gcd(k_1,k_2,k_3)=j} j^2$$

$$\approx t^2 \sum_{n=1}^{\approx\log(\varepsilon^{-1-\delta})} 2^{-4n} \sum_{|k|\approx 2^n/j} \sum_{j=1}^{\approx\varepsilon^{-1-\delta}} \sum_{\gcd(k_1,k_2,k_3)=1} j^2$$

$$\lesssim t^2 \sum_{n=1}^{\approx\log(\varepsilon^{-1-\delta})} \sum_{j=1}^{\approx\varepsilon^{-1-\delta}} 2^{-4n}\frac{2^{3n}}{j^3} j^2$$

$$= t^2 \sum_{n=1}^{\approx\log(\varepsilon^{-1-\delta})} \sum_{j=1}^{\approx\varepsilon^{-1-\delta}} 2^{-n} j^{-1} \lessapprox t^2. \tag{4.3}$$

This completes the three-dimensional proof. We now outline the two-dimensional argument. The determinant of the Hessian matrix of $\Phi_{k,l}$ in two dimensions is given by (3.1). When $\left|\frac{k_1}{k_2}\right| \neq \pm\left|\frac{l_1}{l_2}\right|$, a calculation identical to the one contained in (3.3)–(3.7) does the job. If $\left|\frac{k_1}{k_2}\right| = \pm\left|\frac{l_1}{l_2}\right|$, we repeat the argument in (4.2) and (4.3) as follows:

$$t \sum_{1\le|k_j|,|l_j|\lesssim\varepsilon^{-1-\delta};\left|\frac{k_1}{l_1}\right|=\left|\frac{k_2}{l_2}\right|} |k|^{-\frac{3}{2}}|l|^{-\frac{3}{2}} I_{k,l}(t)$$

$$\lesssim t \sum_{1\le|k_j|,|l_j|\lesssim\varepsilon^{-1-\delta};\left|\frac{k_1}{l_1}\right|=\left|\frac{k_2}{l_2}\right|} |k|^{-1}|l|^{-1} I_{k,l}(t)$$

$$\lesssim t \sum_{1\le|k|\lesssim\varepsilon^{-1-\delta}} \sum_{\alpha=m/\gcd(k_1,k_2)\lesssim\varepsilon^{-1-\delta}} \alpha^{-1}|k|^{-2}$$

$$= t \sum_{1\le|k|\lesssim\varepsilon^{-1-\delta}} \sum_{m=1}^{\approx\varepsilon^{-1-\delta}/\gcd(k_1,k_2)} \frac{\gcd(k_1,k_2)}{m}|k|^{-2}$$

$$\lesssim t \sum_{1\le|k|\lesssim\varepsilon^{-1-\delta}} \gcd(k_1,k_2)|k|^{-2}$$

$$= t \sum_{n=1}^{\approx \log(\varepsilon^{-1-\delta})} 2^{-2n} \sum_{|k| \approx 2^n} \sum_{j=1}^{\approx \varepsilon^{-1-\delta}} \sum_{\gcd(k_1,k_2)=j} j$$

$$\approx t \sum_{n=1}^{\approx \log(\varepsilon^{-1-\delta})} 2^{-2n} \sum_{|k| \approx 2^n/j} \sum_{j=1}^{\approx \varepsilon^{-1-\delta}} \sum_{\gcd(k_1,k_2)=1} j$$

$$\lesssim t \sum_{n=1}^{\approx \log(\varepsilon^{-1-\delta})} \sum_{j=1}^{\approx \varepsilon^{-1-\delta}} 2^{-2n} \frac{2^{2n}}{j^2} j$$

$$= t \sum_{n=1}^{\approx \log(\varepsilon^{-1-\delta})} \sum_{j=1}^{\approx \varepsilon^{-1-\delta}} 2^{-n} j^{-1} \lessapprox t. \tag{4.4}$$

5. Appendix: Oscillatory Integrals of the First Kind

In this paper we made use of the following basic facts about oscillatory integrals of the form

$$I(t) = \int_{\mathbb{R}^d} e^{itf(x)} \psi(x)\, dx, \tag{5.1}$$

where ψ is a smooth cutoff function and f is smooth. See for example [St] or [BNW] for related information.

THEOREM 5.1. *Suppose that f is convex and of finite type, and supose that the Hessian matrix of f contains an $M \times M$ submatrix of determinant $\geq c_0$. Then*

$$|I(t)| \lesssim t^{-\frac{M}{2}} c_0^{-\frac{1}{2}}. \tag{5.2}$$

THEOREM 5.2. *Suppose that $|\nabla f(a)| \gtrsim c_0$. Then*

$$|I(t)| \lesssim t^{-1} c_0^{-1}. \tag{5.3}$$

We note that, in both theorems, the constants may depend on the upper bounds of derivatives of f and ψ.

References

[BNW] J. Bruna, A. Nagel, and S. Wainger, *Convex hypersurfaces and Fourier transforms,* Ann. of Math. (2) 127 (1988), 333–365.

[H-B] D. R. Heath-Brown, *Lattice points in the sphere,* Number theory in progress, vol. 2 (Zakopane-Koscielisko, 1997), pp. 883–892, de Gruyter, Berlin, 1999.

[H] C. Herz, *Fourier transforms related to convex sets,* Ann. of Math. (2) 75 (1962), 81–92.

[Hu] M. N. Huxley, *Area, lattice points, and exponentials sums,* London Math. Soc. Monogr. (N.S.), 13, Oxford Univ. Press, New York, 1996.

[ISS] A. Iosevich, E. Sawyer, and A. Seeger, *Mean square discrepancy bounds for the number of lattice points in large convex bodies,* J. Anal. Math. 87 (2002), 209–230.
[M] W. Muller, *On the average order of the lattice rest of a convex body,* Acta Arith. 80 (1997), 89–100.
[Sz] G. Szegö, *Beitrage zur Theorie der Laguereschen Polynome II: Zahlentheoretische Anwendungen,* Math. Z. 25 (1926), 388–404.
[St] E. M. Stein, *Harmonic analysis,* Princeton Univ. Press, Princeton, NJ, 1993.
[TP] J. Toth and Y. Petridis, *The remainder in Weyl's law for random two-dimensional flat tori,* Geom. Funct. Anal. 12 (2002), 756–775.
[V] I. M. Vinogradov, *On the number of integer points in a sphere,* Izv. Akad. Nauk SSSR Ser. Mat. 27 (1963), 957–968.

S. Hofmann
Department of Mathematics
University of Missouri
Columbia, MO 65211

hofmann@math.missouri.edu

A. Iosevich
Department of Mathematics
University of Missouri
Columbia, MO 65211

iosevich@math.missouri.edu

D. Weidinger
Department of Physics
University of Illinois
Urbana, IL 61801

weidinge@students.uiuc.edu