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$\mu(z)$-Homeomorphisms in the Plane

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0. Introduction

In this paper we consider the concept of $\mu(z)$-homeomorphisms, which are the natural generalization of K-quasiconformal mappings. We establish an existence and uniqueness theorem of $\mu(z)$-homeomorphic solutions of the Beltrami equation,

$$f_{\bar{z}}(z) = \mu(z) f_z(z), \tag{0.1}$$

where $\mu(z) < 1$ a.e. and $\|\mu\|_\infty = 1$.

The theory of quasiconformal mappings in the complex plane is well developed and plays important roles in study of Teichmüller spaces [8], Riemann surfaces [13], Fuchsian groups and complex dynamic systems [22], and more.

The analytic definition of quasiconformality was first presented in a 1938 paper of Morrey [17]. He studied homeomorphic solutions of the Beltrami equation in the case where $\mu(z)$ is a measurable function defined almost everywhere in a plane domain and satisfies

$$\|\mu\|_\infty \le k < 1. \tag{0.2}$$

Definition 1. A topological mapping f of a region Ω is *K-quasiconformal* if:

(i) f is absolutely continuous on almost all lines (ACL) in Ω; and
(ii) $|f_{\bar{z}}| \le k|f_z|$ a.e., where $k = \frac{K-1}{K+1}$.

The classical result on equation (0.1) is that there exists a homeomorphic solution that is a K-quasiconformal mapping and unique up to a conformal mapping under condition (0.2).

The condition (0.2), on which most properties of quasiconformal mappings are based, is essential and important in the theory of quasiconformal mappings. A natural and interesting problem arises: What happens when assumption (0.2) is dropped? This is the starting point of the present paper.

Many mathematicians have been interested in the more general case where $|\mu(z)| < 1$ almost everywhere yet possibly $\|\mu\|_\infty = 1$. In 1968, Lehto treated the case of the plane with following two additional assumptions on $\mu(z)$.

(L_1) The modulus of μ is bounded away from 1 on every compact subset of $\mathbb{C} - E$, where $\mathbb{C}$ denotes the complex plane and E is a closed set in $\mathbb{C}$ that is of σ-finite linear measure.

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(L_2) For any complex z and $0 < r_1 < r_2 < \infty$, the integral

$$\int_{r_1}^{r_2} \left(1 + 2\pi \int_0^{2\pi} \frac{|1 - e^{-2i\theta}\mu(z + re^{i\theta})|^2}{1 - |\mu(z + re^{i\theta})|^2} \, d\theta \right)^{-1} \frac{dr}{r}$$

is strictly positive and tends to ∞ as $r_1 \to 0$ or $r_2 \to \infty$.

Under these conditions, Lehto [14] presented an existence theorem of homeomorphic solutions. However, he did not touch the uniqueness then. For properties of these homeomorphisms, we refer to [4], [5], and [6].

In 1988, David [7] studied the problem in quite a different way and obtained the existence and uniqueness theorems provided that

(D) there exist constant $\alpha > 0$ and $C > 0$ such that, for any $\varepsilon > 0$ sufficiently small,

$$\text{measure}\{z : |\mu(z)| > 1 - \varepsilon\} \le Ce^{-\alpha/\varepsilon}.$$

Tukia [23] as well as Ryazanov and Potyemkin [19] have studied the compactness property of David class. More recently, Petersen and Zakeri [18] use some property of the family of David class to study complex dynamics.

For about ten years, the connections or differences between conditions of Lehto's and of David's were not clear; it is Brakalova and Jenkins who first showed the differences between them. In [3] they proved the existence of a homeomorphic solution to the Beltrami equation under weaker conditions than that of David's. These conditions are of the following form:

$$\iint_B F\left(\frac{1}{1 - |\mu(z)|} \right) dx\, dy < \Phi_B, \tag{A_1}$$

where B is a bounded measurable set, $\Phi_B > 0$ is a constant depending on B, and $F(x) = \exp[x/(1 + \log x)]$ for $x \ge 1$; and

$$\iint_{|z|<R} \frac{1}{1 - |\mu|} \, dx\, dy = O(R^2), \quad R \to \infty. \tag{A_2}$$

In 2001, Iwaniec and Martin [12] studied the Beltrami equation in the case where $\mu(z)$ satisfies

$$\exp \frac{D(z)}{1 + \log D(z)} \in L^p \tag{IM}$$

for some positive p. They obtained fine results both in existence and uniqueness of homeomorphic solutions (see [12] for more details). The relationships among these conditions may be listed as follows.

(R_1) Condition (D) $\Longrightarrow$ (A_1) and (A_2), but not vice versa.
(R_2) Condition (IM) $\Longrightarrow$ condition (L_2), but not vice versa.
(R_3) (A_1) and (A_2) $\Longrightarrow$ condition (L_2), but not vice versa.
(R_4) Condition (D) $\not\Longleftrightarrow$ condition (L_1).

From our previous discussion, the restriction of compactness on the exceptional set E in (L_1) seems stringent and unnecessary for the existence of homeomorphic solutions to the Beltrami equation. In the meantime, condition (L_2) is the weakest and is essential in the study of homeomorphic solutions to equation (0.1).

Much inspired by [3], [7], [12], and [14], this paper shall establish the existence and uniqueness theorems of $\mu(z)$-homeomorphic solutions of the Beltrami equation by removing (L_1) and retaining (L_2) as the weakest condition. This will be done in Section 1.

As a natural generalization of quasiconformal mappings, we give the definition of $\mu(z)$-homeomorphism.

DEFINITION 2. A topological mapping f of a region Ω is *$\mu(z)$-homeomorphic* if

(I) f is ACL in Ω and
(II) $f_{\bar{z}}(z) = \mu(z) f_z(z)$,

where $\mu(z)$ is a measurable function defined a.e. in Ω and $|\mu(z)| < 1$.

By definition, David class and Brakalova–Jenkins class belong to the family of $\mu(z)$-homeomorphisms.

1. $\mu(z)$-Homeomorphic Solutions

In this section, we discuss the $\mu(z)$-homeomorphic solutions of the Beltrami equation in the complex plane $\mathbb{C}$. Let $\mu(z)$ be a measurable function in $\mathbb{C}$. Define the dilatation function $D(z)$ as

$$D(z) = \begin{cases} \frac{1+|\mu(z)|}{1-|\mu(z)|}, & z \notin E, \\ \infty, & z \in E, \end{cases} \tag{1.0}$$

where E is the exceptional set of measure zero. Define the linear integral mean dilatation function as

$$D^*(z, r) = \frac{1}{2\pi} \int_0^{2\pi} D(z + re^{i\theta})\, d\theta.$$

LEMMA 1 [16]. *Let B be a ring domain that separates the points a_1, b_1 from the points a_2, b_2. If $s(a_i, b_i) \geq \eta$ $(i = 1, 2)$, then*

$$\operatorname{mod} B \leq \frac{\pi^2}{2\eta^2}, \tag{1.1}$$

where s is the spherical metric and mod B *is the module of the ring domain B, which is actually defined by extremal length.*

For more details about module and extremal length, see [1] and [16].

LEMMA 2 [13]. *Let $f_n(z)$ with normalization $f_n(a_i) = a_i$ $(i = 1, 2, 3,\ n = 1, 2, \dots)$ be a sequence of quasiconformal mappings of the extended plane that converges uniformly toward a limit function $f(z)$. If, for every annulus A, the numbers* mod $f_n(A)$ *are bounded away from zero, then f is a homeomorphism.*

The following lemma is due to Lemmas 4 and 5 in [3].

LEMMA 3. *Let f_n denote that quasiconformal mappings of the plane onto itself with complex dilatation μ_n satisfying $|\mu_n(z)| \le |\mu(z)| \le 1$ and $\lim_{n\to\infty} \mu_n(z) = \mu(z)$ a.e. Suppose that the functions f_n converge uniformly on any compact subset of the plane to a function f. If $D(z)$ is locally in L^λ for some $\lambda > 1$, then $f(z)$ satisfies the Beltrami equation a.e. and its derivatives belong locally to L^q when $q \le \frac{2\lambda}{1+\lambda}$.*

Now we establish the following existence theorem.

THEOREM 1. *Assume that the function $\mu(z)$ is measurable in the complex plane and that $|\mu(z)| < 1$ almost everywhere. Suppose that, for some $\lambda > 1$ and any $z \in \mathbb{C}$, the following statements hold*:

(B$_1$) *$D(z)$ is locally in $L^\lambda(\mathbb{C})$;*
(B$_2$) *$\int_{r_1}^{r_2} \frac{dr}{rD^*(z,r)}$ tends to ∞ as $r_1 \to 0$ or $r_2 \to \infty$.*

Then there exists a $\mu(z)$-homeomorphism $f_\mu(z)$ of $\mathbb{C}$ solving the Beltrami equation (0.1), *and its derivatives $\frac{\partial}{\partial z} f_\mu(z)$ and $\frac{\partial}{\partial \bar{z}} f_\mu(z)$ belong locally to L^q when $q \le \frac{2\lambda}{1+\lambda}$.*

Proof. Define

$$\mu_n(z) = \begin{cases} \mu(z), & |\mu(z)| \le 1 - \frac{1}{n}, \\ 0, & |\mu(z)| > 1 - \frac{1}{n}. \end{cases} \tag{1.2}$$

From the theory of quasiconformal mappings, there exist quasiconformal mappings f_n of the plane onto itself with complex dilatations $\mu_n(z)$ ($n = 1, 2, \ldots$). We require that the $f_n(z)$ be normalized by the conditions $f_n(a_i) = a_i$, $i = 1, 2, 3$ ($a_1 = 0$, $a_2 = 1$, $a_3 = \infty$).

We first show that the family $\{f_n\}$ is equicontinuous with respect to the spherical metric on any compact subset of the plane. Choose an arbitrary finite z_0 and then a number $\varrho > 0$ so small that at least two of the points a_i lie outside of the disk $\{z : |z - z_0| \le \varrho\}$. Without loss of generality, we may assume that the two points are a_1 and a_2. Fixing $\varrho > 0$, we consider the annulus $A_\delta = \{z : \delta < |z - z_0| < \varrho\}$. If z_1 is an arbitrary point such that $|z_1 - z_0| < \delta$, then $f_n(A_\delta)$ separates the points $f_n(z_0)$ and $f_n(z_1)$ from the points a_1 and ∞. According to Lemma 1, we have

$$\operatorname{mod} f_n(A_\delta) \le \frac{\pi^2}{2\eta^2}, \tag{1.3}$$

where $\eta = \min\{s(a_1, a_2), s(f_n(z_0), f_n(z_1))\}$.

For every $n \in \mathbb{N}$, we have that f_n is n-quasiconformal. In view of a result in [20], it follows that

$$\operatorname{mod} f_n(A_\delta) \ge \int_\delta^\varrho \frac{dr}{rD_n^*(z, r)}. \tag{1.4}$$

From the definition of D^* and (1.2), it is obvious that

$$D_n^*(z, r) \le D^*(z, r). \tag{1.5}$$

Combining (1.3), (1.4), and (1.5), we obtain

$$\int_{\delta}^{\varrho} \frac{dr}{rD^*(z,r)} \le \operatorname{mod} f_n(A_\delta) \le \frac{\pi^2}{2\eta^2}. \tag{1.6}$$

Rearrangement then yields

$$\eta \le \pi \left(\int_{\delta}^{\varrho} \frac{dr}{rD^*(z,r)} \right)^{-1/2}. \tag{1.7}$$

By assumption (B_2), the right side of (1.7) tends to 0 as $\delta \to 0$ independently of n. This implies the equicontinuity of the family $\{f_n\}$ at the point z_0.

An obvious modification of this reasoning yields the result for $z = \infty$. In this case, use is made of the hypothesis that the integral tends to infinity when $\rho \to \infty$. By Arzela–Ascoli's theorem, there exists a subsequence f_{n_j} that converges uniformly to a function $f_\mu(z)$ in the spherical metric in the whole extended plane.

We next show that f is a homeomorphism of the whole plane. Let $A = \{z : r_1 < |z - z_0| < r_2\}$. Using again the left-hand inequality of (1.6), we have

$$\operatorname{mod} f_n(A) \ge \int_{r_1}^{r_2} \frac{dr}{rD^*(z,r)}. \tag{1.8}$$

We obtain that mod $f_n(A)$ are bounded away from zero. Thus, by Lemma 2, f_μ is a homeomorphism of the whole plane.

Note that the limit function $f_\mu(z)$ satisfies the conditions of Lemma 3. Hence $f_\mu(z)$ solves the Beltrami equation, and its derivatives belong to $L^q_{\mathrm{loc}}(\mathbb{C})$ when $q \le \frac{2\lambda}{1+\lambda}$. This concludes our proof of the theorem. $\square$

THEOREM 2. *Let $\mu(z)$ and $f_\mu(z)$ be as in Theorem 1. If $h(z)$ is another $\mu(z)$-homeomorphic solution to the Beltrami equation and if its derivatives belong locally to L^2, then*

$$h(z) = af_\mu(z) + b,$$

where a and b are constants and $a \ne 0$.

Proof. Let $\zeta = h(z)$, $w = f_\mu(z)$, and g be the inverse of f_μ. Then, by local integrability of $D(z)$, we can prove that g is ACL and that g_w is locally in L^2 as in [10, Prop. 3]. According to [16, Chap. III, Thm. 6.1], g is locally absolutely continuous in the w-plane. That is,

$$|g(B)| = \iint_B J_g(w)\, d\sigma_w,$$

where B is a Borel set in the w-plane. This implies that g maps null sets onto null sets. Since h and g have partial derivatives locally in L^2, it follows that $h \circ g$ is absolutely continuous in the sense of Tonelli [16, p. 143] and that, a.e.,

$$(h \circ g)_{\bar{w}} = h_z(g(w))g_{\bar{w}}(w) + h_{\bar{z}}(g(w))\bar{g}_{\bar{w}}. \tag{1.9}$$

Let E be the exceptional set in the z-plane that is of zero measure, and set:

$$F = \{z \mid f_\mu \text{ is not differentiable or does not satisfy (0.1) at } z\};$$

$$G = \{w \mid g \text{ is not differentiable at } w\};$$

$$H = \{z \mid h \text{ is not differentiable or does not satisfy (0.1) at } z\}.$$

For convenience, we use χ to denote $E \cup F \cup H$. When $w \notin f_\mu(\chi) \cup G$,

$$\mu_{h\circ g}(w) = \frac{\mu_h - \mu_f}{1 - \overline{\mu_h}\mu_f} \cdot \frac{\partial f_\mu/\partial z}{\partial \overline{f_\mu}/\partial \bar{z}} = 0.$$

When $w \in f_\mu(\chi)$, there are two cases as follows.

Case 1: $f_\mu(\chi)$ is of zero measure. Then $\mu_{h\circ g}(w) = 0$ a.e. This, together with the ACL property of $h \circ g$, implies that $h \circ g$ is conformal.

Case 2: $f_\mu(\chi)$ is of positive measure. Then, by [16, Chap. III, Lemma 2.1],

$$\begin{aligned} 0 = \iint_\chi \frac{1+|\mu(z)|^2}{1-|\mu(z)|^2} &= \iint_{f_\mu(\chi)} \frac{1+|\mu(g(w))|^2}{1-|\mu(g(w))|^2} J_g(w)\, d\sigma_w \\ &= \iint_{f_\mu(\chi)} (|g_{\bar{w}}(w)|^2 + |g_w(w)|^2)\, d\sigma_w, \end{aligned}$$

where $J_g(w)$ is the Jacobian of g.

Therefore, $g_w = 0$ and $g_{\bar{w}} = 0$ a.e. on $f_\mu(\chi)$. From (1.9) we then have that $\mu_{h\circ g}(w) = 0$ a.e. on $f_\mu(\chi)$; hence, it holds almost everywhere on the w-plane. Thus, $h \circ g$ is 1-quasiconformal and actually is conformal on the w-plane. As a result,

$$h \circ g(w) = aw + b,$$

where a and b are constants and $a \neq 0$. The proof is complete. □

2. Subfamily of $\mu(z)$-Homeomorphisms, $\mathcal{M}(\lambda, *)$

In this section, we discuss properties of the family where $\mu(z)$ satisfies conditions (B_1) and (B_2). Set

$$\mathcal{M}(\lambda, *) = \{f_\mu \mid f_\mu \text{ are } \mu(z)\text{-homeomorphisms and } \mu \text{ satisfies } (B_1) \text{ and } (B_2)\}.$$

PROPOSITION 1. *The Brakalova–Jenkins class is a subfamily of $\mathcal{M}(\lambda, *)$. In other words, conditions (A_1) and (A_2) imply (B_1) and (B_2), but not vice versa.*

Proof.

Step 1: $(A_1) \Longrightarrow (B_1)$. Let $q(z) = \frac{1}{1-|\mu(z)|}$. For $\lambda > 1$, there exists a constant q_0 such that, when $q(z) > q_0$,

$$q^\lambda(z) < \exp \frac{q(z)}{1 + \log q(z)}. \tag{2.1}$$

Therefore, by (2.1) and condition (A_1),

$$\begin{aligned} \iint_B q^\lambda(z)\, dx\, dy &= \iint_{B(q(z)>q_0)} q^\lambda(z)\, dx\, dy + \iint_{B(q(z)\le q_0)} q^\lambda(z)\, dx\, dy \\ &< \iint_{B(q(z)>q_0)} \exp \frac{q(z)}{1+\log q(z)}\, dx\, dy + q_0^\lambda |B| \\ &< \Phi(B) + q_0^\lambda |B| < +\infty. \end{aligned}$$

Since $D^\lambda(z) \le 2^\lambda q^\lambda(z)$, it follows that $D^\lambda(z)$ is locally integrable.

Step 2: (A_1) *and* $(A_2) \Longrightarrow (B_2)$. This can be found in the process of the proofs in Proposition 1 and Proposition 3 in [3].

Step 3: An example. Let $\mu(z)$ satisfy the following condition:

$$\frac{1+|\mu(z)|}{1-|\mu(z)|} = \begin{cases} \log\frac{1}{|z|}, & e^{-(n+1/2)} \le |z| \le e^{-n}, \\ |z|^{-p}, & e^{-(n+1)} < |z| \le e^{-(n+1/2)}, \end{cases} \tag{2.2}$$

where $1 < p < 2/\lambda$.

It is not difficult to show that $\mu(z)$ satisfies (B_1) and (B_2). Now, we prove that $\mu(z)$ in (2.2) does not satisfy (A_1). In fact,

$$\begin{aligned}\iint_\Delta \exp\frac{q(z)}{1+\log q(z)}\,dx\,dy &> \sum_{n=1}^{\infty}\int_0^{2\pi}\int_{e^{-(n+1)}}^{e^{-(n+1/2)}} r\cdot\exp\frac{q(re^{i\theta})}{1+\log q(re^{i\theta})}\,dr\,d\theta \\ &> \sum_{n=1}^{\infty}\int_{e^{-(n+1)}}^{e^{-(n+1/2)}} r\cdot\exp\frac{\frac{1}{2}r^{-p}}{1+\log\frac{1}{2}r^{-p}}\,dr \\ &> \sum_{n=n_0}^{\infty}\int_{e^{-(n+1)}}^{e^{-(n+1/2)}} r\cdot r^{-2}\,dr \\ &= \sum_{n=n_0}^{\infty}\log\frac{e^{n+1}}{e^{n+1/2}} = \infty,\end{aligned}$$

where n_0 is large enough. This finishes the proof. □

REMARK. The proof of (R_2) is analogous to the process just described. Therefore, the results given here are relatively general.

PROPOSITION 2. *The two conditions in* $\mathcal{M}(\lambda, *)$ *are independent of each other.*

Proof. First, we show if $D(z)$ (or $\mu(z)$) satisfies (B_1) then it does not have to satisfy (B_2). Let $D(z) = |z|^{-1}$. Obviously, $D(z)$ satisfies (B_1) as long as $\lambda < 2$, but it does not satisfy (B_2).

Second, if $D(z)$ satisfies (B_2) then it does not have to satisfy (B_1). This will be shown by the following example. Set

$$D(z) = \begin{cases} -\log|z|, & e^{-(n+1/2)} < |z| \le e^{-n}, \\ |z|^{-3}, & e^{-(n+1)} < |z| \le e^{-(n+1/2)}, \\ 1, & \text{elsewhere}, \end{cases}$$

where $n \in \mathbb{N}$.

The origin $\mathcal{O}$ is the exceptional set. Now,

$$\begin{aligned}\int_0^1 \frac{dr}{rD^*(r)} &> \sum_{n=1}^{\infty}\int_{e^{-(n+1/2)}}^{e^{-n}} \frac{-dr}{r\log r} \\ &= \sum_{n=1}^{\infty}\log\left(1+\frac{1}{2n}\right) = +\infty.\end{aligned}$$

Let $\lambda > 1$. We will show that $D^{\lambda}(z)$ is not integrable in the unit disk Δ. In fact,

$$\begin{aligned}\iint_{\Delta} D^{\lambda}(z)\,dx\,dy &> 2\pi \sum_{n=1}^{\infty} \int_{e^{-(n+1)}}^{e^{-(n+1/2)}} r^{-3\lambda+1}\,dr \\ &> 2\pi \sum_{n=1}^{\infty} \int_{e^{-(n+1)}}^{e^{-(n+1/2)}} r^{-2}\,dr \\ &= 2\pi \sum_{n=1}^{\infty} (e^{n+1} - e^{n+1/2}) = +\infty.\end{aligned}$$

Hence, $D(z)$ satisfies (B_2) but does not satisfy (B_1). $\square$

PROPOSITION 3. *The family $\mathcal{M}(\lambda, *)$ is convex. That is, for any $k_1 \geq 0$ and $k_2 \geq 0$ $(k_1 + k_2 = 1)$, if $\mu_1(z)$ and $\mu_2(z)$ satisfy* (B_1) *and* (B_2) *then*

$$\mu(z) = k_1\mu_1(z) + k_2\mu_2(z)$$

also satisfies (B_1) *and* (B_2).

Proof. First, for any $z \in \mathbb{C}$ we have

$$\begin{aligned}\int_{\varepsilon}^{R} \frac{dr}{rD^{*}(z,r)} &= \int_{\varepsilon}^{R} \frac{2\pi\,dr}{r\int_0^{2\pi} \frac{1+|\mu(z+re^{i\theta})|}{1-|\mu(z+re^{i\theta})|}\,d\theta} \\ &\geq \int_{\varepsilon}^{R} \frac{\pi\,dr}{r\int_0^{2\pi} \frac{1}{1-|k_1\mu_1+k_2\mu_2|}\,d\theta}.\end{aligned}$$

Without loss of generality, we assume that $k_1 \geq \frac{1}{2}$. Then

$$\begin{aligned}\int_{\varepsilon}^{R} \frac{\pi\,dr}{r\int_0^{2\pi} \frac{1}{1-|k_1\mu_1+k_2\mu_2|}\,d\theta} &\geq \int_{\varepsilon}^{R} \frac{\pi\,dr}{r\int_0^{2\pi} \frac{1}{k_1(1-|\mu_1|)}\,d\theta} \\ &\geq k_1\int_{\varepsilon}^{R} \frac{\pi\,dr}{r\int_0^{2\pi} \frac{1}{1-|\mu_1|}\,d\theta} \to \infty\end{aligned}$$

as $\varepsilon \to 0$ or as $R \to \infty$.

Second, for any $\lambda > 1$ and $0 \leq x < 1$, the function

$$\eta(x) = \left(\frac{1+x}{1-x}\right)^{\lambda}$$

is convex. Therefore,

$$\begin{aligned}D^{\lambda}(z) &\leq \left(\frac{1+(k_1|\mu_1(z)|+k_2|\mu_2(z)|)}{1-(k_1|\mu_1(z)|+k_2|\mu_2(z)|)}\right)^{\lambda} \\ &\leq k_1D_1^{\lambda}(z) + k_2D_2^{\lambda}(z).\end{aligned}$$

Hence $D^{\lambda}(z)$ is also locally integrable.

The foregoing discussion shows that $\mu(z)$ satisfies (B_1) and (B_2), which completes the proof. $\square$

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