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Homoclinic Orbits for Schrödinger Systems

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1. Introduction

We consider the following Schrödinger system:

$$\begin{cases} \partial_t u - \Delta_x u + V(x)u = H_v(t,x,u,v) \\ -\partial_t v - \Delta_x v + V(x)v = H_u(t,x,u,v) \end{cases} \quad \text{for } (t,x) \in \mathbf{R} \times \mathbf{R}^N, \tag{S}$$

where $V : \mathbf{R}^N \to \mathbf{R}$ and $H : \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}^{2M} \to \mathbf{R}$ are periodic in t and x; $(u,v) \equiv (0,0) \in \mathbf{R}^{2M}$ is a stationary solution. Our purpose is to find a nonstationary solution $z = (u,v) : \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^{2M}$ of (S) satisfying $z(t,x) \to 0$ as $|t| + |x| \to \infty$. In this case, it is called the homoclinic orbit that is homoclinic to the stationary solution.

During the last ten years, the existence of homoclinic solutions has been studied by variational methods (see e.g. [AB1; AB2; CR; ScZ; SZ; WZ] and the references cited therein). Since there is no compactness of imbedding, the problem becomes very complicated. The difficulty also occurs when we consider (S). Before stating the main results, let us recall some well-known results related to (S). Brézis and Nirenberg [BrN] considered the system

$$\begin{cases} \partial_t u - \Delta_x u = -v^5 + f \\ -\partial_t v - \Delta_x v = u^3 + g \end{cases} \quad \text{for } (t,x) \in (0,T) \times \Omega$$

satisfying $u = v = 0$ on $(0,T) \times \partial\Omega$ and $u(0,x) = v(T,x) = 0$ in Ω. Here Ω is a bounded domain of $\mathbf{R}^N$ and $f, g \in L^\infty(\Omega)$. Using Schauder's fixed point theorem, they obtained a solution (u,v) with $u \in L^4((0,T) \times \Omega)$ and $v \in L^6((0,T) \times \Omega)$.

In [CFM], the authors studied the following problem:

$$\begin{cases} \partial_t u - \Delta_x u = |v|^{q-2} v \\ -\partial_t v - \Delta_x v = |u|^{p-2} u \end{cases} \quad \text{for } (t,x) \in (-T,T) \times \Omega,$$

where Ω is a smooth bounded domain in $\mathbf{R}^N$ and $N/(N+2) < 1/p + 1/q < 1$. By the usual mountain pass theorem, they obtained at least one positive solution satisfying $u(t,\cdot)|_{\partial\Omega} = v(t,\cdot)|_{\partial\Omega} = 0$ for all $t \in (-T,T)$ and for $u(-T,\cdot) = u(T,\cdot)$ and $v(-T,\cdot) = v(T,\cdot)$.

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Very little is known for (S). In a more recent paper [BD2], system (S) was explored by a new linking theorem due to [KS] and [BD1]. The authors obtained one nontrivial solution and, moreover, infinitely many solutions if the potential is even in (u, v). In [BD1; BD2] (see also [KS]), the following Ambrosetti–Rabinowitz global superquadratic condition plays an important role:

$$0 < \gamma H(t, x, z) \le H_z(t, x, z)z \quad \forall (t, x) \in \mathbf{R} \times \mathbf{R}^N,\ \forall z \ne 0, \tag{1.1}$$

where $\gamma > 2$ is a constant. In the present paper, we shall study the weak superlinear case without (1.1) and the asymptotically linear case. Without (1.1), the problem becomes quite different and complex. Because of the strong indefinite nature of the energy functional, the main obstacle is the proof of the boundedness of the (PS) (i.e. Palais–Smale) sequence. It is also not easy to derive a (PS) sequence for the asymptotically linear case. It should be mentioned that the methods used in [BD1; BD2; J; KS; Z] cannot be applied to our cases. By virtue of the new theory established in [ScZ], we can easily obtain a bounded (PS) sequence directly from the weak linking theorem for the modified functional and thereby have a sequence of critical points, which provides a nontrivial solution of (S).

Throughout this paper, we always assume that V and H satisfy the following conditions.

(V) $V \in \mathcal{C}(\mathbf{R}^N, \mathbf{R})$, and V is T_j-periodic in x_j for $j = 1, \dots, N$; furthermore, $0 \notin \sigma(-\Delta_x + V)$, where σ denotes the purely continuous spectrum of $-\Delta_x + V$.

(H) $H \in \mathcal{C}^1(\mathbf{R} \times \mathbf{R}^N \times \mathbf{R}^{2M}, \mathbf{R})$ is T_0-periodic in t and T_j-periodic in x_j, $j = 1, \dots, N$; also, $H(t, x, z) \ge 0$ for all (t, x, z), where $z = (u, v) \in \mathbf{R}^{2M}$.

By [RS], $\sigma(-\Delta_x + V)$ is bounded below and consists of closed disjoint intervals. We permit $-\Delta_x + V$ to have essential spectrum below 0.

1.1. The Superlinear Case

From now on, the letter c will be indiscriminately used to denote various constants whose exact values are irrelevant. We need the following assumptions:

(S_1) $H_z(t, x, z) = o(|z|)$ as $z \to 0$ uniformly in t and x;

(S_2) $|H_z(t, x, z)| \le c|z|^\mu$ for all (t, x) and $|z| \ge R_0$, where $R_0 > 0$ and $\mu > 0$ are constants, $1 < \mu < (N + 4)/N$;

(S_3) $\frac{1}{2}H_z(t, x, z)z - H(t, x, z) \ge c|z|^\beta$ for all (t, x, z), where we also have $\beta > \max\{2, (2N + 4)(\mu^2 - 1)/(N\mu + 4\mu - N)\}$.

Our main result is as follows.

THEOREM 1.1. *Assume that* (S_1)–(S_3) *hold. Then* (S) *has at least one nontrivial solution.*

As an immediate consequence, we have the following corollary.

COROLLARY 1.1. *Assume that H is of the form*

$$H(t, x, z) = A_0|z|^{\mu+1} + G(t, x, z), \quad A_0 > 0,\ 1 < \mu < (N + 4)/N,$$

where $|G_z(t,x,z)| \le ((\mu-1)A_0/4)|z|^{\mu}$ *for all* (t,x,z). *Then* (S) *has at least one nontrivial solution*.

Next, we consider the second case. The potential satisfies local conditions at zero and at infinity.

(F_1) There exist $\nu > p > 2$, $\nu < (2N+4)/N$, and $c_1, c_2, c_3 > 0$ such that

$$c_1|z|^{\nu} \le H_z(t,x,z)z \le |H_z(t,x,z)||z| \le c_2|z|^{\nu} + c_3|z|^{p}$$

for all $(t,x,z) \in \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}^{2M}$.

(F_2) $H_z(t,x,z)z - 2H(t,x,z) > 0$ for all $(t,x,z) \ne (0,0,0)$.

(F_3) There exists $\gamma_0 > 2$ such that

$$\liminf_{|z|\to\infty} \frac{H_z(t,x,z)z}{H(t,x,z)} \ge \gamma_0$$

uniformly for $(t,x) \in \mathbf{R} \times \mathbf{R}^N$.

(F_4) There exists an $\alpha > p$ such that

$$\liminf_{z\to 0} \frac{H_z(t,x,z)z - 2H(t,x,z)}{|z|^{\alpha}} \ge c > 0$$

uniformly for $(t,x) \in \mathbf{R} \times \mathbf{R}^N$.

THEOREM 1.2. *Assume that* (F_1)–(F_4) *hold*. *Then* (S) *has at least one nontrivial solution*.

REMARK 1.1. Compared with [BD2] (see also [BD1; KS; SZ]), our assumptions are quite weak. In [BD2], besides (1.1) and others, the following condition was imposed:

$$|H_z(t,x,z)|^{\alpha'} \le cH_z(t,x,z)z \quad \text{for all } (t,x) \text{ and } |z| \ge 1,$$

where $\alpha' = \alpha/(\alpha-1)$ and $\alpha \in (2, (2N+4)/N)$.

1.2. The Asymptotically Linear Case

We make the following assumptions.

(T_1) $H(t,x,z) = \frac{1}{2}\beta_0|z|^2 + K(t,x,z)$, where $K_z(t,x,z) = o(|z|)$ as $|z| \to \infty$ uniformly for all (t,x); moreover, $\beta_0 > \mu_1$, where μ_1 is the smallest positive point in the spectrum of $-\Delta_x + V$.

(T_2) There exist $m \in (2, (2N+4)/N)$ and $R_0 > 0$ such that

$$c|z|^m \le H_z(t,x,z)z \le |H_z(t,x,z)||z| \le c|z|^m$$

for all $(t,x) \in \mathbf{R} \times \mathbf{R}^N$ and $|z| \le R_0$.

(T_3) $K_z(t,x,z)z - 2K(t,x,z) > 0$ for all $(t,x,z) \ne (0,0,0)$.

(T_4) There exists a $\mu > 2$ such that

$$\liminf_{z\to 0} \frac{H_z(t,x,z)z}{H(t,x,z)} = \mu$$

uniformly for $(t,x) \in \mathbf{R} \times \mathbf{R}^N$.

(T_5) There exists an $\alpha \in (0, 2)$ such that

$$\liminf_{|z|\to\infty} \frac{K_z(t, x, z)z - 2K(t, x, z)}{|z|^\alpha} \geq c > 0$$

uniformly for $(t, x) \in \mathbf{R} \times \mathbf{R}^N$.

REMARK 1.2. Conditions (T_1)–(T_5) imply that (S) is asymptotically linear at infinity and superlinear at the origin.

We shall prove the following result.

THEOREM 1.3. *Assume* (T_1)–(T_5). *Then* (S) *has at least one nontrivial solution.*

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2. The Superlinear Case

Let

$$\mathcal{J} := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \qquad \mathcal{J}_0 := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

and $\mathcal{A} := \mathcal{J}_0(-\Delta_x + V)$. Then (S) can be rewritten as $J\partial_t z = -Az + H_z(t, x, z)$ for $z = (u, v)$. In this way, (S) can be regarded as an unbounded infinite-dimensional Hamiltonian system in $L^2(\mathbf{R}^N, \mathbf{R}^{2M})$.

Let $\tilde{H}_0 := L^2(\mathbf{R}^N, \mathbf{R}^{2M})$; then $D(A) = \mathcal{D}(JA) = W^{2,2}(\mathbf{R}^N, \mathbf{R}^{2M})$ and $\tilde{H} := L^2(\mathbf{R}, \tilde{H}_0) \cong L^2(\mathbf{R} \times \mathbf{R}^N, \mathbf{R}^{2M})$ (cf. [BD2]). By (V), there is an associated orthogonal decomposition $\tilde{H} = \mathcal{H}^- \oplus \mathcal{H}^+$ with $z = z^- + z^+$, where $z^\pm \in \mathcal{H}^\pm$. Let $E := \mathcal{D}(|L|^{1/2})$ be equipped with the inner product

$$\langle z_1, z_2 \rangle = \langle |L|^{1/2} z_1, |L|^{1/2} z_2 \rangle_{L^2}$$

and norm $\|z\| = \langle z, z \rangle^{1/2}$, where $L = J\partial_t + A$. We then have the decomposition $E = E^+ \oplus E^-$, where $E^\pm = E \cap \mathcal{H}^\pm$ are orthogonal with respect to both $\langle \cdot, \cdot \rangle_{L^2}$ and $\langle \cdot, \cdot \rangle$. By [BD2], E is continuously embedded in $L^r(\mathbf{R} \times \mathbf{R}^N, \mathbf{R}^{2M})$ for any $r \geq 2$ if $N = 1$ and for $r \in [2, 2(N + 2)/N]$ if $N \geq 2$. In particular, E is compactly embedded in $L^r_{\text{loc}}(\mathbf{R} \times \mathbf{R}^N, \mathbf{R}^{2M})$ for any $r \geq 2$ if $N = 1$ and for $r \in [2, 2(N + 2)/N)$ if $N \geq 2$.

Let

$$\Phi(z) := \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2) - \int_{\mathbf{R} \times \mathbf{R}^N} H(t, x, z).$$

Then, under the assumptions of Theorems 1.1–1.3, $\Phi \in \mathcal{C}^1(E, \mathbf{R})$ and the critical points of Φ are weak solutions of (S).

Our results stated in Section 1 shall be proved with the help of the following critical point theorem (cf. [ScZ]).

Let E be a Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$, and let $N \subset E$ be a separable subspace, $E = N \oplus N^\perp$. Since N is separable, we can define a new norm $|v|_w$ satisfying $|v|_w \le \|v\|$ for all $v \in N$ and such that the topology induced by this norm is equivalent to the weak topology of N on a bounded subset of N (cf. [DS]). For $z = v + w \in E = N \oplus N^\perp$ with $v \in N$ and $w \in N^\perp$, we define $|z|_w^2 = |v|_w^2 + \|w\|^2$. Then $|z|_w \le \|z\|$ for all $u \in E$.

In particular, if $z_n = v_n + w_n$ is bounded and $z_n \to z$ is in the norm $|\cdot|_w$, then: $v_n \rightharpoonup v$ weakly in N; $w_n \to w$ strongly in $N^\perp$; and $z_n \rightharpoonup v + w$ weakly in E (cf. [DS]).

Let $Q \subset N$ be a bounded open convex subset and let $p_0 \in Q$ be a fixed point. Let F be a $|\cdot|_w$-continuous map from E onto N that satisfies the following conditions:

(1) $F|_Q = \mathrm{id}$ and F maps bounded sets to bounded sets;

(2) there exists a fixed finite-dimensional subspace E_0 of E such that

$$F(u-v) - (F(u) - F(v)) \subset E_0 \quad \forall v, u \in E;$$

(3) F maps finite-dimensional subspaces of E to finite-dimensional subspaces of E.

Set

$$A := \partial Q, \qquad B := F^{-1}(p_0).$$

REMARK 2.1. There are many examples.

(i) Let $N = E^-$ and $N^\perp = E^+$ for $E = E^- \oplus E^+$, and let $Q := \{u \in E^- : \|u\| < R\}$ with $p_0 = 0 \in Q$. For any $u = u^- \oplus u^+ \in E$, define $F : E \mapsto N$ by $Fu := u^-$. Then $A := \partial Q$ and $B := F^{-1}(p_0) = E^+$ satisfy conditions (1)–(3).

(ii) Let $E = E^- \oplus E^+$ and $z_0 \in E^+$ with $\|z_0\| = 1$. For any $u \in E$, we write $u = u^- \oplus s z_0 \oplus w^+$ with $u^- \in E^-$, $s \in \mathbf{R}$, and $w^+ \in (E^- \oplus \mathbf{R} z_0)^\perp := E_1^+$. Let $N := E^- \oplus \mathbf{R} z_0$. For $R > 0$, let $Q := \{u := u^- + s z_0 : s \in \mathbf{R},\ u^- \in E^-,\ \|u\| < R\}$ with $p_0 = s_0 z_0 \in Q$. Let $F : E \mapsto N$ be defined by $Fu := u^- + s\|z_0 + w^+\| z_0$. Then F, Q, p_0 satisfy the conditions (1)–(3) with

$$B = F^{-1}(s_0 z_0) = \{u := s z_0 + s w^+ : s \ge 0,\ w^+ \in E_1^+,\ \|u\| = s_0\}.$$

For $\Phi \in \mathcal{C}^1(E, \mathbf{R})$, we define

$$\begin{aligned}\Gamma := \Big\{ h : [0,1] \times \bar{Q} \mapsto E \text{ and } h \text{ is } |\cdot|_w\text{-continuous; for any } (s_0, u_0) \in [0,1] \times \bar{Q},\\ \text{there is a } |\cdot|_w\text{-neighborhood } U_{(s_0,u_0)} \text{ such that}\\ \{u - h(t,u) : (t,u) \in U_{(s_0,u_0)} \cap ([0,1] \times \bar{Q})\} \subset E_{\mathrm{fin}},\\ h(0,u) = u,\ \Phi(h(s,u)) \le \Phi(u)\ \forall u \in \bar{Q} \Big\}.\end{aligned}$$

Here and henceforth, we use E_{fin} to denote various finite-dimensional subspaces of E whose exact dimensions are irrelevant. Note that $\Gamma \ne \emptyset$ since $\mathrm{id} \in \Gamma$.

The variant weak linking theorem may be stated as follows.

THEOREM 2.1 [ScZ]. *Let*

$$\Phi_\lambda(u) := I(u) - \lambda J(u) \quad \forall (\lambda \in [1,2],\ u \in E)$$

be a family of $\mathcal{C}^1$-functionals and assume that

(a) $J(u) \ge 0$ *for all* $u \in E$, *where* $\Phi_1 := \Phi$;

(b) $I(u) \to \infty$ *or* $J(u) \to \infty$ *as* $\|u\| \to \infty$;

(c) Φ_λ *is* $|\cdot|_w$*-upper semicontinuous,* Φ'_λ *is weakly sequentially continuous, and* Φ_λ *maps bounded sets to bounded sets*;

(d) $\sup_A \Phi_\lambda < \inf_B \Phi_\lambda$ *for all* $\lambda \in [1,2]$.

Then, for almost all $\lambda \in [1,2]$, *there exists a sequence* (u_n) *such that*

$$\sup_n \|u_n\| < \infty, \qquad \Phi'_\lambda(u_n) \to 0, \quad \Phi_\lambda(u_n) \to C_\lambda,$$

where

$$C_\lambda := \inf_{h\in\Gamma} \sup_{u\in\bar{Q}} \Phi_\lambda(h(1,u)) \in \Big[\inf_B \Phi_\lambda, \sup_{\bar{Q}} \Phi\Big].$$

In order to study (S), we consider

$$\Phi_\lambda(z) := \frac{1}{2}\|z^+\|^2 - \lambda\left(\frac{1}{2}\|z^-\|^2 + \int_{\mathbf{R}\times\mathbf{R}^N} H(t,x,z)\right), \quad u \in E.$$

By (S_1) and (S_2), for any $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that $H(t,x,z) \le \varepsilon|z|^2 + C_\varepsilon |z|^{\mu+1}$. Therefore,

$$\Phi_\lambda(z^+) \ge \tfrac{1}{2}\|z^+\|^2 - \lambda\varepsilon\|z^+\|_2^2 - C_\varepsilon\|z^+\|_{\mu+1}^{\mu+1} \ge b > 0 \quad \forall\lambda \in [1,2]$$

for $b > 0$ and $z \in B := \{z : z \in E^+,\ \|z\| = r_0\}$. The constants b and r_0 are independent of λ. Here and in the sequel, $\|\cdot\|_r$ denotes the usual norm of $L^r(\mathbf{R}^{1+N})$. On the other hand, by (S_1) and (S_3), $H(t,x,z) \ge c|z|^\beta$ for all (t,x,z). Then, for fixed $z_0 \in E^+$ with $\|z_0\| = 1$ and $z = z^- + sz_0$, we have

$$\Phi_\lambda(z) \le \tfrac{1}{2}s^2 - \tfrac{1}{2}\|z^-\|^2 - c\|z\|_\beta^\beta \le 0$$

for $z \in A := \partial\{z = z^- + sz_0 : \|z\| \le R,\ R > 0,\ s \in \mathbf{R}\}$, given R large enough. Moreover, it is easy to check that Φ_λ is $|\cdot|_w$-upper semicontinuous and that Φ'_λ is weakly sequentially continuous. Combining the argument here with Remark 2.1 and Theorem 2.1, we have the following lemma.

LEMMA 2.1. *For almost all* $\lambda \in [1,2]$, *there exist* $\{z_n\} \subset E$ *such that, as* $n \to \infty$,

$$\sup_n \|z_n\| < \infty, \qquad \Phi'_\lambda(z_n) \to 0, \quad \Phi_\lambda(z_n) \to C_\lambda \in [b,d],$$

where $d := \sup_{\bar{Q}} \Phi$ *and* $\bar{Q} := \{z = z^- + sz_0 : s \in \mathbf{R},\ z^- \in E^-,\ \|z\| \le R_0\}$.

LEMMA 2.2. *For almost all* $\lambda \in [1,2]$, *there exists a* w_λ *such that* $\Phi'_\lambda(w_\lambda) = 0$ *and* $\Phi_\lambda(w_\lambda) \le d$.

Proof. For $\{z_n\}$ in Lemma 2.1, write $z_n := z_n^+ + z_n^-$ with $z_n^\pm \in E^\pm$. We claim that there exist $\alpha > 0$ and a sequence $\{y_n\} \in \mathbf{R}^{1+N}$ such that $\lim_{n\to\infty} \int_{B(y_n,1)} |z_n^+|^2 \ge \alpha > 0$, where $B(y,r)$ denotes the ball centered at y with radius r. In fact, if this is not true then, by a variation of Lions's concentration compactness lemma [L], we have that $z_n^+ \to 0$ in $L^t(\mathbf{R}^{1+N})$ for $2 < t < (2N+4)/N$. By (S_1) and (S_2), for any $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that

$$\left|\int_{\mathbf{R}^{1+N}} H_z(t,x,z_n)z_n^+\right| \le \varepsilon \int_{\mathbf{R}^{1+N}} |z_n||z_n^+| + C_\varepsilon \int_{\mathbf{R}^{1+N}} |z_n|^\mu |z_n^+|,$$

which implies that the left-hand side converges to 0. Consequently,

$$2\Phi_\lambda(z_n) \le \|z_n^+\|^2 = \Phi'_\lambda(z_n)z_n^+ + \lambda \int_{\mathbf{R}^{1+N}} H_z(t,x,z_n)z_n^+ \to 0,$$

a contradiction. Therefore, our claim is true, and we may assume that there exist $r > 0$ (independent of n) and $y^* := (\bar{y}_0, \bar{y}_1, \dots, \bar{y}_N) \in T_0\mathbf{Z} \times \cdots \times T_N\mathbf{Z}$ such that $\int_{B(0,r)} |\bar{z}_n^+|^2 \ge \alpha/2$, where $\bar{z}_n := \bar{z}_n^+ + \bar{z}_n^- = z_n(t + \bar{y}_0, x_1 + \bar{y}_1, \dots, x_N + \bar{y}_N)$. By periodicity, $\{\bar{z}_n\}$ is still bounded and we have $\lim_{n\to\infty} \Phi_\lambda(\bar{z}_n) \in [b, d]$ and $\lim_{n\to\infty} \Phi'_\lambda(\bar{z}_n) = 0$. Without loss of generality, we may suppose that $\bar{z}_n^+ \to w_\lambda^+$ and $\bar{z}_n^- \to w_\lambda^-$. The compactness of the embedding of E^+ into $L^t_{\mathrm{loc}}(\mathbf{R}^{1+N})$ for $2 \le t < 2(N+2)/N$ implies that $w_\lambda^+ \ne 0$, and consequently $w_\lambda := w_\lambda^+ + w_\lambda^- \ne 0$. Evidently, $\Phi'_\lambda(w_\lambda) = 0$. Finally, by ($S_3$) and Fatous's lemma, we have

$$\begin{aligned}
\Phi_\lambda(w_\lambda) &= \Phi_\lambda(w_\lambda) - \tfrac{1}{2}\langle \Phi'_\lambda(w_\lambda), w_\lambda\rangle \\
&= \lambda \int_{\mathbf{R}^{1+N}} \big(\tfrac{1}{2} H_z(t,x,w_\lambda)\cdot w_\lambda - H(t,x,w_\lambda)\big) \\
&= \lambda \int_{\mathbf{R}^{1+N}} \lim_{n\to\infty} \big(\tfrac{1}{2} H_z(t,x,\bar{z}_n)\cdot \bar{z}_n - H(t,x,\bar{z}_n)\big) \\
&\le \lim_{n\to\infty} \big(\Phi_\lambda(\bar{z}_n) - \tfrac{1}{2}\langle \Phi'_\lambda(\bar{z}_n), \bar{z}_n\rangle\big) \\
&= \lim_{n\to\infty} \Phi_\lambda(\bar{z}_n) \\
&\le d.
\end{aligned}$$

□

LEMMA 2.3. *There exist $\lambda_n \to 1$ and $w_n \ne 0$ such that $\Phi'_{\lambda_n}(w_n) = 0$, $\Phi_{\lambda_n}(w_n) \le d$, and $\{w_n\}$ is bounded.*

Proof. By Lemma 2.2, we need only prove the boundedness of $\{w_n\}$. Since $\Phi_{\lambda_n}(w_n) - \frac{1}{2}\langle \Phi'_{\lambda_n}(w_n), w_n\rangle \le d$, condition ($S_3$) implies that $\int_{\mathbf{R}^{1+N}} |w_n|^\beta \le c$. By ($S_2$)–($S_3$), we may assume that $\beta \le \mu + 1$. We have

$$\begin{aligned}
&\int_{\mathbf{R}^{1+N}} |w_n|^{\mu+1} \\
&\quad \le \left(\int_{\mathbf{R}^{1+N}} |w_n|^\beta\right)^{(1-t)(1+\mu)/\beta} \left(\int_{\mathbf{R}^{1+N}} |w_n|^{(2N+4)/N)}\right)^{Nt(\mu+1)/(2N+4)} \\
&\quad \le c\|w_n\|^{t(\mu+1)}_{(2N+4)/N} \\
&\quad \le c\|w_n\|^{(\mu+1)t},
\end{aligned}$$

where $t := \frac{(2N+4)(1+\mu-\beta)}{(1+\mu)(2N+4-\beta N)} \in [0,1)$. Therefore,

$$\begin{aligned}
\|w_n^+\|^2 &= \lambda_n \int_{\mathbf{R}^{1+N}} H_z(t,x,w_n)w_n^+ \\
&\le c\varepsilon \int_{\mathbf{R}^{1+N}} |w_n||w_n^+| + c\int_{\mathbf{R}^{1+N}} |w_n|^\mu |w_n^+| \\
&\le c\varepsilon\|w_n^+\|^2 + \|w_n^+\|^{t\mu+1}.
\end{aligned}$$

From the arbitrariness of ε and the fact that $t\mu + 1 < 2$, we conclude that $\{w_n^+\}$, and hence $\{w_n\}$, is bounded. □

Proof of Theorem 1.1. Since $\Phi'_{\lambda_n}(w_n) = 0$, we see that

$$\begin{aligned}\|w_n^+\|^2 &= \lambda_n \int_{\mathbf{R}^{1+N}} H_z(t,x,w_n)w_n^+ \\ &\le c\int_{\mathbf{R}^{1+N}} (\varepsilon|w_n| + C_\varepsilon|w_n|^\mu)|w_n^+| \\ &\le c\varepsilon\|w_n^+\|^2 + c\|w_n^+\|^{\mu+1}.\end{aligned}$$

Therefore, $\|w_n^+\| \ge c > 0$. We know that there exist $\varepsilon_0 > 0$ and a sequence $\{y_n\} \subset \mathbf{R}^{1+N}$ such that $\lim_{n\to\infty} \int_{B(y_n,1)} |w_n^+| \ge \varepsilon_0 > 0$. Otherwise, by Lions's concentration compactness lemma, $w_n^+ \to 0$ in $L^t(\mathbf{R}^{1+N})$ for $2 < t < (2N+4)/N$. This is impossible because $\|w_n^+\| \ge c$. Therefore, by standard arguments, there exists a $z^* = z^+ + z^-$ such that $z^+ \ne 0$ and $\Phi'(z^*) = 0$. □

Proof of Theorem 1.2. Under the assumptions (F_1)–(F_4), the conclusions of Lemmas 2.1–2.3 are still true. It suffices to prove the boundedness of $\{w_n\}$ in Lemma 2.3. Now

$$\|w_n^+\|^2 - \lambda_n\|w_n^-\|^2 = \lambda_n \int_{\mathbf{R}^{1+N}} H_z(t,x,w_n)w_n \ge c\|w_n\|_\nu^\nu. \tag{2.1}$$

By (F_3), there exist $R_0 > 0$ and $\varepsilon_0 > 0$ such that $r_0 - \varepsilon_0 > 2$ and

$$H_z(t,x,w_n)w_n \ge (r_0-\varepsilon_0)H(t,x,w_n) \quad \text{for } |w_n| \ge R_0. \tag{2.2}$$

By (F_2) and (F_4), there exists a $c > 0$ such that

$$H_z(t,x,w_n)w_n - 2H(t,x,w_n) \ge c|w_n|^\alpha \quad \text{for } |w_n| \le R_0. \tag{2.3}$$

Since $\Phi_{\lambda_n}(w_n) \le d$ and $\Phi'_{\lambda_n}(w_n) = 0$, it follows that

$$\begin{aligned}&\Big(\frac{1}{2} - \frac{1}{r_0-\varepsilon_0}\Big)(\|w_n^+\|^2 - \lambda_n\|w_n^-\|^2) \\ &\qquad + \lambda_n \int_{\mathbf{R}^{1+N}} \Big(\frac{1}{r_0-\varepsilon}\Big)(H_z(t,x,w_n)w_n - H(t,x,w_n)) \le d.\end{aligned}$$

Therefore, by (F_1),

$$\begin{aligned}&\|w_n^+\|^2 - \lambda_n\|w_n^-\|^2 \\ &\quad\le c + c\Big(\int_{|w_n|\le R_0} + \int_{|w_n|\ge R_0}\Big)\Big(H(t,x,w_n) - \frac{1}{(r_0-\varepsilon)}H_z(t,x,w_n)w_n\Big) \\ &\quad\le c + c\int_{|w_n|\le R_0}\Big(H(t,x,w_n) - \frac{1}{r_0-\varepsilon}H_z(t,x,w_n)w_n\Big) \\ &\quad\le c + c\int_{|w_n|\le R_0} H_z(t,x,w_n)w_n \\ &\quad\le c + c\int_{|w_n|\le R_0} (|w_n|^\nu + |w_n|^p).\end{aligned}$$

On the other hand, $\Phi_{\lambda_n}(w_n) - \frac{1}{2}\Phi'_{\lambda_n}(w_n)w_n \le d$ and hence—by (2.2), (2.3), and (F_1)—we see that

$$c \ge \int_{\mathbf{R}^{1+N}} \big(\tfrac{1}{2}H_z(t,x,w_n)w_n - H(t,x,w_n)\big) \ge c\int_{|w_n|\le R_0} |w_n|^{\alpha} \tag{2.4}$$

and

$$\begin{aligned} c &\ge \int_{\mathbf{R}^{1+N}} \big(\tfrac{1}{2}H_z(t,x,w_n)w_n - H(t,x,w_n)\big) \\ &\ge c\int_{|w_n|\ge R_0} \left(\frac{r_0-\varepsilon_0}{2} - 1\right) H(t,x,w_n) \\ &\ge c\int_{|w_n|\ge R_0} |w_n|^{\nu}. \end{aligned} \tag{2.5}$$

Consequently, $\int_{|w_n|\ge R_0} |w_n|^p \le c$. Assumptions (F_1) and (F_4) imply that either $\nu > \alpha > p$ or $\alpha \ge \nu > p$.

If $p < \alpha < \nu$, then by (2.4) we have $\int_{|w_n|\le R_0} |w_n|^{\nu} \le c$ and, for t small enough,

$$\begin{aligned} \int_{|w_n|\le R_0} |w_n|^p &= \int_{|w_n|\le R_0} |w_n|^{(1-t)p}|w_n|^{tp} \\ &\le \left(\int_{|w_n|\le R_0} |w_n|^{\alpha}\right)^{(1-t)p/\alpha} \left(\int_{|w_n|\le R_0} |w_n|^{tpq}\right)^{1/q} \\ &\le c\left(\int_{|w_n|\le R_0} |w_n|^{tpq}\right)^{1/q} \\ &\le c\|w_n\|^{tp}, \end{aligned} \tag{2.6}$$

where $1/q + (1-t)p/\alpha = 1$.

If $p < \nu \le \alpha$ then, by (2.4) and (2.5), we have that

$$\int_{|w_n|\le R_0} |w_n|^p \le c\|w_n\|^{tp}, \qquad \int_{|w_n|\le R_0} |w_n|^{\nu} \le c\|w_n\|^{t\nu}. \tag{2.7}$$

Combining (2.4)–(2.7) yields the following estimates:

$$\begin{aligned} \|w_n^+\|^2 &= \lambda_n \int_{\mathbf{R}^{1+N}} H_z(t,x,w_n)w_n^+ \\ &\le c\int_{\mathbf{R}^{1+N}} (|w_n|^{\nu-1} + |w_n|^{p-1})|w_n^+| \\ &\le c\|w_n^+\|\left(\int_{|w_n|\ge R_0} |w_n|^{\nu} + \int_{|w_n|< R_0} |w_n|^{\nu}\right)^{(\nu-1)/\nu} \\ &\quad + c\|w_n^+\|\left(\int_{|w_n|\ge R_0} |w_n|^{p} + \int_{|w_n|< R_0} |w_n|^{p}\right)^{(p-1)/p} \\ &\le c\|w_n^+\|(c + \|w_n\|^{t(\nu-1)} + \|w_n\|^{t(p-1)}). \end{aligned}$$

Since t can be taken arbitrarily small, it follows that $\{\|w_n^+\|\}$, and hence $\{\|w_n\|\}$, is bounded. $\square$

3. Asymptotically Linear Case

In order to prove Theorem 1.3, we first check the conditions of Theorem 2.1.

LEMMA 3.1. *There exist* $r_0 > 0$ *and* $b > 0$ (*independent of* λ) *such that* $\Phi_\lambda|_B \geq b$ *for all* $\lambda \in [1,2]$, *where* $B = \{z : z \in E^+,\ \|z\| = r_0\}$.

Proof. The proof is similar to the argument in the previous section. □

LEMMA 3.2. *There exist* $z_0 \in E^+$, $\|z_0\| = 1$, *and* $R > r_0$ (*independent of* λ) *such that* $\Phi_\lambda|_A \leq 0$, *where* $A = \partial\{z = z^- + sz_0 : z^- \in E^-,\ \|z\| \leq R,\ R > 0\}$.

Proof. Since $\beta_0 > \mu_1$, we can find a $\bar{z}_0 \in E^+\backslash\{0\}$ such that the quadratic form corresponding to $-\Delta_x + V - \beta_0$ is negative on $\mathbf{R}\bar{z}_0 \oplus E^-$. Hence, $\|\bar{z}_0\|^2 - \beta_0 \int_{\mathbf{R}^{1+N}} \bar{z}_0^2 < 0$. We choose $z_0 = \bar{z}_0/\|\bar{z}_0\|$. Now we need only prove $\Phi|_A \leq 0$ for large R, since H is positive. If this is not true, then we may find $w_n = s_n z_0 + w_n^-$ with $\|w_n\| \to \infty$ such that $\Phi(w_n) \geq 0$. Setting $t_n = s_n/\|w_n\|$ and $u_n^- = w_n^-/\|w_n\|$, it follows that $t_n \geq \|u_n^-\|$. Since $t_n^2 + \|u_n^-\|^2 = 1$, we may assume $t_n \to t_* > 0$ and $u_n^- \rightharpoonup u^-$ weakly in E. Denote $u = t_* z_0 + u^-$. Since $\langle z_0, u^-\rangle_{L^2} = 0$, we have

$$
\begin{aligned}
t_*^2 - \|u^-\|^2 - \beta_0 \int_{\mathbf{R}^{1+N}} u \cdot u &= t_*^2 - \|u^-\|^2 - \beta_0 \int_{\mathbf{R}^{1+N}} (t_* z_0 + u^-)(t_* z_0 + u^-) \\
&\leq t_*^2 - \|u^-\|^2 - \beta_0 t_*^2 \int_{\mathbf{R}^{1+N}} z_0^2 - \beta_0 \int_{\mathbf{R}^{1+N}} (u^-)^2 \\
&\leq t_*^2 \left(1 - \beta_0 \int_{\mathbf{R}^{1+N}} z_0^2\right) \\
&< 0.
\end{aligned}
$$

Hence, there exists a bounded set Ω such that $t_*^2 - \|u^-\|^2 - \beta_0 \int_\Omega u^2 < 0$. On the other hand, $\Phi(w_n) \geq 0$ implies that

$$
\begin{aligned}
0 &\leq \frac{1}{2} t_n^2 - \frac{1}{2}\|u_n^-\|^2 - \int_{\mathbf{R}^{1+N}} \frac{H(t,x,w_n)}{\|w_n\|^2} \\
&\leq \frac{1}{2} t_n^2 - \frac{1}{2}\|u_n^-\|^2 - \int_\Omega \frac{H(t,x,w_n)}{\|w_n\|^2} \\
&= \frac{1}{2} t_n^2 - \frac{1}{2}\|u_n^-\|^2 - \int_\Omega \frac{\frac{1}{2}\beta_0 |w_n|^2 + K(t,x,w_n)}{\|w_n\|^2}.
\end{aligned}
$$

By (T_1) and the Lebesgue dominated convergence theorem,

$$
\lim_{n\to\infty} \int_\Omega \frac{K(t,x,w_n)}{\|w_n\|^2} = 0.
$$

Hence $t_*^2 - \|u^-\|^2 - \beta_0 \int_\Omega u^2 \geq 0$, and we have a contradiction. Consequently, there exists an $R > 0$ such that $\Phi_\lambda(z) \leq \Phi(z) \leq 0$ for all $z \in A$. □

LEMMA 3.3. *There exist $\lambda_n \in [1, 2]$ and $w_n \in E\backslash\{0\}$ such that $\lambda_n \to 1$, $\Phi_{\lambda_n}(w_n) \le d$, and $\Phi'_{\lambda_n}(w_n) = 0$. In particular, $\{w_n\}$ is bounded.*

Proof. We only prove the boundedness of $\{w_n\}$; the proofs of the existence of w_n and λ_n are similar to those in Section 2. Since $\Phi_{\lambda_n}(w_n) \le d$ and $\Phi'_{\lambda_n}(w_n) = 0$, we have

$$\begin{aligned}&\left(\tfrac{1}{2} - \tfrac{1}{\mu}\right)(\|w_n^+\|^2 - \lambda_n\|w_n^-\|^2)\\ &\qquad + \lambda_n \int_{\mathbf{R}\times\mathbf{R}^N}\left(\tfrac{1}{\mu}H_z(t, x, w_n)w_n - H(t, x, w_n)\right) \le d. \qquad (3.1)\end{aligned}$$

On the other hand, by (T_2)–(T_5) we may assume that

$$H_z(t, x, z)z \ge \mu H(t, x, z) \quad \text{for } |z| \le R_0, \qquad (3.2)$$

$$K_z(t, x, z)z - 2K(t, x, z) \ge c|z|^\alpha \quad \text{for } |z| \ge R_0. \qquad (3.3)$$

Therefore, by (3.1) and (3.2),

$$\begin{aligned}&\|w_n^+\|^2 - \lambda_n\|w_n^-\|^2\\ &\quad \le c + c\int_{\mathbf{R}\times\mathbf{R}^N}\left(H(t, x, w_n) - \tfrac{1}{\mu}H_z(t, x, w_n)w_n\right)\\ &\quad = c + c\left(\int_{|w_n|\le R_0} + \int_{|w_n|\ge R_0}\right)\left(H(t, x, w_n) - \tfrac{1}{\mu}H_z(t, x, w_n)w_n\right)\\ &\quad \le c + \int_{|w_n|\ge R_0}\left(H(t, x, w_n) - \tfrac{1}{\mu}H_z(t, x, w_n)w_n\right)\\ &\quad \le c + \int_{|w_n|\ge R_0}\left(\tfrac{1}{2} - \tfrac{1}{\mu}\right)H_z(t, x, w_n)w_n\\ &\quad \le c + c\int_{|w_n|\ge R_0}|w_n|^2.\end{aligned}$$

Since $\Phi_{\lambda_n}(w_n) - \frac{1}{2}\langle\Phi'_{\lambda_n}(w_n), w_n\rangle \le d$, it follows that

$$\begin{aligned}c &\ge \int_{\mathbf{R}\times\mathbf{R}^N}(H_z(t, x, w_n)w_n - 2H(t, x, w_n))\\ &= \left(\int_{|w_n|\le R_0} + \int_{|w_n|\ge R_0}\right)(H_z(t, x, w_n)w_n - 2H(t, x, w_n))\\ &\ge c\int_{|w_n|\le R_0}|w_n|^m + c\int_{|w_n|\ge R_0}|w_n|^\alpha. \qquad (3.4)\end{aligned}$$

Choose $t := (N + 2)(2 - \alpha)/(2(N + 2) - \alpha N)$. Then $t \in (0, 1)$ and

$$\begin{aligned}\int_{|w_n|\ge R_0}|w_n|^2 &= \int_{|w_n|\ge R_0}|w_n|^{(1-t)2}|w_n|^{2t}\\ &\le \left(\int_{|w_n|\ge R_0}|w_n|^\alpha\right)^{(1-t)2/\alpha}\left(\int_{|w_n|\ge R_0}|w_n|^{(2N+4)/N}\right)^{2tN/(2N+4)}\\ &\le c\|w_n\|^{2t}. \qquad (3.5)\end{aligned}$$

Combining (3.1)–(3.5), we have the following estimates:

$$
\begin{aligned}
\|w_n^+\|^2 &= \lambda_n \int_{\mathbf{R}\times\mathbf{R}^N} H_z(t,x,w_n)w_n^+ \\
&\le c\left(\int_{|w_n|\ge R_0} + \int_{|w_n|\le R_0}\right)|H_z(t,x,w_n)||w_n^+| \\
&\le c\int_{|w_n|\ge R_0} |w_n||w_n^+| + c\int_{|w_n|\le R_0} |w_n|^{m-1}|w_n^+| \\
&\le c\left(\int_{|w_n|\le R_0} |w_n|^m\right)^{(m-1)/m}\left(\int_{|w_n|\le R_0} |w_n^+|^m\right)^{1/m} \\
&\quad + c\left(\int_{|w_n|\ge R_0} |w_n|^2\right)^{1/2}\left(\int_{|w_n|\ge R_0} |w_n^+|^2\right)^{1/2} \\
&\le c\|w_n^+\| + c\|w_n\|^t\|w_n^+\|,
\end{aligned}
$$

which imply that $\{w_n^+\}$, and hence $\{w_n\}$, is bounded. □

Proof of Theorem 1.3. By the assumptions of Theorem 1.3, for any $\varepsilon > 0$ we have

$$
\|w_n^+\|^2 = \lambda_n \int_{\mathbf{R}^{1+N}} H_z(t,x,w_n)w_n^+ \le \varepsilon\|w_n\|^2 + c\|w_n\|^p,
$$

$$
\|w_n^-\|^2 = -\int_{\mathbf{R}^{1+N}} H_z(t,x,w_n)w_n^- \le \varepsilon\|w_n\|^2 + c\|w_n\|^p,
$$

where $p > 2$. It follows that $\|w_n\| \ge c > 0$. Similarly, there exists a $z^* \ne 0$ such that $\Phi(z^*) = 0$. □

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