Michigan Math. J. 50 (2002)

Deformation Theory of 5-Dimensional CR Structures and the Rumin Complex

Takao Akahori, Peter M. Garfield, & John M. Lee

1. Introduction

A natural problem in several complex variables is that of classifying the deformations of an isolated singularity in a complex analytic variety. The problem is solved by constructing a "versal family" of deformations of the singularity, which is, roughly speaking, a minimal family of deformations that includes biholomorphic representatives of all other deformations. (See Section 8 for a precise definition.)

Versal families for isolated singularities were first constructed from an algebraic point of view in the late 1960s and early 1970s by Tjurina, Grauert, and Donin [Tj; Gr; D]. Shortly thereafter, Kuranishi [K] outlined a program for relating deformations of an isolated singularity to deformations of the CR structure on a real hypersurface obtained by intersecting the variety with a small sphere surrounding the singular point (the "link" of the singularity). His idea was to construct a versal family of deformations of the CR structure on the link (versal modulo "wiggles" of the link within the ambient complex space, not just modulo changes in the CR structure). Kuranishi's construction was extended and simplified by subsequent work of the first author and others [A3; A4; A5; M1; M2; BM].

A fundamental limitation of all of these results has been a dimensional restriction: Because the deformation complex that was introduced in [A3; A4; A5] failed to be subelliptic in low dimensions, these results applied only to CR manifolds of dimension 7 or more (and therefore to singularities of varieties whose complex dimension is at least 4).

The purpose of this paper is to extend the Kuranishi construction of versal families of CR structures to the case of 5-dimensional CR manifolds. The new idea here is a subelliptic estimate and consequent Hodge theory for a certain subcomplex of the standard deformation complex inspired by recent work of M. Rumin on contact manifolds.

Miyajima [M3], following an idea introduced in [Be], has introduced an alternative approach to constructing versal families in all dimensions that is based on analyzing deformations not only of the CR structure but also of the CR structure together with its embedding into $\mathbf{C}^N$. The present approach is of independent interest, however, because it represents a completion of the original Kuranishi

Received June 13, 2001. Revision received March 8, 2002.

program of constructing an intrinsically defined versal family of deformations of the CR structure itself. There appears to be little hope for extending this intrinsic approach to the case of 3-dimensional CR manifolds, because the relevant cohomology groups in that case are infinite-dimensional.

Let $(M, {}^0T'')$ be a compact strictly pseudoconvex CR manifold of real dimension 5. Deformations of the CR structure of M can be represented as T'-valued $(0,1)$-forms, where $T' = (\mathbf{C} \otimes TM)/{}^0T''$ (which we identify, in a non-CR-invariant way, with a 3-dimensional complex subbundle of $\mathbf{C} \otimes TM$ transverse to the antiholomorphic tangent bundle ${}^0T''$; see Section 2 for precise definitions). The space of such forms fits into a complex $(\Gamma(M, T' \otimes \Lambda^j({}^0T'')^*, \bar\partial^{(j)}_{T'})$, the *standard deformation complex* [A3; BM]. In earlier work on higher-dimensional CR deformation theory, the first author defined a subcomplex $(\Gamma(M, E_j), \bar\partial_j)$ of the standard deformation complex corresponding to deformations of the CR structure that leave the contact structure fixed. When $\dim M = 2n - 1 \geq 7$, there is a subelliptic estimate on $\Gamma(M, E_2)$ that leads to the construction of a versal family [A3; A4]. But if $\dim M = 5$, there is no such estimate.

In this paper, inspired by the differential-form complex introduced by Rumin [R] for studying de Rham theory on contact manifolds, we extend the E_i complex by defining a new second-order operator D:

$$0 \to \Gamma(M, F) \xrightarrow{D} \Gamma(M, E_1) \xrightarrow{\bar\partial_1} \Gamma(M, E_2),$$

where F is a 1-dimensional subbundle of $\mathbf{C} \otimes TM$ transverse to ${}^0T'' \oplus \overline{{}^0T''}$. This is closely related to Rumin's complex, as explained in Section 4. A similar complex has also been used in [BM].

Once we have proved an a priori estimate on $\Gamma(M, E_1)$, it follows that there is a Kodaira–Hodge decomposition theorem on $\Gamma(M, E_1)$. Using techniques similar to those in [A3; A4], this leads to a construction of the versal family in the 5-dimensional case. We remark that, since this paper was accepted for publication, the second author (following a suggestion of Rumin) has proved a subellipticity result [G] that can be used to extend the results of this paper to all dimensions ≥ 5, thus giving an alternative approach to the results of [A3; A4].

2. Background and Notation

Let $(M, {}^0T'')$ be a CR manifold. By this we mean that M is a smooth manifold of dimension $2n-1$ and ${}^0T''$ is a complex subbundle of the complexified tangent bundle $\mathbf{C} \otimes TM$ satisfying

$$\begin{gathered} {}^0T'' \cap \overline{{}^0T''} = 0, \quad \dim_{\mathbf{C}} {}^0T'' = n-1, \\ [X, Y] \in \Gamma(M, {}^0T'') \quad \text{for all } X, Y \in \Gamma(M, {}^0T''), \end{gathered}$$

where by $\Gamma(M, E)$ we mean the space of C^∞ sections of the bundle E. For convenience we will write ${}^0T'$ for $\overline{{}^0T''}$ and H for the real bundle $\mathrm{Re}({}^0T'' \oplus {}^0T')$. We assume that there is a global nonvanishing real 1-form θ that annihilates H, that is, such that $\theta(X) = \theta(\bar X) = 0$ for all $X \in \Gamma(M, {}^0T'')$. Since H is oriented by its complex structure, the existence of such a form is equivalent to M being orientable.

We define the *Levi form* L_θ by

$$L_\theta(X, \bar{Y}) = -i\theta([X, \bar{Y}]) \quad \text{for } X, Y \in {}^0T'. \tag{2.1}$$

If this Levi form L_θ is positive definite or negative definite, then $(M, {}^0T'')$ is called *strictly* (or *strongly*) *pseudoconvex*. (After this section, we will always assume that our CR structure is strictly pseudoconvex.) In this case, we call a choice of nonvanishing 1-form θ annihilating H a *pseudohermitian structure.* Let ξ be the unique real vector field satisfying $\theta(\xi) = 1$ and $d\theta(\xi, X) = 0$ for all $X \in H$. Notice that this implies, for every point p of M, that $\xi_p \notin H_p$. The Levi form gives us a metric on H that extends to a Riemannian metric on all of TM by declaring that ξ is of unit length and orthogonal to H. We will call this metric the *Webster metric* (see [W]).

Let F denote the complex line bundle $\mathbf{C}\xi$, and define $T' := {}^0T' + \mathbf{C}\xi$. It is easy to check that the projection $\mathbf{C} \otimes TM \to (\mathbf{C} \otimes TM)/{}^0T''$ restricts to an isomorphism $T' \cong (\mathbf{C} \otimes TM)/{}^0T''$. The latter quotient bundle (often denoted also by T'), though invariantly defined, is less convenient for computations, so throughout this paper we consider T' as the subbundle of $\mathbf{C} \otimes TM$ just defined.

We then obtain vector bundle decompositions

$$\mathbf{C}TM = T' + {}^0T'' \tag{2.2}$$

and

$$\mathbf{C}TM = {}^0T' + {}^0T'' + F. \tag{2.3}$$

Note that these decompositions depend on the choice of θ (and thus ξ), and hence they are not CR-invariant. We will often take advantage of these decompositions to project onto various components. For a vector X, let us write $\pi_F(X)$ for the F-component of X, $\pi'(X)$ for the T'-component, ${}^0\pi'(X)$ for the ${}^0T'$-component, and ${}^0\pi''(X)$ for the ${}^0T''$-component, according to these decompositions. Moreover, since we will often be dealing with vector-valued forms, let us use the same notation for the projection of, say, $\mathbf{C} \otimes TM \otimes \Lambda^j({}^0T'')^*$ into component parts $F \otimes \Lambda^j({}^0T'')^*$, $T' \otimes \Lambda^j({}^0T'')^*$, ${}^0T' \otimes \Lambda^j({}^0T'')^*$, and ${}^0T'' \otimes \Lambda^j({}^0T'')^*$ via equations (2.2) and (2.3).

It is often useful to identify $\mathbf{C} \otimes \Lambda^k M$ with $\mathbf{C} \otimes \Lambda^k H^* \oplus \theta \wedge \mathbf{C} \otimes \Lambda^{k-1} H^*$. Notice that this identification depends on the choice of θ. The CR structure defines a natural bigrading on $\mathbf{C} \otimes \Lambda^k H^*$, so we may make the further identification

$$\mathbf{C} \otimes \Lambda^k M = \sum_{p+q=k} \Lambda^{p,q} H^* + \theta \wedge \sum_{p+q=k-1} \Lambda^{p,q} H^*. \tag{2.4}$$

This allows us to identify, for example, $\Lambda^q({}^0T'')^* = \Lambda^{0,q} H^*$ with honest forms on M.

Finally, we note that we will use the Einstein summation convention whenever possible. We will use Roman indices (j, k, for example) to indicate sums from 1 to $2n - 1$ and will use Greek indices (α, β, and so on) for sums from 1 to $n - 1$.

3. Review of CR Deformation Theory

In this section we survey previous work on the deformation theory of CR structures. This work was initiated by Kuranishi [K] as a CR analogue of his work on complex manifolds. Most of the work reviewed here was done by the first author [A1; A2; A3; A4].

Following [A2], we introduce a first-order differential operator $\bar\partial_{T'}\colon \Gamma(M, T') \to \Gamma(M, T' \otimes ({}^0T'')^*)$ by

$$\bar\partial_{T'} Y(\bar X) = \pi'[\bar X, Y] \quad \text{for } Y \in \Gamma(M, T') \text{ and } \bar X \in \Gamma(M, {}^0T''). \tag{3.1}$$

(This definition reflects the fact that T' has a natural structure as a CR vector bundle; if M is a real hypersurface in a complex manifold U, then T' is naturally isomorphic to $T^{1,0}U|_M$.) As in the case of scalar-valued differential forms, this generalizes to the operators $\bar\partial^{(p)}\colon \Gamma(M, T' \otimes \Lambda^p({}^0T'')^*) \to \Gamma(M, T' \otimes \Lambda^{p+1}({}^0T'')^*)$ $(p = 1, 2, \dots)$ given by

$$\begin{aligned}\bar\partial^{(p)}\phi(\bar X_1, \dots, \bar X_{p+1}) &= \sum_{j=1}^{p+1}(-1)^{j+1}\pi'[\bar X_j, \phi(\bar X_1, \dots, \hat{\bar X}_j, \dots, \bar X_{p+1})] \\ &\quad + \sum_{j<k}(-1)^{j+k}\phi([\bar X_j, \bar X_k], \bar X_1, \dots, \hat{\bar X}_j, \dots, \hat{\bar X}_k, \dots, \bar X_{p+1})\end{aligned} \tag{3.2}$$

for $\phi \in \Gamma(M, T' \otimes \Lambda^p({}^0T'')^*)$ and $\bar X_k \in \Gamma(M, {}^0T'')$. We then have a differential complex

$$\begin{aligned}0 \to \Gamma(M, T') &\xrightarrow{\bar\partial_{T'}} \Gamma(M, T' \otimes ({}^0T'')^*) \xrightarrow{\bar\partial^{(1)}} \Gamma(M, T' \otimes \Lambda^2({}^0T'')^*) \xrightarrow{\bar\partial^{(2)}} \\ &\cdots \to \Gamma(M, T' \otimes \Lambda^p({}^0T'')^*) \xrightarrow{\bar\partial^{(p)}} \Gamma(M, T' \otimes \Lambda^{p+1}({}^0T'')^*) \to \cdots\end{aligned} \tag{3.3}$$

with $\bar\partial^{(p+1)}\bar\partial^{(p)} = 0$ (see [A2]). This complex is called the *standard deformation complex*.

A complex subbundle $E \subset \mathbf{C} \otimes TM$ is an *almost CR structure* (and the pair (M, E) is an *almost CR manifold*) if $E \cap \bar E = 0$ and $\dim_{\mathbf{C}} E = n - 1$. An almost CR structure E is *at finite distance from* ${}^0T''$ if ${}^0\pi''|_E\colon E \to {}^0T''$ is a bundle isomorphism. These almost CR structures are characterized by the fact that they are graphs over ${}^0T''$: there is a bijective correspondence between elements $\phi \in \Gamma(M, \mathrm{Hom}({}^0T'', T')) = \Gamma(M, T' \otimes ({}^0T'')^*)$ and almost CR structures

$${}^\phi T'' := \{\bar X + \phi(\bar X) : \bar X \in {}^0T''\}$$

at finite distance from ${}^0T''$ (see e.g. [A1, Prop. 1.1, p. 618]). The almost CR structure ${}^\phi T''$ is a CR structure exactly when it satisfies the integrability condition, which can be written as the nonlinear partial differential equation

$$P(\phi) := \bar\partial^{(1)}\phi + R_2(\phi) + R_3(\phi) = 0,$$

where $R_k(\phi) \in \Gamma(M, T' \otimes \Lambda^2({}^0T'')^*)$ $(k = 2, 3)$ are the parts of $P(\phi)$ that are of degree k in ϕ. They are given by

$$R_2(\phi)(\bar{X}, \bar{Y}) = \pi'[\phi(\bar{X}), \phi(\bar{Y})] - \phi({}^0\pi''[\bar{X}, \phi(\bar{Y})] + {}^0\pi''[\phi(\bar{X}), \bar{Y}]) \quad (3.4)$$

and

$$R_3(\phi)(\bar{X}, \bar{Y}) = -\phi({}^0\pi''[\phi(\bar{X}), \phi(\bar{Y})]). \quad (3.5)$$

See [A1, Thm. 2.1, p. 619] and the proof given therein for details.

If we consider only deformations ϕ that preserve the contact structure (i.e., for which ${}^\phi T'' \oplus \overline{{}^\phi T''} = {}^0T'' \oplus \overline{{}^0T''}$), then we are simply restricting to $\phi \in \Gamma(M, {}^0T' \otimes ({}^0T'')^*)$. For such ϕ, we notice that $R_3(\phi) = 0$ and that $\pi_F R_2(\phi) = 0$ (so ${}^0\pi' R_2(\phi) = R_2(\phi)$). Hence $P(\phi) = \pi_F \bar{\partial}^{(1)}\phi + {}^0\pi'\bar{\partial}^{(1)}\phi + R_2(\phi)$. Our integrability condition $P(\phi) = 0$ is thus equivalent in this case to $\pi_F \bar{\partial}^{(1)}\phi = 0$ and ${}^0\pi'\bar{\partial}^{(1)}\phi + R_2(\phi) = 0$ (cf. [A2, Prop. 1.7.3, p. 797]). This, in part, motivates the definition of the following subspaces of $\Gamma(M, {}^0T' \otimes \Lambda^p({}^0T'')^*)$:

$$\Gamma_p = \{u \in \Gamma(M, {}^0T' \otimes \Lambda^p({}^0T'')^*) : \pi_F \bar{\partial}^{(p)} u = 0\}. \quad (3.6)$$

For $\phi \in \Gamma_1 \subset \Gamma(M, {}^0T' \otimes ({}^0T'')^*)$, then, the integrability condition becomes $P(\phi) = {}^0\pi'\bar{\partial}^{(1)}\phi + R_2(\phi) = 0$.

We remark that, contrary to appearances, the definition of Γ_p is an algebraic condition on u, not a differential one. To see this, apply the 1-form θ to both sides of equation (3.2). By the definition of Γ_p, the left-hand side is zero and so

$$\begin{aligned} 0 = &\sum_{j=1}^{p+1} (-1)^{p+1}\theta([\bar{X}_j, u(\bar{X}_1, \dots, \hat{\bar{X}}_j, \dots, \bar{X}_{p+1})]) \\ &+ \sum_{j<k} (-1)^{j+k}\theta(u([\bar{X}_j, \bar{X}_k], \bar{X}_1, \dots, \hat{\bar{X}}_j, \dots, \hat{\bar{X}}_k, \dots, \bar{X}_{p+1})). \end{aligned}$$

Because u maps into ${}^0T'$, which is annihilated by θ, the second sum is a sum of zeros. Using $\theta([X, Y]) = -d\theta(X, Y)$ for $X, Y \in \mathbf{C} \otimes H = {}^0T' \oplus {}^0T''$, the first sum becomes

$$0 = \sum_{j=1}^{p+1} (-1)^p \, d\theta(\bar{X}_j, u(\bar{X}_1, \dots, \hat{\bar{X}}_j, \dots, \bar{X}_{p+1})). \quad (3.7)$$

This is an algebraic condition on u.

In fact, the spaces Γ_p are smooth sections of vector bundles. There exist [A3, Prop. 2.1, p. 313] subbundles $E_p \subset T' \otimes \Lambda^p({}^0T'')^*$ such that $\Gamma_p = \Gamma(M, E_p)$. Restricting $\bar{\partial}^{(p)}$ to E_p yields a sequence of maps $\bar{\partial}_p$,

$$0 \to \Gamma(M, E_0) \xrightarrow{\bar{\partial}_0} \Gamma(M, E_1) \xrightarrow{\bar{\partial}_1} \Gamma(M, E_2) \xrightarrow{\bar{\partial}_2} \Gamma(M, E_3) \xrightarrow{\bar{\partial}_3} \cdots, \quad (3.8)$$

and ${}^\phi T''$ is integrable for $\phi \in \Gamma_1$ if and only if $P(\phi) = \bar{\partial}_1\phi + R_2(\phi) = 0$.

It turns out that $E_0 = 0$ and that the resulting complex

$$0 \to 0 \to \Gamma(M, E_1) \xrightarrow{\bar{\partial}_1} \Gamma(M, E_2) \xrightarrow{\bar{\partial}_2} \Gamma(M, E_3) \xrightarrow{\bar{\partial}_3} \cdots \quad (3.9)$$

is a differential subcomplex of the standard deformation complex (see [A3, Thm. 2.2, p. 314]). This subcomplex still contains enough information to be useful; for example, the inclusion map $\iota\colon \Gamma(M, E_p) \to \Gamma(M, {}^0T' \otimes \Lambda^p({}^0T'')^*)$ induces a map

$$\iota^*\colon \frac{\ker \bar\partial_p}{\operatorname{im} \bar\partial_{p-1}} \to \frac{\ker \bar\partial^{(p)}}{\operatorname{im} \bar\partial^{(p-1)}}$$

that is an isomorphism if $p \geq 2$ and surjective if $p = 1$ [A3, Thm. 2.4, p. 315].

Furthermore, there exist both a subelliptic estimate for this complex [A3, Thm. 4.1, p. 319] and a Kodaira–Hodge decomposition theorem for $\Gamma(M, E_2)$ [A3, Thm. 4.1, p. 328], provided $\dim M = 2n - 1 \geq 7$. That is, if we define the Laplacian $\Box = \bar\partial_2^*\bar\partial_2 + \bar\partial_1\bar\partial_1^*$, then there is a harmonic projector H such that $\Box Hu = 0$ for all $u \in \Gamma(M, E_2)$ and a Neumann operator N such that $NHu = HNu = 0$ and $u = \Box Nu + Hu$ for all $u \in \Gamma(M, E_2)$. This construction fails if $\dim M = 5$, since there is no subelliptic estimate for this complex.

4. The New Complex

In this section, we introduce a new complex as a replacement for the differential subcomplex (3.9) of the standard differential complex. Set

$$H_0 = \{v \in \Gamma(M, T') : \pi_F \bar\partial_{T'} v = 0\}. \tag{4.1}$$

We then obtain a new differential subcomplex of the standard differential complex (3.3):

$$0 \to H_0 \xrightarrow{\bar\partial_0} \Gamma(M, E_1) \xrightarrow{\bar\partial_1} \Gamma(M, E_2) \xrightarrow{\bar\partial_2} \cdots . \tag{4.2}$$

This complex is a generalization of ideas of the first author. Versions of it have been used by Bland and Epstein [BE, pp. 353–355] (in the 3-dimensional case) and by Buchweitz and Millson [BM, p. 82] (based in part on ideas of the third author). It is straightforward to see that this is a complex: the definition of H_0 ensures that $\bar\partial_0 u \in \Gamma(M, {}^0T' \otimes ({}^0T'')^*)$, and that (4.2) is a subcomplex of the standard differential complex (3.3) means that, in fact, $\bar\partial_0 u \in \Gamma(M, E_1)$.

We would like to make a few remarks about H_0. It is not the space of smooth sections of a vector bundle over M; rather, it is the image of a first-order differential operator. We define this operator $\Gamma(M, F) \to \Gamma(M, T')$ as follows. For $Z \in \Gamma(M, F)$, we may write $Z = u \cdot \xi$ for some smooth function u (namely, $u = \theta(Z)$). We then get an element $X_u \in \Gamma(M, {}^0T')$ by requiring that $u\xi + X_u \in H_0$; thus, $\pi_F \bar\partial_{T'}(X_u + u\xi) = 0$. This is equivalent to $\theta([\bar Y, X_u + u\xi]) = 0$ for all $\bar Y \in \Gamma(M, {}^0T'')$. Another way to write this is

$$d\theta(\bar Y, X_u) = \bar Y u, \tag{4.3}$$

because $\theta(\bar Y) = \theta(X_u) = 0$ and $d\theta(\xi, \cdot) = 0$. Since our CR structure is strictly pseudoconvex, equation (4.3) uniquely determines X_u. Thus H_0 is the image of the first-order differential operator $\rho\colon \Gamma(M, F) \to \Gamma(M, T')$ defined by $\rho(u\xi) = X_u + u\xi$.

Define a second-order operator $D\colon \Gamma(M, F) \to \Gamma(M, E_1)$ as the composition $D = \bar{\partial}_{T'} \circ \rho$. We then clearly have a complex

$$0 \to \Gamma(M, F) \xrightarrow{D} \Gamma(M, E_1) \xrightarrow{\bar{\partial}_1} \Gamma(M, E_2) \xrightarrow{\bar{\partial}_2} \cdots . \tag{4.4}$$

It is this complex that we will use to derive our subelliptic estimate and thence our decomposition theorems.

Notice that X_u includes a first derivative of u. Using a local moving frame $\{e_1, \dots, e_{n-1}\}$ for ${}^0T'$ satisfying

$$L_\theta(e_\alpha, \bar{e}_\beta) = \delta_{\alpha\beta}, \tag{4.5}$$

we set $X_u = X^\alpha e_\alpha$ (note the implicit sum). Expanding $\theta([\bar{e}_\beta, X^\alpha e_\alpha + u\xi]) = 0$ yields

$$\theta((\bar{e}_\beta X^\alpha)e_\alpha + X^\alpha[\bar{e}_\beta, e_\alpha] + (\bar{e}_\beta u)\xi + u[\bar{e}_\beta, \xi]) = 0. \tag{4.6}$$

This simplifies to $X^\alpha(-i\delta_{\beta\alpha}) + \bar{e}_\beta u = 0$, so $X^\alpha = i\delta^{\alpha\beta}\bar{e}_\beta u$. Thus ρ is indeed a first-order operator, and our composition $D = \bar{\partial}_{T'} \circ \rho$ is a second-order operator.

Finally, we would like to relate our operator D to that of Rumin [R]. Define, for $p + q \geq n$,

$$F^{p,q} = \{u \in \theta \wedge \Lambda^{p-1,q}H^* : d\theta \wedge u = 0\}, \tag{4.7}$$

and set $F^k = \bigoplus_{p+q=k} F^{p,q}$ for $k \geq n$. Although definition (4.7) seems to depend on the noninvariant decomposition (2.4), we may actually express F^k invariantly as

$$F^k = \{u \in \mathbf{C} \otimes \Lambda^k M : v \wedge u = 0 \text{ for all } v \in \langle \theta, d\theta \rangle\},$$

where $\langle \theta, d\theta \rangle$ is the ideal generated by θ and $d\theta$. Since this ideal is CR-invariant, the definition of F^k is as well. Below the middle dimension, we define a slightly different space. For $p + q = k \leq n - 1$, set

$$E^{p,q} = \Lambda^{p,q}H^*/\langle d\theta \rangle$$

and $E^k = \bigoplus_{p+q=k} E^{p,q}$, so that

$$E^k = \mathbf{C} \otimes \Lambda^k M/\langle \theta, d\theta \rangle$$

is CR-invariant as well. Rumin's D operator is a map $D\colon E^{n-1} \to F^n$ given by $D[u] = d\tilde{u}$, where the representative $\tilde{u}$ of $[u] \in E^{n-1}$ is chosen so that $d\tilde{u}$ will be in F^n. There is then a complex

$$\cdots \xrightarrow{d} E^{n-1} \xrightarrow{D} F^n \xrightarrow{d} F^{n+1} \xrightarrow{d} \cdots, \tag{4.8}$$

which decomposes into subcomplexes

$$\cdots \xrightarrow{d''} E^{p,n-p-1} \xrightarrow{D''} F^{p,n-p} \xrightarrow{d''} F^{p,n-p+1} \xrightarrow{d''} \cdots . \tag{4.9}$$

We hope to provide more details on these complexes in another paper.

The relation between our complex (4.4) and Rumin's complex (4.9) occurs when $p = n - 1$ in Rumin's complex, in which case (4.9) is

$$0 \to E^{n-1,0} \xrightarrow{D''} F^{n-1,1} \xrightarrow{d''} F^{n-1,2} \xrightarrow{d''} \cdots \tag{4.10}$$

and we note that $E^{n-1,0} = \Lambda^{n-1,0}H^* = \Lambda^{n-1}({}^0T')^*$. Let K_M denote a nonvanishing closed $(n,0)$-form (i.e., an element of $\theta \wedge \Lambda^{n-1,0}H^*$), if one exists. For any positive k, we obtain a map $P_k \colon \Gamma(M, E_k) \to F^{n-1,k}$ by interior multiplying the vector part of $u \in \Gamma(M, E_k)$ into K_M and then wedging the remainder with the form part of u. Let $P_0 \colon \Gamma(M, F) \to E^{n-1,0}$ be given by $P_0(u\xi) = u \cdot K_M$. The claim is that each P_k is an isomorphism and that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(M,F) & \xrightarrow{D} & \Gamma(M,E_1) & \xrightarrow{\bar\partial_1} & \Gamma(M,E_2) & \xrightarrow{\bar\partial_2} & \cdots \\ & & \downarrow{\scriptstyle P_0} & & \downarrow{\scriptstyle P_1} & & \downarrow{\scriptstyle P_2} & & \\ 0 & \longrightarrow & E^{n-1,0} & \xrightarrow{D''} & F^{n-1,1} & \xrightarrow{d''} & F^{n-1,2} & \xrightarrow{d''} & \cdots . \end{array}$$

Because K_M always exists locally, the two complexes are locally isomorphic. If the canonical line bundle is trivial, then this complex version (4.9) of the Rumin complex is isomorphic to our new complex (4.4).

5. A Subelliptic Estimate and Decomposition Theorem

In this section, we state two of our main results. First, we produce a subelliptic estimate at $\Gamma(M, E_1)$ for our complex (4.4) in the 5-dimensional case. Using this, we get a Hodge–Kodaira decomposition theorem for elements of $\Gamma(M, E_1)$. As remarked in Section 1, these results can be extended to higher dimensions using the subellipticity results of [G]. We concentrate here on the 5-dimensional case because it is new.

We begin with some preliminaries. Our choice of pseudohermitian structure θ determines the *pseudohermitian connection* ∇ (see [W; T]): this is the unique connection that is compatible with H and its complex structure, for which θ and $d\theta$ are parallel, and that satisfies an additional torsion condition. For any tensor field u on M, the total covariant derivative ∇u can be decomposed as

$$\nabla u = \nabla' u + \nabla'' u + \nabla_\xi u \otimes \theta,$$

where $\nabla' u$ (resp. $\nabla'' u$) involves derivatives only with respect to vector fields in ${}^0T'$ (resp. ${}^0T''$). Writing $\nabla_H u = \nabla' u + \nabla'' u$, the *Folland–Stein norms* $\|\cdot\|_k$ are defined by

$$\|u\|_k^2 = \sum_{j=0}^{k} \|\nabla_H^j u\|^2,$$

where $\|\cdot\|$ denotes the L^2 norm defined with respect to the Webster metric. (Note that, in [A3], the $\|\cdot\|_1$ and $\|\cdot\|_2$ norms were called $\|\cdot\|'$ and $\|\cdot\|''$, respectively.) We will write $(\cdot,\cdot)$ for the hermitian inner product that corresponds to the norm $\|\cdot\|$, and for any bundle E we will let $\Gamma_2(M, E)$ denote the completion of $\Gamma(M, E)$ with respect to the L^2 norm.

Define a second-order operator $L = 1 + \nabla'^*\nabla' + \nabla''^*\nabla''$. We then define our Laplacian $\Box \colon \Gamma(M, E_1) \to \Gamma(M, E_1)$ by $\Box u = DD^*u + \bar\partial_1^* L \bar\partial_1 u$, where the adjoints are defined with respect to the complex (4.4). We use this operator and the norms defined previously to express our subelliptic estimate in the following theorem.

THEOREM 5.1 (Main Estimate). *Let* $(M, {}^0T'')$ *be a compact, strictly pseudoconvex CR manifold of dimension* 5. *Then there exists a constant* $c > 0$ *such that*

$$(\phi, \Box\phi) = \|D^*\phi\|^2 + \|\bar\partial_1\phi\|_1^2 \geq c\|\phi\|_2^2 - \|\phi\|_1^2 \tag{5.1}$$

for all $\phi \in \Gamma(M, E_1)$.

The details of the proof of this estimate will be confined to the next section.

We define new norms that are Sobolev extensions of the Folland–Stein norms $\|\cdot\|_k$ as follows. Set

$$\|u\|_{k,m}^2 = \sum_{l=0}^{m}\sum_{j=0}^{k}\|\nabla^l\nabla_H^j u\|^2.$$

The first parameter, k, specifies the number of derivatives in the H directions, whereas the second parameter, m, is the number of unconstrained derivatives. (We remark that, in [A3], these norms were written slightly differently; for example, $\|\cdot\|_{2,m}$ was $\|\cdot\|''_{(m)}$.) Then our Main Estimate (Theorem 5.1), together with standard integration-by-parts techniques, gives us the following Sobolev estimate.

COROLLARY 5.2. *Let* $(M, {}^0T'')$ *be a compact, strictly pseudoconvex CR manifold of dimension* 5. *For each positive integer* m, *there exists a constant* $c_m > 0$ *such that*

$$\|D^*\phi\|_{0,m}^2 + \|\bar\partial_1\phi\|_{1,m}^2 \geq c_m\|\phi\|_{2,m}^2 - \|\phi\|_{1,m}^2 \tag{5.2}$$

for all $\phi \in \Gamma(M, E_1)$.

Let us write $\mathcal{H}$ for the harmonic elements of $\Gamma(M, E_1)$ with respect to the Laplacian $\Box$. In order to find a useful expression for $\mathcal{H}$, we use the following lemma to express the adjoint of D in simpler terms.

LEMMA 5.3. *Let* $\tilde H_0$ *be the completion of* H_0 *under the* L^2 *norm, where* $\pi_{\tilde H_0}\colon \Gamma_2(M, T') \to \tilde H_0$ *is orthogonal projection. Then we have the following relations*:

(a) $\bar\partial_0^* = \pi_{\tilde H_0} \circ \bar\partial_{T'}^*$, *where* $\bar\partial_{T'}^*$ *is the formal adjoint of* $\bar\partial_{T'}$;

(b) $\ker D^* = \ker \bar\partial_0^*$.

Proof. The first conclusion follows from the relation between the standard deformation complex (3.3) and the complex (4.2) involving H_0. Since $H_0 \subset \Gamma(M, T')$ and $\Gamma(M, E_1) \subset \Gamma(M, T' \otimes ({}^0T'')^*)$, we may write $\bar\partial_0 = \bar\partial_{T'} \circ \pi_{\tilde H_0}$, from which it follows that $\bar\partial_0^* = \pi_{\tilde H_0} \circ \bar\partial_{T'}^*$ on $\Gamma(M, E_1)$. That $\ker D^* = \ker \bar\partial_0^*$ is due to two simple facts: first, that $D^* = \rho^* \circ \bar\partial_0^*$; second, that $\rho\colon \Gamma(M, F) \to H_0$ is an isomorphism. $\Box$

This lemma then implies that we may write $\mathcal{H}$ as

$$\mathcal{H} = \ker\Box = \{\phi \in \Gamma(M, E_1) : \bar\partial_0^*\phi = 0 \text{ and } \bar\partial_1\phi = 0\}.$$

The subelliptic estimate in Theorem 5.1 gives us the following Hodge–Kodaira decomposition theorem.

THEOREM 5.4. *Let* $(M, {}^0T'')$ *be a compact, strictly pseudoconvex CR manifold of dimension* 5. *Then*

$$\mathcal{H} \cong \frac{\ker \bar\partial_1}{\operatorname{im} D}.$$

Moreover, there exist both a Neumann operator $N\colon \Gamma_2(M, E_1) \to \Gamma_2(M, E_1)$ *and a harmonic projector* $H\colon \Gamma_2(M, E_1) \to \mathcal{H}$ *satisfying* $NH = HN = 0$, $[N, DD^*] = 0 = [N, \bar\partial_1^* L \bar\partial_1]$, *and* $u = Hu + \Box Nu = Hu + N\Box u$ *for all* $u \in \Gamma_2(M, E_1)$.

We will construct the Neumann operator N and the harmonic projector H by considering the differential equation

$$\Box u = f. \tag{5.3}$$

Let us write $\mathcal{H}^\perp$ for elements of $\Gamma_2(M, E_1)$ that are orthogonal to $\mathcal{H}$ with respect to the L^2 norm. We begin with a fairly standard lemma.

LEMMA 5.5. *There is a constant* $c > 0$ *for which*

$$\|D^* u\|^2 + \|\bar\partial_1 u\|_1^2 \ge c\|u\|_1^2$$

for all $u \in \mathcal{H}^\perp \subset \Gamma_2(M, E_1)$.

Proof. We assume the conclusion is false. That is, for each integer $k > 0$, we assume that there is an element $u_k \in \mathcal{H}^\perp$ satisfying $\|D^* u_k\|^2 + \|\bar\partial_1 u_k\|_1^2 \le \frac{1}{k}\|u_k\|_1^2$. Rescaling these u_k if necessary, we may assume that $\|u_k\|_1 = 1$ and hence $\|D^* u_k\|^2 + \|\bar\partial_1 u_k\|_1^2 \le \frac{1}{k}$. By our estimate (5.1) (extended by continuity to $\Gamma_2(M, E_1)$), we have

$$\begin{aligned} c\|u_k\|_2^2 &\le \|D^* u_k\|^2 + \|\bar\partial_1 u_k\|_1^2 + \|u_k\|_1^2 \\ &\le \left(\frac{1}{k} + 1\right) \le 2. \end{aligned}$$

The sequence $\{u_k\}$ is thus bounded with respect to $\|\cdot\|_2$, the Folland–Stein 2-norm. Any such set is precompact with respect to $\|\cdot\|_1$; this means there is a subsequence $\{u_{k_j}\}$ that converges weakly in $\Gamma_2(M, E_1)$ and strongly in the Folland–Stein 1-norm. Let u be its limit. On the one hand, $u \in \mathcal{H}^\perp$ because each element u_{k_j} is. On the other hand, the closedness of the differential operator $\Box$ implies that $u \in \operatorname{Dom}\Box$ and $\Box u = 0$. Thus $u \in \mathcal{H}$ and so $u = 0$. But $\|u\|_1 = 1$, so this is a contradiction. $\Box$

Proof of Theorem 5.4. By Lemma 5.5 and Theorem 5.1, the quadratic form

$$Q(u, u) = \|D^* u\|^2 + \|\bar\partial_1 u\|_1^2$$

defines a norm that is equivalent to $\|\cdot\|_2$. We endow $\mathcal{H}^\perp$ with this norm and let $Q(u, v)$ denote the associated symmetric bilinear form. Note that if u and v are smooth, then $Q(u, v) = (\Box u, v)$.

By Lemma 5.5, the linear functional $v \mapsto (f, v)$ is bounded on $\mathcal{H}^\perp$ for any $f \in \Gamma_2(M, E_1)$. The Riesz representation theorem then implies that there is a unique $u \in \mathcal{H}^\perp$ such that $Q(u, v) = (f, v)$ for all $v \in \mathcal{H}^\perp$. Thus we have solved (5.3) for $f \in \mathcal{H}^\perp$.

The Neumann operator is given by $Nf = u$, the solution $u \in \mathcal{H}^\perp$ to $\Box u = f$ in the sense just described. This makes sense for $f \in \mathcal{H}^\perp$, so under the orthogonal decomposition $\Gamma_2(M, E_1) = \mathcal{H} \oplus \mathcal{H}^\perp$ we can extend N to all of $\Gamma_2(M, E_1)$ by declaring that it is identically zero on $\mathcal{H}$. We define the harmonic projector H as orthogonal projection onto $\mathcal{H}$ under this decomposition. The operators H and N project onto orthogonal spaces, so $HN = 0 = NH$. On the other hand, the decompositions $u = Hu + \Box Nu = Hu + N\Box u$ follow immediately from the construction of N and H.

To see that $[\bar\partial_1^* L \bar\partial_1, N] = 0 = [DD^*, N]$ takes a bit more work. From $[\Box, N] = 0$ it follows directly that $[\bar\partial_1^* L \bar\partial_1, N] + [DD^*, N] = 0$, so we need only show that, say, $[DD^*, N] = 0$. This follows easily by considering separately $u \in \mathcal{H}$ (on which DD^* and N are separately zero) and $u = \Box v \in \mathcal{H}^\perp$, in which case $[DD^*, N]\Box v = 0$ is a straightforward computation based on the formulas $N\Box v = v - Hv$, $[DD^*, \Box] = 0$, and $HDD^* = DD^*H = 0$.

Finally, the isomorphism $\mathcal{H} \cong \ker \bar\partial_1 / \operatorname{im} D$ follows as usual from the existence of the Neumann operator, since the harmonic projector H restricts to a map $H\colon \ker \bar\partial_1 \to \mathcal{H}$ whose kernel is exactly $\operatorname{im} D$ by the preceding arguments. □

6. Proof of the Subelliptic Estimate

In this section we prove Theorem 5.1, our subelliptic estimate. Since our manifold M is assumed to be compact, it will suffice to show that (5.1) holds for ϕ supported in a neighborhood of each point; assuming this, we can choose a locally finite collection $\{\alpha_i\}$ of smooth nonnegative functions satisfying $\sum_i \alpha_i^2 = 1$, apply (5.1) to $\alpha_i \phi$ and then sum over i, yielding (5.1) plus some lower-order terms that can be absorbed into the right-hand side.

Let $\{e_1, e_2\}$ be a local moving frame for ${}^0T'$ satisfying (4.5), from which it follows that

$$\pi_F[e_\alpha, \bar e_\beta] = -i\delta_{\alpha\beta}\xi, \tag{6.1}$$

and let $\{\theta^1, \theta^2\}$ be the dual sections of $({}^0T')^*$, viewed as 1-forms according to the decomposition (2.3). We may then write $\phi \in \Gamma(M, {}^0T' \otimes \Lambda^j({}^0T'')^*)$ in coordinates as

$$\phi = \phi^\alpha_{\beta_1, \dots, \beta_j} e_\alpha \otimes \bar\theta^{\beta_1} \wedge \cdots \wedge \bar\theta^{\beta_j}. \tag{6.2}$$

(Notice the implicit sums over α and β_1 through β_j.) Throughout this section, we will assume that ϕ is supported in the neighborhood on which our moving frame is defined, so that

$$\|\phi\|^2 = \sum_{\alpha, \beta_1, \dots, \beta_j} \|\phi^\alpha_{\beta_1, \dots, \beta_j}\|^2. \tag{6.3}$$

We will often find it useful to look only at the top-order derivatives. In light of the commutation relation (6.1), this unfortunately is not possible. Instead, we will look at only the top-*weight* derivatives, where we allocate a weight of 1 to vector fields in H and a weight of 2 to ξ. We will then write $\sim$ for "equal modulo lower-weight terms". This generalizes to $\gtrsim$ and $\lesssim$, meaning greater than or less than modulo negligible terms. Our main estimate (5.1) can thus be written

$$(u, \Box u) = \|D^*\phi\|^2 + \|\bar\partial_1\phi\|_1^2 \gtrsim c\|\phi\|_2^2$$

for all $\phi \in \Gamma(M, E_1)$. To prove this estimate, we will need a local expression for $\|\phi\|_2^2$ rather than $\|\phi\|^2$. Modulo lower-weight terms, this expression is

$$\|\phi\|_2^2 \sim \sum_{k,l,\alpha,\beta_1,\dots,\beta_j} \|e_k e_l \phi^\alpha_{\beta_1,\dots,\beta_j}\|^2,$$

where $j, k \in \{1, \dots, n-1, \bar 1, \dots, \overline{n-1}\}$.

We begin the actual proof of Theorem 5.1 by describing ϕ, $\bar\partial_1\phi$, and $D^*\phi$ in terms of our local moving frame (cf. [A3, Lemmas 3.2 and 3.3, p. 317]).

LEMMA 6.1. *Suppose* $\phi \in \Gamma(M, E_1)$. *Then* $\phi^1_{\bar 2} = \phi^2_{\bar 1}$, $(\bar\partial_1\phi)^\alpha_{\bar 1,\bar 2} \sim \bar e_1\phi^\alpha_{\bar 2} - \bar e_2\phi^\alpha_{\bar 1}$, *and*

$$D^*\phi \sim -i(e_1e_1\phi^1_{\bar 1} + e_1e_2\phi^1_{\bar 2} + e_2e_1\phi^2_{\bar 1} + e_2e_2\phi^2_{\bar 2})\xi. \tag{6.4}$$

Proof. In our local frame, we may write $\phi = \phi^\alpha_{\bar\beta} e_\alpha \otimes \bar\theta^\beta$. (Since $\Gamma(M, E_1) \subset \Gamma(M, {}^0T' \otimes ({}^0T'')^*)$, there are no $\xi \otimes \bar\theta^\beta$ terms.) In this case $\bar\partial^{(1)}\phi(\bar e_2, \bar e_2)$ is (see equation (3.2))

$$\begin{aligned}
&\bar\partial^{(1)}\phi(\bar e_1, \bar e_2)\\
&\quad = \pi'[\bar e_1, \phi(\bar e_2)] - \pi'[\bar e_2, \phi(\bar e_1)] - \phi([\bar e_1, \bar e_2])\\
&\quad = (\bar e_1\phi^\alpha_{\bar 2})e_\alpha + \phi^\alpha_{\bar 2}\pi'[\bar e_1, e_\alpha] - (\bar e_2\phi^\alpha_{\bar 1})e_\alpha - \phi^\alpha_{\bar 1}\pi'[\bar e_2, e_\alpha] - \phi([\bar e_1, \bar e_2])\\
&\quad \sim (\bar e_1\phi^\alpha_{\bar 2})e_\alpha - (\bar e_2\phi^\alpha_{\bar 1})e_\alpha,
\end{aligned} \tag{6.5}$$

where we have discarded all the terms without a derivative of a component of ϕ. This proves the second claim; the first claim follows from applying the 1-form θ to both sides of (6.5):

$$0 = \phi^\alpha_{\bar 2}\theta([\bar e_1, e_\alpha]) - \phi^\alpha_{\bar 1}\theta([\bar e_2, e_\alpha]) = i\phi^\alpha_{\bar 2}\delta_{1\alpha} - i\phi^\alpha_{\bar 1}\delta_{2\alpha} = i(\phi^1_{\bar 2} - \phi^2_{\bar 1}),$$

where we have simplified using $\theta([e_\alpha, \bar e_\beta]) = -i\delta_{\alpha\beta}$.

Finally, we prove equation (6.4). To compute this adjoint, we take the inner product of $D^*\phi$ with an element $u\xi$ of $\Gamma(M, F)$ and then integrate by parts:

$$(u\xi, D^*\phi) = (D(u\xi), \phi) \sim \big(\bar\partial_0(u\xi + i(\bar e_1u)e_1 + i(\bar e_2u)e_2), \phi\big).$$

If we write $\psi = \psi^\alpha_{\bar\beta} e_\alpha \otimes \bar\theta^\beta$ for $D(u\xi) = \bar\partial_0(u\xi + i(\bar e_1u)e_1 + i(\bar e_2u)e_2)$ (again, there is no $\xi \otimes \bar\theta^\beta$ term as $D(u\xi) \in \Gamma(M, E_1)$), then we can compute $\psi^\alpha_{\bar\beta} = \theta^\alpha(\psi(\bar e_\beta))$. The inside term is not difficult to compute, and we obtain $\psi(\bar e_\beta) \sim \pi'[\bar e_\beta, u\xi + i(\bar e_1u)e_1 + i(\bar e_2u)e_2]$, so $\psi^\alpha_{\bar\beta} \sim i\bar e_\beta \bar e_\alpha u$. Undoing the integration by parts (just displayed) yields equation (6.4). □

The primary tool in our proof of Theorem 5.1 is the following lemma. This follows at least in part from the local expressions computed in Lemma 6.1.

LEMMA 6.2 (Key Estimate). *For all* $\phi \in \Gamma(M, E_1)$,

$$\|D^*\phi\|^2 + 2\|\bar\partial_1\phi\|_1^2 \gtrsim \|e_1e_1\phi_1^1\|^2 + \|e_2e_2\phi_2^2\|^2 + 4\|e_1\bar e_2\phi_2^1\|^2 + 4\|e_2\bar e_1\phi_2^1\|^2 + \|\bar e_1\bar e_1\phi_2^2\|^2 + \|\bar e_2\bar e_2\phi_1^1\|^2. \tag{6.6}$$

Proof. We begin by computing $\|D^*\phi\|^2$. From (6.4), we have

$$\|D^*\phi\|^2 \sim \|e_1e_1\phi_1^1 + e_1e_2\phi_2^1 + e_2e_1\phi_1^2 + e_2e_2\phi_2^2\|^2.$$

We expand this to get

$$\begin{aligned}\|D^*\phi\|^2 \sim\ & \|e_1e_1\phi_1^1\|^2 + \|e_1e_2\phi_2^1\|^2 + \|e_2e_1\phi_1^2\|^2 + \|e_2e_2\phi_2^2\|^2\\ &+ 2\operatorname{Re}(e_1e_1\phi_1^1, e_1e_2\phi_2^1) + 2\operatorname{Re}(e_1e_1\phi_1^1, e_2e_1\phi_1^2)\\ &+ 2\operatorname{Re}(e_1e_1\phi_1^1, e_2e_2\phi_2^2) + 2\operatorname{Re}(e_1e_2\phi_2^1, e_2e_1\phi_1^2)\\ &+ 2\operatorname{Re}(e_1e_2\phi_2^1, e_2e_2\phi_2^2) + 2\operatorname{Re}(e_2e_1\phi_1^2, e_2e_2\phi_2^2).\end{aligned} \tag{6.7}$$

Since $\phi_2^1 = \phi_1^2$ (by Lemma 6.1) and $[e_\alpha, e_\beta] \sim 0$ for all α and β, one of the cross terms simplifies: $2\operatorname{Re}(e_1e_2\phi_2^1, e_2e_1\phi_1^2) = 2\operatorname{Re}(e_1e_2\phi_2^1, e_1e_2\phi_2^1) = 2\|e_1e_2\phi_2^1\|^2$. Four of the other cross terms combine, and (6.7) simplifies to

$$\begin{aligned}\|D^*\phi\|^2 \sim\ & \|e_1e_1\phi_1^1\|^2 + \|e_1e_2\phi_2^1\|^2 + \|e_2e_1\phi_2^1\|^2 + \|e_2e_2\phi_2^2\|^2\\ &+ 4\operatorname{Re}(e_1e_1\phi_1^1, e_1e_2\phi_2^1) + 2\operatorname{Re}(e_1e_1\phi_1^1, e_2e_2\phi_2^2)\\ &+ 2\|e_1e_2\phi_2^1\|^2 + 4\operatorname{Re}(e_1e_2\phi_2^1, e_2e_2\phi_2^2).\end{aligned} \tag{6.8}$$

We will deal with the remaining cross terms by adding $2\|\bar\partial_1\phi\|_1^2$. By Lemma 6.1,

$$\begin{aligned}2\|\bar\partial_1\phi\|_1^2 &\sim 2\|\bar e_1\phi_2^1 - \bar e_2\phi_1^1\|_1^2 + 2\|\bar e_1\phi_2^2 - \bar e_2\phi_2^1\|_1^2\\ &\sim 2\|e_1(\bar e_1\phi_2^1 - \bar e_2\phi_1^1)\|^2 + 2\|e_2(\bar e_1\phi_2^1 - \bar e_2\phi_1^1)\|^2\\ &\quad + 2\|\bar e_1(\bar e_1\phi_2^1 - \bar e_2\phi_1^1)\|^2 + 2\|\bar e_2(\bar e_1\phi_2^1 - \bar e_2\phi_1^1)\|^2\\ &\quad + 2\|e_1(\bar e_1\phi_2^2 - \bar e_2\phi_2^1)\|^2 + 2\|e_2(\bar e_1\phi_2^2 - \bar e_2\phi_2^1)\|^2\\ &\quad + 2\|\bar e_1(\bar e_1\phi_2^2 - \bar e_2\phi_2^1)\|^2 + 2\|\bar e_2(\bar e_1\phi_2^2 - \bar e_2\phi_2^1)\|^2\\ &\gtrsim 2\|e_1(\bar e_1\phi_2^1 - \bar e_2\phi_1^1)\|^2 + 2\|e_2(\bar e_1\phi_2^2 - \bar e_2\phi_2^1)\|^2\\ &\quad + 2\|\bar e_2(\bar e_1\phi_2^1 - \bar e_2\phi_1^1)\|^2 + 2\|\bar e_1(\bar e_1\phi_2^2 - \bar e_2\phi_2^1)\|^2.\end{aligned}$$

Since $[e_\alpha, \overline{e_\beta}] \sim -i\delta_{\alpha\beta}\xi$, we have

$$\begin{aligned}e_1(\bar e_1\phi_2^1 - \bar e_2\phi_1^1) &\sim -i\xi\phi_2^1 + \bar e_1e_1\phi_2^1 - e_1\bar e_2\phi_1^1,\\ e_2(\bar e_1\phi_2^2 - \bar e_2\phi_2^1) &\sim e_2\bar e_1\phi_2^2 - \bar e_2e_2\phi_2^1 + i\xi\phi_2^1.\end{aligned}$$

Moreover,

$$\begin{aligned}&2\|\bar e_2(\bar e_1\phi_2^1 - \bar e_2\phi_1^1)\|^2 + 2\|\bar e_1(\bar e_1\phi_2^2 - \bar e_2\phi_2^1)\|^2\\ &\quad\geq \|\bar e_2(\bar e_1\phi_2^1 - \bar e_2\phi_1^1) + \bar e_1(\bar e_1\phi_2^2 - \bar e_2\phi_2^1)\|^2\\ &\quad\sim \|\bar e_1\bar e_1\phi_2^2 - \bar e_2\bar e_2\phi_1^1\|^2\end{aligned}$$

as $[\bar{e}_\alpha, \bar{e}_\beta] \sim 0$. Hence

$$\begin{aligned}
2\|\bar\partial_1\phi\|_1^2 \gtrsim{}& 2\|-i\xi\phi_2^1 + \bar{e}_1 e_1\phi_2^1 - e_1\bar{e}_2\phi_1^1\|^2 \\
&+ 2\|e_2\bar{e}_1\phi_2^2 - \bar{e}_2 e_2\phi_2^1 + i\xi\phi_2^1\|^2 \\
&+ \|\bar{e}_1\bar{e}_1\phi_2^2 - \bar{e}_2\bar{e}_2\phi_1^1\|^2 \\
\sim{}& 2\|\xi\phi_2^1\|^2 + 2\|\bar{e}_1 e_1\phi_2^1\|^2 + 2\|e_1\bar{e}_2\phi_1^1\|^2 \\
&- 4\operatorname{Re}(i\xi\phi_2^1, \bar{e}_1 e_1\phi_2^1) + 4\operatorname{Re}(i\xi\phi_2^1, e_1\bar{e}_2\phi_1^1) \\
&- 4\operatorname{Re}(\bar{e}_1 e_1\phi_2^1, e_1\bar{e}_2\phi_1^1) + 2\|e_2\bar{e}_1\phi_2^2\|^2 \\
&+ 2\|\bar{e}_2 e_2\phi_2^1\|^2 + 2\|\xi\phi_2^1\|^2 - 4\operatorname{Re}(e_2\bar{e}_1\phi_2^2, \bar{e}_2 e_2\phi_2^1) \\
&+ 4\operatorname{Re}(e_2\bar{e}_1\phi_2^2, i\xi\phi_2^1) - 4\operatorname{Re}(\bar{e}_2 e_2\phi_2^1, i\xi\phi_2^1) \\
&+ \|\bar{e}_1\bar{e}_1\phi_2^2\|^2 + \|\bar{e}_2\bar{e}_2\phi_1^1\|^2 - 2\operatorname{Re}(\bar{e}_1\bar{e}_1\phi_2^2, \bar{e}_2\bar{e}_2\phi_1^1).
\end{aligned} \tag{6.9}$$

To cancel the cross terms in (6.8), we make use of the fact that e_2 commutes with $\bar{e}_1$ and e_1 modulo lower-weight terms; therefore, integrating by parts yields

$$\begin{aligned}
-4\operatorname{Re}(e_2\bar{e}_1\phi_2^2, \bar{e}_2 e_2\phi_2^1) &\sim -4\operatorname{Re}(\bar{e}_1 e_2\phi_2^2, \bar{e}_2 e_2\phi_2^1) \\
&\sim 4\operatorname{Re}(e_2\bar{e}_1 e_2\phi_2^2, e_2\phi_2^1) \\
&\sim 4\operatorname{Re}(\bar{e}_1 e_2 e_2\phi_2^2, e_2\phi_2^1) \\
&\sim -4\operatorname{Re}(e_2 e_2\phi_2^2, e_1 e_2\phi_2^1) \\
&\sim -4\operatorname{Re}(e_1 e_2\phi_2^1, e_2 e_2\phi_2^2).
\end{aligned}$$

A similar argument shows that three of the cross terms on the right-hand side of (6.9) cancel all the cross terms of (6.8):

$$\begin{aligned}
\|D^*\phi\|^2 + 2\|\bar\partial_1\phi\|_1^2 \gtrsim{}& \|e_1 e_1\phi_1^1\|^2 + \|e_1 e_2\phi_2^1\|^2 + \|e_2 e_1\phi_2^1\|^2 + \|e_2 e_2\phi_2^2\|^2 \\
&+ 2\|e_1 e_2\phi_2^1\|^2 + 2\|\xi\phi_2^1\|^2 + 2\|\bar{e}_1 e_1\phi_2^1\|^2 + 2\|e_1\bar{e}_2\phi_1^1\|^2 \\
&- 4\operatorname{Re}(i\xi\phi_2^1, \bar{e}_1 e_1\phi_2^1) + 4\operatorname{Re}(i\xi\phi_2^1, e_1\bar{e}_2\phi_1^1) \\
&+ 2\|e_2\bar{e}_1\phi_2^2\|^2 + 2\|\bar{e}_2 e_2\phi_2^1\|^2 + 2\|\xi\phi_2^1\|^2 \\
&+ 4\operatorname{Re}(e_2\bar{e}_1\phi_2^2, i\xi\phi_2^1) - 4\operatorname{Re}(\bar{e}_2 e_2\phi_2^1, i\xi\phi_2^1) \\
&+ \|\bar{e}_1\bar{e}_1\phi_2^2\|^2 + \|\bar{e}_2\bar{e}_2\phi_1^1\|^2.
\end{aligned}$$

We now have more cross terms, this time involving ξ.

We will deal with some of these cross terms using integration by parts. The adjoint of e_α is $-\bar{e}_\alpha$, and so (using $[e_2, \bar{e}_2] \sim -i\xi$ and other commutation relations) we have

$$\begin{aligned}-4\operatorname{Re}(i\xi\phi_2^1, \bar{e}_1 e_1\phi_2^1) &\sim +4\operatorname{Re}(e_2\bar{e}_2\phi_2^1, \bar{e}_1 e_1\phi_2^1) - 4\operatorname{Re}(\bar{e}_2 e_2\phi_2^1, \bar{e}_1 e_1\phi_2^1)\\ &\sim -4\operatorname{Re}(\bar{e}_2\phi_2^1, \bar{e}_2\bar{e}_1 e_1\phi_2^1) - 4\operatorname{Re}(\bar{e}_2 e_2\phi_2^1, \bar{e}_1 e_1\phi_2^1)\\ &\sim -4\operatorname{Re}(\bar{e}_2\phi_2^1, \bar{e}_1 e_1\bar{e}_2\phi_2^1) - 4\operatorname{Re}(\bar{e}_2 e_2\phi_2^1, \bar{e}_1 e_1\phi_2^1)\\ &\sim +4\|e_1\bar{e}_2\phi_2^1\|^2 - 4\operatorname{Re}(\bar{e}_2 e_2\phi_2^1, \bar{e}_1 e_1\phi_2^1).\end{aligned}$$

Similarly,

$$\begin{aligned}-4\operatorname{Re}(i\xi\phi_2^1, \bar{e}_2 e_2\phi_2^1) &\sim +4\operatorname{Re}(e_1\bar{e}_1\phi_2^1, \bar{e}_2 e_2\phi_2^1) - 4\operatorname{Re}(\bar{e}_1 e_1\phi_2^1, \bar{e}_2 e_2\phi_2^1)\\ &\sim +4\|e_2\bar{e}_1\phi_2^1\|^2 - 4\|e_2 e_1\phi_2^1\|^2.\end{aligned}$$

Thus

$$\begin{aligned}&\|D^*\phi\|^2 + 2\|\bar{\partial}_1\phi\|_1^2\\ &\quad\gtrsim \|e_1e_1\phi_1^1\|^2 + \|e_1e_2\phi_2^1\|^2 + \|e_2e_1\phi_2^1\|^2 + \|e_2e_2\phi_2^2\|^2\\ &\qquad + 2\|e_1e_2\phi_2^1\|^2 + 2\|\xi\phi_2^1\|^2 + 2\|\bar{e}_1e_1\phi_2^1\|^2 + 2\|e_1\bar{e}_2\phi_1^1\|^2\\ &\qquad + 4\|e_1\bar{e}_2\phi_2^1\|^2 - 4\operatorname{Re}(\bar{e}_2e_2\phi_2^1, \bar{e}_1e_1\phi_2^1) + 4\operatorname{Re}(i\xi\phi_2^1, e_1\bar{e}_2\phi_1^1)\\ &\qquad + 2\|e_2\bar{e}_1\phi_2^2\|^2 + 2\|\bar{e}_2e_2\phi_2^1\|^2 + 2\|\xi\phi_2^1\|^2\\ &\qquad + 4\operatorname{Re}(e_2\bar{e}_1\phi_2^2, i\xi\phi_2^1) + 4\|e_2\bar{e}_1\phi_2^1\|^2 - 4\|e_2e_1\phi_2^1\|^2\\ &\qquad + \|\bar{e}_1\bar{e}_1\phi_2^2\|^2 + \|\bar{e}_2\bar{e}_2\phi_1^1\|^2\\ &\quad\sim \|e_1e_1\phi_1^1\|^2 + \|e_2e_2\phi_2^2\|^2 + 4\|e_1\bar{e}_2\phi_2^1\|^2\\ &\qquad + 4\|e_2\bar{e}_1\phi_2^1\|^2 + \|\bar{e}_1\bar{e}_1\phi_2^2\|^2 + \|\bar{e}_2\bar{e}_2\phi_1^1\|^2\\ &\qquad + \left(2\|\bar{e}_2e_2\phi_2^1\|^2 - 4\operatorname{Re}(\bar{e}_2e_2\phi_2^1, \bar{e}_1e_1\phi_2^1) + 2\|\bar{e}_1e_1\phi_2^1\|^2\right)\\ &\qquad + \left(2\|\xi\phi_2^1\|^2 + 4\operatorname{Re}(i\xi\phi_2^1, e_1\bar{e}_2\phi_1^1) + 2\|e_1\bar{e}_2\phi_1^1\|^2\right)\\ &\qquad + \left(2\|e_2\bar{e}_1\phi_2^2\|^2 + 4\operatorname{Re}(e_2\bar{e}_1\phi_2^2, i\xi\phi_2^1) + 2\|\xi\phi_2^1\|^2\right).\end{aligned} \tag{6.10}$$

Now the three parts grouped in parentheses can be removed by the Schwarz inequality. This gives us

$$\begin{aligned}\|D^*\phi\|^2 + 2\|\bar{\partial}_1\phi\|_1^2 \gtrsim{}& \|e_1e_1\phi_1^1\|^2 + \|e_2e_2\phi_2^2\|^2 + 4\|e_1\bar{e}_2\phi_2^1\|^2\\ &+ 4\|e_2\bar{e}_1\phi_2^1\|^2 + \|\bar{e}_1\bar{e}_1\phi_2^2\|^2 + \|\bar{e}_2\bar{e}_2\phi_1^1\|^2,\end{aligned}$$

which is equation (6.6). This concludes the proof of the Key Estimate. □

Now, to prove Theorem 5.1 we need an estimate:

$$\|D^*\phi\|^2 + \|\bar{\partial}_1\phi\|_1^2 \gtrsim c\|\phi\|_2^2. \tag{6.11}$$

In our local frame, the right-hand side of this equation can be written as

$$c\|\phi\|_2^2 \sim c\sum_{\alpha,\beta,j,k}\|e_je_k\phi_\beta^\alpha\|^2,$$

where α and β run from 1 to 2 and where $j, k \in \{1, 2, \bar{1}, \bar{2}\}$. We construct each of these estimates individually and then organize them in the following lemma.

LEMMA 6.3. *There exists a positive constant C such that*

$$\|D^*\phi\|^2 + \|\bar{\partial}_1\phi\|_1^2 \gtrsim C\|e_j e_k \phi_\beta^\alpha\|^2 \tag{6.12}$$

for all $j, k \in \{1, 2, \bar{1}, \bar{2}\}$, all $\alpha, \beta \in \{1, 2\}$, and all $\phi \in \Gamma(M, E_2)$.

Proof. What we will show, in fact, is that for each j, k and each $\varepsilon > 0$ there exists a constant $C > 0$ such that

$$\|D^*\phi\|^2 + \|\bar{\partial}_1\phi\|_1^2 + \varepsilon\|\phi\|_2^2 \gtrsim C\|e_j e_k \phi_\beta^\alpha\|^2. \tag{6.13}$$

The constant ε can be chosen to be dominated by all the different constants C, so that the sum of the various individual estimates (6.12) and (6.13) yields the subelliptic estimate (6.11).

We prove this lemma in stages: we produce the estimate (6.12) for each of the components ϕ_2^1, ϕ_1^2, ϕ_1^1, and ϕ_2^2 in turn.

The ϕ_2^1 Case. We begin by noting that we already have the estimate (6.12) for $\|\bar{e}_2 e_1 \phi_2^1\|^2 \sim \|e_1 \bar{e}_2 \phi_2^1\|^2$ and $\|\bar{e}_1 e_2 \phi_2^1\|^2 \sim \|e_2 \bar{e}_1 \phi_2^1\|^2$ by the Key Estimate, Lemma 6.2.

Now consider the part of inequality (6.10) that we discarded in the last step of the proof of the Key Estimate:

$$\begin{aligned} &\|D^*\phi\|^2 + 2\|\bar{\partial}_1\phi\|_1^2 \\ &\quad\gtrsim \left(2\|\xi\phi_2^1\|^2 + 4\operatorname{Re}(i\xi\phi_2^1, e_1\bar{e}_2\phi_1^1) + 2\|e_1\bar{e}_2\phi_1^1\|^2\right) \\ &\qquad + \left(2\|e_2\bar{e}_1\phi_2^2\|^2 + 4\operatorname{Re}(e_2\bar{e}_1\phi_2^2, i\xi\phi_2^1) + 2\|\xi\phi_2^1\|^2\right). \end{aligned} \tag{6.14}$$

Notice that, since $[\xi, e_j] \sim 0$ and $(i\xi)^* \sim i\xi$,

$$\begin{aligned} |{+4}\operatorname{Re}(i\xi\phi_2^1, e_1\bar{e}_2\phi_1^1)| &\sim |{+4}\operatorname{Re}(e_2\bar{e}_1\phi_2^1, i\xi\phi_1^1)| \\ &\lesssim 2\left(\frac{1}{\varepsilon}\|e_2\bar{e}_1\phi_2^1\|^2 + \varepsilon\|i\xi\phi_1^1\|^2\right) \\ &\lesssim 2\left(\frac{1}{\varepsilon}\|e_2\bar{e}_1\phi_2^1\|^2 + \varepsilon\|\phi\|_2^2\right) \end{aligned}$$

for any $\varepsilon > 0$. Since we have already estimated $\|e_2\bar{e}_1\phi_2^1\|^2$, this allows us to obtain an estimate

$$c\operatorname{Re}(i\xi\phi_2^1, e_1\bar{e}_2\phi_1^1) \lesssim \|D^*\phi\|^2 + \|\bar{\partial}_1\phi\|_1^2 + \varepsilon\|\phi\|_2^2$$

for some $c > 0$. Similarly, we can obtain an estimate

$$c\operatorname{Re}(i\xi\phi_2^1, e_2\bar{e}_1\phi_2^2) \lesssim \|D^*\phi\|^2 + \|\bar{\partial}_1\phi\|_1^2 + \varepsilon\|\phi\|_2^2$$

for some $c > 0$. From these estimates and inequality (6.14), we obtain estimates for $\|i\xi\phi_2^1\|^2$, $\|e_1\bar{e}_2\phi_1^1\|^2 \sim \|\bar{e}_2 e_1\phi_1^1\|^2$, and $\|e_2\bar{e}_1\phi_2^2\|^2 \sim \|\bar{e}_1 e_2\phi_2^2\|^2$.

We again return to a term that was discarded at the end of the proof of the Key Estimate: we have

$$\|D^*\phi\|^2 + 2\|\bar\partial_1\phi\|_1^2 \gtrsim 2\|\bar e_2 e_2 \phi_2^1\|^2 - 4\,\mathrm{Re}(\bar e_2 e_2 \phi_2^1, \bar e_1 e_1 \phi_2^1) + 2\|\bar e_1 e_1 \phi_2^1\|^2.$$

We may rewrite part of this as

$$\begin{aligned}|-4\,\mathrm{Re}(\bar e_2 e_2 \phi_2^1, \bar e_1 e_1 \phi_2^1)| &\sim |-4\,\mathrm{Re}(i\xi\phi_2^1, \bar e_1 e_1 \phi_2^1) - 4\,\mathrm{Re}(e_2 \bar e_2 \phi_2^1, \bar e_1 e_1 \phi_2^1)|\\ &\sim |-4\,\mathrm{Re}(i\xi\phi_2^1, \bar e_1 e_1 \phi_2^1) - 4\|e_1 \bar e_2 \phi_2^1\|^2|\\ &\lesssim |4\,\mathrm{Re}(i\xi\phi_2^1, \bar e_1 e_1 \phi_2^1)| + 4\|e_1 \bar e_2 \phi_2^1\|^2.\end{aligned}$$

In the same way as before, we can control the inner product on the right. Because we already have an estimate for $\|e_1\bar e_2\phi_2^1\|^2$, we proceed to get estimates for $\|\bar e_2 e_2 \phi_2^1\|^2$ and $\|\bar e_1 e_1 \phi_2^1\|^2$.

We can integrate by parts to write

$$\begin{aligned}\|e_1 e_2 \phi_2^1\|^2 &\sim \|e_1 \bar e_2 \phi_2^1\|^2 - \mathrm{Re}(i\xi\phi_2^1, \bar e_1 e_1 \phi_2^1)\\ &\lesssim \|e_1 \bar e_2 \phi_2^1\|^2 + \|i\xi\phi_2^1\|^2 + \|\bar e_1 e_1 \phi_2^1\|^2.\end{aligned}$$

The previous estimates for the terms on the the right-hand side of this inequality then establish estimates for $\|e_1 e_2 \phi_2^1\|^2 \sim \|e_2 e_1 \phi_2^1\|^2$.

We use that $e_\alpha \bar e_\alpha \sim \bar e_\alpha e_\alpha - i\xi$ to obtain

$$\|e_\alpha \bar e_\alpha \phi_2^1\|^2 \lesssim 2(\|\bar e_\alpha e_\alpha \phi_2^1\|^2 + \|i\xi\phi_2^1\|^2),$$

which gives us estimates for $\|e_1 \bar e_1 \phi_2^1\|^2$ and $\|e_2 \bar e_2 \phi_2^1\|^2$.

Using integration by parts, we derive the equality

$$\|\bar e_1 e_2 \phi_2^1\|^2 + \|e_2 e_2 \phi_2^1\|^2 \sim \|e_1 e_2 \phi_2^1\|^2 + \|\bar e_2 e_2 \phi_2^1\|^2.$$

We thus obtain an estimate on $\|e_2 e_2 \phi_2^1\|^2$ from the estimates on $\|e_1 e_2 \phi_2^1\|^2$ and $\|\bar e_2 e_2 \phi_2^1\|^2$. Using this same trick, we have

$$\|e_1 e_1 \phi_2^1\|^2 + \|\bar e_2 e_1 \phi_2^1\|^2 \sim \|\bar e_1 e_1 \phi_2^1\|^2 + \|e_2 e_1 \phi_2^1\|^2,$$

and we obtain an estimate on $\|e_1 e_1 \phi_2^1\|^2$.

Using Lemma 6.1 for the local expression of $\bar\partial_1\phi$, we have

$$\begin{aligned}\|\bar\partial_1\phi\|_1^2 &\gtrsim \|\bar e_1(\bar e_1 \phi_2^1 - \bar e_2 \phi_1^1)\|^2\\ &\sim \|\bar e_1 \bar e_1 \phi_2^1\|^2 - 2\,\mathrm{Re}(\bar e_1 \bar e_1 \phi_2^1, \bar e_1 \bar e_2 \phi_1^1) + \|\bar e_1 \bar e_2 \phi_1^1\|^2.\end{aligned}$$

On the other hand,

$$\begin{aligned}|-2\,\mathrm{Re}(\bar e_1 \bar e_1 \phi_2^1, \bar e_1 \bar e_2 \phi_1^1)| &\sim |-2\,\mathrm{Re}(\bar e_1 e_2 \phi_2^1, e_1 \bar e_1 \phi_1^1)|\\ &\lesssim \varepsilon\|\phi\|_2^2 + \frac{1}{\varepsilon}\|\bar e_1 e_2 \phi_2^1\|^2.\end{aligned}$$

Since we've already estimated $\|\bar e_1 e_2 \phi_2^1\|^2$, this gives us an estimate on $\|\bar e_1 \bar e_1 \phi_2^1\|^2$ and $\|\bar e_1 \bar e_2 \phi_1^1\|^2 \sim \|\bar e_2 \bar e_1 \phi_1^1\|^2$. Similarly, we may use

$$\|\bar\partial_1\phi\|_1^2 \gtrsim \|\bar e_2(\bar e_1 \phi_2^1 - \bar e_2 \phi_1^1)\|^2$$

and

$$|-2\,\mathrm{Re}(\bar e_2\bar e_1\phi^1_2, \bar e_2\bar e_2\phi^1_1)| \lesssim \varepsilon\|\phi\|_2^2 + \frac{1}{\varepsilon}\|e_1\bar e_2\phi^1_2\|^2$$

to obtain estimates on $\|\bar e_1\bar e_2\phi^1_2\|^2 \sim \|\bar e_2\bar e_1\phi^1_2\|^2$. This completes the proof of the ϕ^1_2 case of Lemma 6.3.

The ϕ^2_1 Case. Recall that $\phi^2_1 = \phi^1_2$ by Lemma 6.1, so this case follows from the ϕ^1_2 case.

The ϕ^1_1 Case. We begin by recalling that we have our estimate for $\|e_1e_1\phi^1_1\|^2$ and $\|\bar e_2\bar e_2\phi^1_1\|^2$ by the Key Estimate, Lemma 6.2. We also remark that we have estimated $\|\bar e_1\bar e_2\phi^1_1\|^2 \sim \|\bar e_2\bar e_1\phi^1_1\|^2$ in our proof of the ϕ^1_2 case of Lemma 6.3.

In our proof of the Key Estimate (Lemma 6.2), we did not use the fact that

$$\|\bar\partial_1\phi\|_1^2 \gtrsim \|e_1(\bar e_1\phi^1_2 - \bar e_2\phi^1_1)\|^2 + \|e_2(\bar e_1\phi^1_2 - \bar e_2\phi^1_1)\|^2,$$

from which it follows that

$$\begin{aligned}\|D^*\phi\|^2 + \|\bar\partial_1\phi\|_1^2 \gtrsim{}& \|e_1\bar e_1\phi^1_2\|^2 - 2\,\mathrm{Re}(e_1\bar e_1\phi^1_2, e_1\bar e_2\phi^1_1)\\ &+ \|e_1\bar e_2\phi^1_1\|^2 + \|e_2\bar e_1\phi^1_2\|^2\\ &- 2\,\mathrm{Re}(e_2\bar e_1\phi^1_2, e_2\bar e_2\phi^1_1) + \|e_2\bar e_2\phi^1_1\|^2.\end{aligned}$$

Using the same method as in the proof of the ϕ^1_2 case—and noting that we have estimates for all of the ϕ^1_2 terms—we obtain estimates for $\|e_1\bar e_2\phi^1_1\|^2$ and $\|e_2\bar e_2\phi^1_1\|^2$.

Using our integration-by-parts trick, we see that

$$\|\bar e_1e_1\phi^1_1\|^2 + \|e_2e_1\phi^1_1\|^2 = \|e_1e_1\phi^1_1\|^2 + \|\bar e_2e_1\phi^1_1\|^2.$$

We have estimates for both terms on the right-hand side, so this gives us estimates for $\|\bar e_1e_1\phi^1_1\|^2$ and $\|e_2e_1\phi^1_1\|^2 \sim \|e_1e_2\phi^1_1\|^2$.

Now we produce an estimate for $\|i\xi\phi^1_1\|^2$. We can write $i\xi \sim [\bar e_\alpha, e_\alpha]$ for $\alpha = 1, 2$, so integration by parts yields

$$\begin{aligned}\|i\xi\phi^1_1\|^2 &\sim (\bar e_1e_1\phi^1_1 - e_1\bar e_1\phi^1_1, \bar e_2e_2\phi^1_1 - e_2\bar e_2\phi^1_1)\\ &\sim (\bar e_1e_1\phi^1_1, \bar e_2e_2\phi^1_1) - (e_1\bar e_1\phi^1_1, e_2\bar e_2\phi^1_1)\\ &\quad - (\bar e_1e_1\phi^1_1, \bar e_2e_2\phi^1_1) + (e_1\bar e_1\phi^1_1, e_2\bar e_2\phi^1_1)\\ &\sim \|e_1e_2\phi^1_1\|^2 - \|e_1\bar e_2\phi^1_1\|^2 - \|\bar e_1e_2\phi^1_1\|^2 + \|\bar e_1\bar e_2\phi^1_1\|^2\\ &\lesssim \|e_1e_2\phi^1_1\|^2 + \|\bar e_1\bar e_2\phi^1_1\|^2.\end{aligned}$$

This gives us an estimate on $\|i\xi\phi^1_1\|^2$.

Since $e_1\bar e_1\phi^1_1 \sim \bar e_1e_1\phi^1_1 - i\xi\phi^1_1$, we have $\|e_1\bar e_1\phi^1_1\|^2 \lesssim 2(\|\bar e_1e_1\phi^1_1\|^2 + \|i\xi\phi^1_1\|^2)$ and an estimate on $\|e_1\bar e_1\phi^1_1\|^2$. Similarly, $\|\bar e_2e_2\phi^1_1\|^2 \lesssim 2(\|e_2\bar e_2\phi^1_1\|^2 + \|i\xi\phi^1_1\|^2)$ and we may estimate $\|\bar e_2e_2\phi^1_1\|^2$.

Finally, integration by parts gives us the equalities

$$\|\bar e_1\bar e_1\phi^1_1\|^2 + \|e_2\bar e_1\phi^1_1\|^2 \sim \|e_1\bar e_1\phi^1_1\|^2 + \|\bar e_2\bar e_1\phi^1_1\|^2$$

and

$$\|e_2e_2\phi_1^1\|^2 + \|\bar{e}_1e_2\phi_1^1\|^2 \sim \|\bar{e}_2e_2\phi_1^1\|^2 + \|e_1e_2\phi_1^1\|^2,$$

which allow us to estimate $\|\bar{e}_1\bar{e}_1\phi_1^1\|^2$, $\|e_2\bar{e}_1\phi_1^1\|^2 \sim \|\bar{e}_1e_2\phi_1^1\|^2$, and $\|e_2e_2\phi_1^1\|^2$. This is the last of the required ϕ_1^1 estimates and so completes the proof of the ϕ_1^1 case of Lemma 6.3.

The ϕ_2^2 Case. It is simplest to notice the symmetry between the ϕ_2^2 case and the ϕ_1^1 case. For example, the Key Estimate gives us an estimate on $\|e_1e_1\phi_1^1\|^2$ and $\|e_2e_2\phi_2^2\|^2$ as well as on $\|\bar{e}_2\bar{e}_2\phi_1^1\|^2$ and $\|\bar{e}_1\bar{e}_1\phi_2^2\|^2$. Making the appropriate changes in the proof of the ϕ_1^1 case will then give us a proof in this case as well. As this is the final case, we have now completed the proof of Lemma 6.3. □

7. A Family of CR Structures

In this section, we introduce an explicit family of CR structures parameterized by a finite-dimensional analytic set and show that it gives a local family of solutions to the deformation problem

$$\left.\begin{aligned} P(\phi) &= 0, \\ \bar{\partial}_0^*\phi &= 0. \end{aligned}\right\} \tag{7.1}$$

We begin by saying precisely what we mean by a family of CR structures. Let $(M, {}^0T'')$ be a compact strictly pseudoconvex CR manifold of real dimension $2n-1$. By a *family of deformations* of a given CR structure ${}^0T''$ we mean a triple $(M, {}^{\phi(t)}T'', T)$, where $T \subset \mathbf{C}^k$ is a complex analytic subset containing the origin o and where $\phi\colon T \to \Gamma(M, T' \otimes ({}^0T'')^*)$ is a complex analytic map such that, for each $t \in T$, $\phi(t)$ determines an integrable CR structure ${}^{\phi(t)}T''$ on M. Recall that this means $P(\phi(t)) = 0$ for all $t \in T$, since P is the integrability condition for CR structures at finite distance from ${}^0T''$. Finally, we require that $\phi(o) = 0$; in other words, that $\phi(o)$ corresponds to the original CR structure ${}^0T''$. Then our main result of this section is the following theorem.

THEOREM 7.1. *Let $(M, {}^0T'')$ be a compact, strictly pseudoconvex CR manifold of real dimension 5, and write $\mathcal{H} = \ker\Box$ for the set of harmonic elements of $\Gamma(M, E_1)$. Then there is a complex analytic map $\phi\colon \Gamma(M, E_1) \to \Gamma(M, E_1)$ defined in a neighborhood of zero such that, if*

$$T = \{t \in \mathcal{H} : R_2(\phi(t)) = \bar{\partial}_1 N \bar{\partial}_1^* L R_2(\phi(t))\}, \tag{7.2}$$

then $(M, {}^{\phi(t)}T'', T)$ is a family of deformations of ${}^0T''$.

We will prove this theorem by constructing a locally complex analytic family of solutions to the deformation problem (7.1). We begin by producing some useful Sobolev estimates.

Our Laplacian $\Box$ is a fourth-order differential operator, so we can expect that the Neumann operator gains four derivatives in the directions of $\mathbf{C} \otimes H = {}^0T' \oplus {}^0T''$. This is the content of the following lemma.

LEMMA 7.2. *Let $(M, {}^0T'')$ be a compact, strictly pseudoconvex CR manifold of dimension 5. For each integer $m \geq 0$, there exists a constant $c_m > 0$ such that*

$$\|N\psi\|_{4,m} \le c_m \|\psi\|_{0,m}$$

for all $\psi \in \Gamma(M, E_1)$.

Proof. We will show that

$$\|u\|_{4,m} \le c_m \|\Box u\|_{0,m} \tag{7.3}$$

whenever $u \in \mathcal{H}^\perp \cap \Gamma(M, E_1)$. Because $\Box$ is subelliptic, Nu is smooth whenever u is smooth, so the required estimate follows by approximating with smooth sections.

The proof of (7.3) is by induction on m. By using a partition of unity we may assume that u is supported in the domain of a frame satisfying (6.1). Observe that Lemma 5.5 and the Cauchy–Schwartz inequality imply that $\|u\| \le \|\Box u\|$. As usual, we will let $\sim$ and $\lesssim$ denote equality and inequality modulo lower-weight terms, which can be absorbed by using standard interpolation inequalities.

We begin by considering derivatives in the ξ direction. By Lemma 5.5 and Theorem 5.1,

$$\begin{aligned}\|\xi u\|_2^2 &\lesssim (\xi u, \Box \xi u)\\ &\sim (\xi u, \xi \Box u + [\Box, \xi] u).\end{aligned}$$

Because ξ commutes with e_α and $\bar{e}_\beta$ modulo terms of weight 1, it follows that $[\Box, \xi]$ is an operator of weight at most 4. Therefore, after integrating by parts, the second term just displayed can be absorbed to yield

$$\|\xi u\|_2^2 \lesssim \|u\|_4 \|\Box u\| \lesssim \varepsilon \|u\|_4^2 + \frac{1}{\varepsilon}\|\Box u\|^2. \tag{7.4}$$

Now we can prove (7.3) for the case $m = 0$. Observe that the commutation relations for e_α and $\bar{e}_\beta$ imply that $[e_\alpha, L]$ is equal to a constant multiple of $e_\alpha \xi$ modulo lower-weight terms. Therefore, using Lemma 5.5 and Theorem 5.1 again, we have

$$\begin{aligned}\|u\|_4^2 &\sim \|Lu\|_2^2\\ &\lesssim (Lu, \Box Lu)\\ &\lesssim (Lu, L\Box u) + (Lu, P_4 \xi u),\end{aligned}$$

where P_4 is some operator of weight 4. Integrating by parts and using (7.4), we find

$$\|u\|_4^2 \lesssim \|u\|_4 \|\Box u\| + \|u\|_4 \|\xi u\|_2,$$

so

$$\begin{aligned}\|u\|_4 &\lesssim \|\Box u\| + \|\xi u\|_2\\ &\lesssim \varepsilon \|u\|_4 + \frac{1}{\varepsilon}\|\Box u\|.\end{aligned}$$

Choosing ε small enough, we can absorb the $\|u\|_4$ term and obtain (7.3) when $m = 0$.

Now assume that (7.3) holds for some $m > 0$. By induction, we have

$$\begin{aligned}\|\xi u\|_{4,m} &\lesssim \|\Box \xi u\|_{0,m}\\ &\lesssim \|\xi \Box u\|_{0,m} + \|[\Box, \xi]u\|_{0,m}\\ &\lesssim \|\Box u\|_{0,m+1} + \|u\|_{4,m}\\ &\lesssim \|\Box u\|_{0,m+1}.\end{aligned}$$

If e denotes any of the vector fields e_α or $\bar{e}_\beta$, then $[\Box, e] = P_3\xi + P_4$, where P_3 and P_4 are operators of weight 3 and 4, respectively. Thus

$$\begin{aligned}\|eu\|_{4,m} &\lesssim \|\Box eu\|_{0,m}\\ &\lesssim \|e\Box u\|_{0,m} + \|[\Box, e]u\|_{0,m}\\ &\lesssim \|\Box u\|_{0,m+1} + \|\xi u\|_{3,m} + \|u\|_{4,m}\\ &\lesssim \|\Box u\|_{0,m+1}.\end{aligned}$$

Since $\|u\|_{4,m+1}$ is a sum of terms of the form $\|\xi u\|_{4,m}$ and $\|eu\|_{4,m}$, this completes the induction. □

Recall that, for $\phi \in \Gamma(M, E_1)$, the almost CR structure ${}^\phi T''$ is integrable exactly when $P(\phi) = \bar\partial_1\phi + R_2(\phi) = 0$. With this in mind, we state the following proposition (cf. [A2, Prop. 3.12, p. 813]).

PROPOSITION 7.3. *Let $(M, {}^0T'')$ be a compact, strictly pseudoconvex CR manifold of dimension* 5. *Then, for each positive integer $m \geq n$, there exists a positive constant $\tilde{c}_m$ such that*

$$\|\bar\partial_1^* L R_2(\phi)\|_{0,m} \leq \tilde{c}_m \|\phi\|_{4,m}^2$$

for all $\phi \in \Gamma(M, E_1)$.

Proof. The proof of this proposition is simply the fact that $\bar\partial_1^*$, L, and R_2 take derivatives only in the $\mathbf{C} \otimes H$ directions; thus $\bar\partial_1^* L R_2(\phi)$ can be written in a local frame for ${}^0T'$ as a homogeneous quadratic polynomial (in the coefficients of ϕ and their derivatives) in which each monomial has a total of no more than four $\mathbf{C} \otimes H$ derivatives. The assumption that $m \geq n$ and the Sobolev embedding theorem then yield the result. □

Thus Proposition 7.3 combined with Lemma 7.2 in the case $\psi = \bar\partial_1^* L R_2(\phi)$ yields the following theorem.

THEOREM 7.4. *Let $(M, {}^0T'')$ be a compact, strictly pseudoconvex CR manifold of dimension* 5. *For each integer $m \geq n$, there exists a constant $\hat{c}_m > 0$ such that*

$$\|N\bar\partial_1^* L R_2(\phi)\|_{4,m} \leq \hat{c}_m \|\phi\|_{4,m}^2$$

for all $\phi \in \Gamma(M, E_1)$.

We now use Theorem 7.4 to prove the main theorem of this section, Theorem 7.1.

Proof of Theorem 7.1. We will solve this problem first in a Banach space. Complete $\Gamma(M, E_1)$ with respect to the norm $\|\cdot\|_{2,m}$ for some integer $m \geq n$ to obtain a Banach space, which we denote by $\Gamma_{2,m}(M, E_1)$. Consider the Banach analytic map from $\Gamma_{2,m}(M, E_1)$ to itself given by

$$\phi \mapsto \phi + N\bar{\partial}_1^* LR_2(\phi).$$

Theorem 7.4 implies that $\phi \in \Gamma_{2,m}(M, E_1)$ is actually mapped to another element of $\Gamma_{2,m}(M, E_1)$. This is clearly an analytic local isomorphism. The Banach inverse mapping theorem then gives us an analytic inverse map, that is, an analytic function $s \mapsto \phi(s)$ from $\Gamma_{2,m}(M, E_1)$ to itself such that

$$\phi(s) + N\bar{\partial}_1^* LR_2(\phi(s)) = s, \quad s \in \Gamma_{2,m}(M, E_1). \tag{7.5}$$

Our family (7.5) is locally (near the origin o) parameterized by the analytic set T defined in (7.2). To see this precisely, notice that equation (7.5) implies that, for $t \in \mathcal{H}$,

$$\bar{\partial}_1\phi(t) + \bar{\partial}_1 N\bar{\partial}_1^* LR_2(\phi(t)) = 0 \tag{7.6}$$

(because $\bar{\partial}_1 = 0$ on $\mathcal{H}$). Combining this with the definition of T, we see that

$$T = \{t \in \mathcal{H} : P(\phi(t)) = 0\}.$$

Since $\phi(t)$ depends complex analytically on $t \in T$, our T is a complex analytic subset of $\mathcal{H}$. □

8. Proof of Versality

In this section we prove that the family of CR structures constructed in Theorem 7.1 is versal—at least with respect to deformations of the complex structure parameterized by smooth complex manifolds. In order to define the notion of versality, we first make clear our definition of deformations of a complex manifold U. (In practice, U will be a complex neighborhood of our CR manifold M, which is embedded as a hypersurface in a complex manifold N.) A *family of deformations* of the complex manifold U is a triple $(\mathcal{U}, \pi, S)$—with $S \subset \mathbf{C}^k$ a complex analytic subset containing the origin o, $\mathcal{U}$ a complex analytic space, and $\pi : \mathcal{U} \to S$ a complex analytic mapping—such that there exists a diffeomorphism $q : U \times S \to \mathcal{U}$ satisfying $\pi \circ q = \pi_2 : U \times S \to S$, where π_2 is the projection onto the second factor.

Such a family of deformations gives rise to a family of smooth embeddings $\varepsilon_s : U \to \mathcal{U}$ defined by $\varepsilon_s(x) = q(x, s)$ for each $s \in S$. The image of ε_s is the fiber $\pi^{-1}(s)$, which is a complex analytic submanifold of $\mathcal{U}$. Therefore, each such embedding in turn induces an integrable complex structure on U, which we denote by ${}^{\omega(s)}T''$, and (provided s is sufficiently near o) a corresponding $T'U$-valued 1-form $\omega(s) \in \Gamma(U, T'U \otimes (T''U)^*)$ that depends complex analytically on S and is defined by

$${}^{\omega(s)}T'' = \{\bar{X} + \omega(s)(\bar{X}) : \bar{X} \in T''U\}.$$

Conversely, by the Newlander–Nirenberg theorem, if such an $\omega(s)$ is given (at least in the case in which S is nonsingular) then we can construct a family of deformations $(\mathcal{U}, \pi, S)$ of the complex manifold U.

Now suppose $(M, {}^0T'')$ is a strictly pseudoconvex CR manifold. A family of deformations $(M, {}^{\phi(t)}T'', T)$ of CR structures over M is said to be *versal* if, whenever $(M, {}^0T'')$ is embedded as a real hypersurface in an n-dimensional complex manifold N and $(\mathcal{U}, \pi, S)$ is any deformation of the complex structure on a neighborhood U of M in N, we have the following two conditions. First, there exists a neighborhood of the origin $S' \subset S$ for which there is a holomorphic map $h\colon S' \to T$ and smooth embeddings $f(s)\colon M \to \pi^{-1}(s)$ for all $s \in S'$ such that $h(o) = o$ and $f(o)$ is the identity map. Second, we note that $\omega(s)$ induces a CR structure over M when we consider M embedded in U via $f(s)$. Let us denote this CR structure by ${}^{\omega(s)\cdot f(s)}T''$. If s is sufficiently close to the origin, this defines a unique deformation tensor $\omega(s) \cdot f(s) \in \Gamma(M, T' \otimes ({}^0T'')^*)$ by

$$ {}^{\omega(s)\cdot f(s)}T'' = \{\bar{X} + (\omega(s) \cdot f(s))(\bar{X}) : \bar{X} \in {}^0T''\}. \tag{8.1} $$

Our requirement is that this CR structure be the same as the one induced by ϕ at the point $h(s) \in T$:

$$ \omega(s) \cdot f(s) = \phi(h(s)) \quad \text{for all } \ s \in S'. $$

We will deal only with smooth deformations—that is, deformations in which the analytic space S is actually a complex manifold rather than a variety with singularities.

We now state our main theorem of this section.

THEOREM 8.1. *Suppose* $(M, {}^0T'')$ *is a compact strictly pseudoconvex CR manifold of real dimension* $2n - 1 = 5$ *that is embedded as a real hypersurface in a complex manifold* N *of complex dimension* $n = 3$. *If the family of CR deformations* $(M, {}^{\phi(t)}T'', T)$ *is a smooth family of deformations, then it is versal with respect to smooth deformations* (*that is, with respect to deformations* $(\mathcal{U}, \pi, S)$ *of a neighborhood* U *of* M *in* N, *where the analytic space* S *is a complex manifold*).

Our proof can be modified to work for when S has a singularity, in which case the claim would be that the family of CR deformations is versal. We leave this claim to another paper.

Proof. We must construct $h(s)$ and $f(s)$. Suppose we are given a family of deformations of a neighborhood U of M, $(\mathcal{U}, \pi, S)$. Let $\{U_j\}$ be a covering of U by coordinate domains, indexed by some finite set. Let $\{z_j^1, z_j^2, z_j^3\}$ be local holomorphic coordinates on U_j, and let $\tau_{jk}^l(z_k^1, z_k^2, z_k^3)$ be transition functions:

$$ z_j^l = \tau_{jk}^l(z_k^1, z_k^2, z_k^3), \quad l = 1, 2, 3 \ \text{ on } U_j \cap U_k. $$

For brevity, we will write this as

$$ z_j = \tau_{jk}(z_k) \quad \text{on } \ U_j \cap U_k. $$

We can extend this to a local coordinate covering $\{U_j \times S\}$ for $(\mathcal{U}, \pi, S)$ with transition functions $\tau^l_{jk}(z^1_k, z^2_k, z^3_k, s)$ defined on $U_j \times S \cap U_k \times S$, holomorphic in z^j_k and smooth in s. We then use a similar abbreviation as before:

$$z_j = \tau_{jk}(z_k, s) \quad \text{on } U_j \times S \cap U_k \times S,$$

with the requirement that $\tau_{jk}(z_k, o) = \tau_{jk}(z_k)$. For simplicity, we use local complex coordinates $\{z^1_j(s), z^2_j(s), z^3_j(s)\}$ depending complex analytically on the parameter s. That is, each function $z^k_j(s)$ is a smooth function on U_j and complex analytic on S, and the corresponding complex structure on $\pi^{-1}(s)$ (as an element of $\Gamma(U, T'U \otimes (T''U)^*)$) is determined by

$$(\bar{X} + \omega(s)(\bar{X}))z^k_j(s) = 0 \quad \text{for all } \bar{X} \in T''U.$$

Similarly, the induced CR structure defined in equation (8.1) is also determined locally by

$$(\bar{X} + \omega(s) \cdot f_j(s)(\bar{X}))f^l_j(s) = 0 \quad \text{for all } \bar{X} \in {}^0T'', \tag{8.2}$$

where $f^l_j(s) = z^l_j \circ f(s)$. This equality also means that the map $f(s)$ is a CR embedding from $(M, {}^{\omega(s)\cdot f(s)}T'')$ to $\pi^{-1}(s)$, with the complex structure $\omega(s)$.

We must now construct $f(s)$, locally expressed by $f_j(s) = (f^1_j(s), f^2_j(s), f^3_j(s))$ on U_j, which depends complex analytically on S, as well as a holomorphic map h from S to $T \subset \mathcal{H}$ satisfying

$$f_j(s) = \tau_{jk}(f_k(s), s),$$
$$\omega(s) \cdot f_j(s) = \phi(h(s)),$$

for all $s \in S$ (where, if necessary, we may shrink S to a smaller neighborhood of o). The proof of the existence of such functions is a standard formal power series argument. Consider the power series expansions

$$f_j(s) = \sum_{|\alpha|=0}^{\infty} f_{j|\alpha} s^\alpha \quad \text{and} \quad h(s) = \sum_{|\alpha|=0}^{\infty} h_{|\alpha} s^\alpha.$$

We are using multi-index notation, so if $s = (s_1, \dots, s_r)$ and $\alpha = (\alpha_1, \dots, \alpha_r)$ then $|\alpha| = \alpha_1 + \cdots + \alpha_r$ and $s^\alpha = s_1^{\alpha_1} \cdots s_r^{\alpha_r}$. In general, if F is any vector-bundle-valued function of s, then we will use the notation $\kappa_m F$ to mean the part of the power series for $F(s)$ about $s = 0$ that is homogeneous of order m in s. For such homogeneous polynomials, we will use a subscript (k) to indicate the degree in s. Similarly, a superscript (k) will indicate a (not usually homogeneous) polynomial of degree k in s.

First we formally construct these power series; then we prove convergence. Let $f^{(m)}_j$ and $h^{(m)}$ be the mth partial sums in the preceding power series expansions:

$$f^{(m)}_j(s) = \sum_{|\alpha|=0}^{m} f_{j|\alpha} s^\alpha \quad \text{and} \quad h^{(m)}(s) = \sum_{|\alpha|=0}^{m} h_{|\alpha} s^\alpha.$$

We construct $f^{(m)}_j(s)$ and $h^{(m)}(s)$ formally by induction on m.

At any step m, we wish to have $f_j^{(m)}$ and $h^{(m)}$ satisfy

$$\left.\begin{aligned} f_j^{(m)}(s) &= \tau_{jk}(f_k^{(m)}(s), s) + O(|s|^{m+1}), \\ \omega(s)\cdot f_j^{(m)}(s) &= \phi(h^{(m)}(s)) + O(|s|^{m+1}), \end{aligned}\right\} \tag{8.3}$$

for $s \in S$ near o.

At our initial step (i.e., at $m = 0$), we define $f_j^{(0)}(s) = z_j(s)$ and $h^{(0)}(s) = 0$. These obviously satisfy our criterion (8.3).

Now we assume that we have already constructed $f_j^{(m)}$ and $h^{(m)}$ satisfying (8.3). To begin our construction of $f_j^{(m+1)}$ and $h^{(m+1)}$, we define a polynomial $g_{j|(m+1)}$ on U_j, homogeneous of degree $m+1$ in s, such that

$$f_j^{(m)}(s) + g_{j|(m+1)}(s) = \tau_{jk}(f_k^{(m)}(s) + g_{k|(m+1)}(s), s) + O(|s|^{m+2}). \tag{8.4}$$

(In this way, $g_{j|(m+1)}$ is a rough first approximation of $\kappa_{m+1}(f_j^{(m+1)})$, the homogeneous part of $f_j^{(m+1)}$ in degree $m+1$.) To do this, we construct vector-valued polynomials $\sigma_{jk|(m+1)}$ on $U_j \cap U_k$, again homogeneous of degree $m+1$ in s, by the relation

$$\sigma_{jk|(m+1)}(s) = \tau_{jk}(f_k^{(m)}(s), s) - f_j^{(m)}(s) + O(|s|^{m+2}). \tag{8.5}$$

This definition of $\sigma_{jk|(m+1)}$ makes sense because the induction hypothesis (8.3) implies that the right-hand side of equation (8.5) has only terms of order $m+1$ and higher in s. We use these $\sigma_{jk|(m+1)}$ and a partition of unity $\{\rho_j\}$ subordinate to the covering $\{U_j\}$ to define

$$g_{j|(m+1)}(s) = \sum_k \rho_k \sigma_{jk|(m+1)}(s). \tag{8.6}$$

We will show that such $g_{j|(m+1)}$ satisfy (8.4). To do this, we need to know how $g_{j|(m+1)}$ (or $\sigma_{jk|(m+1)}$) transforms over different coordinate charts. We have the following lemma (cf. [AM1, Lemma 3.2, p. 828]).

LEMMA 8.2. *On $U_j \cap U_k \cap U_l$,*

$$\sigma_{jk|(m+1)}(s) + \frac{\partial \tau_{jk}}{\partial z_k}(f_k^{(m)}(s), s)\sigma_{kl|(m+1)}(s) = \sigma_{jl|(m+1)}(s) + O(|s|^{m+2}). \tag{8.7}$$

Proof. By the definition of $\sigma_{jk|(m+1)}$,

$$\sigma_{jk|(m+1)}(s) = \tau_{jk}(f_k^{(m)}(s), s) - f_j^{(m)}(s) + O(|s|^{m+2}).$$

We replace $f_k^{(m)}(s)$ with $\tau_{kl}(f_l^{(m)}(s), s) - \sigma_{kl|(m+1)}(s)$ to obtain

$$\sigma_{jk|(m+1)}(s) = \tau_{jk}(\tau_{kl}(f_l^{(m)}(s), s) - \sigma_{kl|(m+1)}(s), s) - f_j^{(m)}(s) + O(|s|^{m+2}).$$

We expand the first term on the right-hand side in a power series about the point $(z_k, s) = (\tau_{kl}(f_l^{(m)}(s), s), s)$; this implies

$$\begin{aligned}
\sigma_{jk|(m+1)}(s) &= \tau_{jk}(\tau_{kl}(f_l^{(m)}(s), s), s) - \frac{\partial \tau_{jk}}{\partial z_k}(\tau_{kl}(f_l^{(m)}(s), s), s)\sigma_{kl|(m+1)}(s) \\
&\quad - f_j^{(m)}(s) + O(|s|^{m+2}) \\
&= \tau_{jl}(f_l^{(m)}(s), s) - \frac{\partial \tau_{jk}}{\partial z_k}(f_k^{(m)}(s), s)\sigma_{kl|(m+1)}(s) \\
&\quad - f_j^{(m)}(s) + O(|s|^{m+2}). \qquad (8.8)
\end{aligned}$$

(In the last line we have used the inductive hypothesis (8.3) and Taylor's theorem applied to $\partial\tau_{jk}/\partial z_k$; any error term involving $\sigma_{kl|(m+1)}(s)$ multiplied either by itself or by $O(|s|^{m+1})$ can be absorbed into $O(|s|^{m+2})$.) The first and third terms simplify to $\sigma_{jl|(m+1)}(s)$ modulo $O(|s|^{m+2})$ and so equation (8.8) reduces to equation (8.7). This proves the lemma. □

LEMMA 8.3. *With* $g_{j|(m+1)}$ *defined by* (8.6), $f_j^{(m)} + g_{j|(m+1)}$ *transforms as in equation* (8.4).

Proof. From the definition of $g_{j|(m+1)}$ and (8.7),

$$\begin{aligned}
g_{j|(m+1)}(s) &= \sum_l \rho_l \sigma_{jl|(m+1)}(s) \\
&= \sigma_{jk|(m+1)}(s) + \sum_l \rho_l \frac{\partial \tau_{jk}}{\partial z_k}(f_k^{(m)}(s), s)\sigma_{kl|(m+1)}(s) + O(|s|^{m+2}) \\
&= \sigma_{jk|(m+1)}(s) + \frac{\partial \tau_{jk}}{\partial z_k}(f_k^{(m)}(s), s) g_{k|(m+1)}(s) + O(|s|^{m+2}). \qquad (8.9)
\end{aligned}$$

Thus

$$\begin{aligned}
&f_j^{(m)}(s) + g_{j|(m+1)}(s) \\
&\quad = f_j^{(m)}(s) + \sigma_{jk|(m+1)}(s) + \frac{\partial \tau_{jk}}{\partial z_k}(f_k^{(m)}(s), s) g_{k|(m+1)}(s) + O(|s|^{m+2}) \\
&\quad = \tau_{jk}(f_k^{(m)}(s), s) + \frac{\partial \tau_{jk}}{\partial z_k}(f_k^{(m)}(s), s) g_{k|(m+1)}(s) + O(|s|^{m+2}).
\end{aligned}$$

By Taylor's theorem, this is equivalent to (8.4). □

To define the next term in our formal power series, we will write locally

$$\begin{aligned}
f_j^{(m+1)}(s) &= f_j^{(m)}(s) + g_{j|(m+1)}(s) + \zeta_{j|(m+1)}(s), \\
h^{(m+1)}(s) &= h^{(m)}(s) + h_{(m+1)}(s),
\end{aligned} \qquad (8.10)$$

where $\zeta_{j|(m+1)}$ is the local expression for a homogeneous polynomial $\zeta_{(m+1)}$ of degree $m+1$ in s with values in $\Gamma(M, T')$ and where $h_{(m+1)}$ is a homogeneous polynomial of degree $m+1$ with values in $\mathcal{H}$. Since the transformation law for sections of T' is

$$\zeta_{j|(m+1)} = \frac{\partial \tau_{jk}}{\partial z_k}\zeta_{k|(m+1)},$$

it follows that our prospective $f^{(m+1)}(s)$ transforms the correct way:

$$f_j^{(m)}(s) + g_{j|(m+1)}(s) + \zeta_{j|(m+1)}(s) \\ = \tau_{jk}(f_k^{(m)}(s) + g_{k|(m+1)}(s) + \zeta_{k|(m+1)}(s), s) + O(|s|^{m+2}).$$

We still must construct $\zeta_{(m+1)}$ and $h_{(m+1)}$ so that $f_j^{(m+1)}(s)$ and $h^{(m+1)}(s)$, defined as in equation (8.10), satisfy the inductive hypothesis (8.3). Note first that, by equation (8.2), the CR structure defined by $f^{(m+1)}(s)$ must satisfy

$$\big(\bar{X} + (\omega(s) \cdot (f_j^{(m)}(s) + g_{j|(m+1)}(s) + \zeta_{j|(m+1)}(s))(\bar{X})\big)(f_j^{(m)}(s) \\ + g_{j|(m+1)}(s) + \zeta_{j|(m+1)}(s)) = 0.$$

From this it follows that

$$\omega(s) \cdot (f_j^{(m)}(s) + g_{j|(m+1)}(s) + \zeta_{j|(m+1)}(s)) \\ = \omega(s) \cdot (f_j^{(m)}(s) + g_{j|(m+1)}(s)) + \bar{\partial}_{T'}\zeta_{j|(m+1)}(s) + O(|s|^{m+2}).$$

On the other hand, from the definition it is clear (see equation (7.5)) that the map ϕ linearizes to the identity, so

$$\phi(h^{(m)}(s) + h_{(m+1)}(s)) = \phi(h^{(m)}(s)) + h_{(m+1)}(s) + O(|s|^{m+2}).$$

Finding solutions to the second equation in (8.3) is thus reduced to the following theorem.

THEOREM 8.4. *There are vector-valued polynomials $\zeta_{(m+1)}$ and $h_{(m+1)}$, homogeneous in s of degree $m+1$, solving*

$$\omega(s) \cdot (f_j^{(m)}(s) + g_{j|(m+1)}(s)) + \bar{\partial}_{T'}\zeta_{(m+1)}(s) \\ = \phi(h^{(m)}(s)) + h_{(m+1)}(s) + O(|s|^{m+2}), \quad (8.11)$$

where $\zeta_{(m+1)}$ takes values in $\Gamma(M, T')$ and where $h_{(m+1)}$ takes values in the finite-dimensional harmonic space $\mathcal{H} \subset \Gamma(M, E_1)$.

The proof of this theorem will follow from several lemmas and propositions.

PROPOSITION 8.5. *There is a homogeneous polynomial $\theta_{(m+1)}$ of degree $m+1$ in s, with values in $\Gamma(M, {}^0T')$, such that*

$$\omega(s) \cdot (f_j^{(m)}(s) + g_{j|(m+1)}(s) + \theta_{j|(m+1)}(s)) \in \Gamma(M, E_1),$$

where we have written $\theta_{j|(m+1)}$ for $\theta_{(m+1)}|_{U_j}$.

Proof. Because our CR structure is strictly pseudoconvex, the map

$$\Gamma(M, {}^0T') \to \Gamma(M, F \otimes ({}^0T'')^*), \\ u \mapsto \pi_F \bar{\partial}_{T'} u,$$

is an isomorphism. Hence there is a $\Gamma(M,{}^0\,T')$-valued polynomial θ such that $\kappa_{m+1}(\omega(s) \cdot (f_j^{(m)}(s) + g_{j|(m+1)}(s)) + \bar{\partial}_{T'}\theta(s))$ is a polynomial that takes values in $\Gamma(M, {}^0T' \otimes ({}^0T'')^*)$. By the inductive hypothesis, for each $l < m$ the polynomial $\kappa_l\big(\omega(s) \cdot (f_j^{(m)}(s) + g_{j|(m+1)}(s))\big) = \kappa_l(\omega(s) \cdot f_j^{(m)}(s))$ already takes values in $\Gamma(M, {}^0T' \otimes ({}^0T'')^*)$; thus we may assume $\theta = O(|s|^{m+1})$. Writing $\theta_{(m+1)}$ for $\kappa_{m+1}\theta$ and $\theta_{j|(m+1)}$ for $\theta_{(m+1)}|_{U_j}$, we thus have

$$\kappa_{m+1}\big(\omega(s)\cdot(f_j^{(m)}(s) + g_{j|(m+1)}(s) + \theta_{j|(m+1)}(s))\big) \in \Gamma(M, {}^0T' \otimes ({}^0T'')^*).$$

To prove the proposition, it suffices to show

$$\kappa_{m+1}\big(\pi_F\bar\partial^{(1)}\big(\omega(s)\cdot(f_j^{(m)}(s) + g_{j|(m+1)}(s) + \theta_{j|(m+1)}(s))\big)\big) = 0. \tag{8.12}$$

In order to show this, we first prove the next lemma.

LEMMA 8.6.

$$R_2\big(\phi(h^{(m)}(s))\big) = R_2\big(\omega(s)\cdot(f_j^{(m)}(s))\big) + O(|s|^{m+2}) \tag{8.13}$$

and

$$R_3\big(\phi(h^{(m)}(s))\big) = R_3\big(\omega(s)\cdot(f_j^{(m)}(s))\big) + O(|s|^{m+2}) \tag{8.14}$$

hold. In particular, $R_3\big(\omega(s)\cdot(f_j^{(m)}(s))\big) = O(|s|^{m+2})$ *as* $\phi(t) \in \Gamma(M, E_1)$.

Proof. For $\psi \in \Gamma(M, T' \otimes ({}^0T'')^*)$, we have that $R_k(\psi)$ $(k = 2, 3)$ are the parts of the deformation equation that are of order k in ψ. (Of course, each $R_k(\psi)$ includes first derivatives of ψ.) The expressions for R_k are given in equations (3.4) and (3.5). Since R_2 is quadratic, we may replace each $\phi(h^{(m)}(s))$ with $\omega(s)\cdot(f_j^{(m)}(s))$ in turn. On the one hand, $\phi(h^{(m)}(s)) = \omega(s)\cdot(f_j^{(m)}(s)) + O(|s|^{m+1})$ by the induction hypothesis (8.3). On the other hand, $\phi(h^{(m)}(s))$ itself satisfies $\phi(h^{(m)}(s)) = O(|s|)$. Together, these facts imply that $R_2\big(\phi(h^{(m)}(s))\big) = R_2\big(\omega(s)\cdot(f_j^{(m)}(s))\big) + O(|s|^{m+2})$. The proof for R_3 is similar. □

Continuing our proof of Proposition 8.5, we remark that $\omega(s)$ is, for each s, an integrable complex structure. Since $(f_j^{(m)}(s) + g_{j|(m+1)} + \theta_{j|(m+1)})$ is a CR embedding for each s modulo terms of order $m+2$ and higher, it follows that the CR structure induced by $\omega(s)$ is also integrable:

$$(\bar\partial^{(1)} + R_2 + R_3)\omega(s)\cdot(f_j^{(m)}(s) + g_{j|(m+1)}(s) + \theta_{j|(m+1)}(s)) = O(|s|^{m+2}).$$

Obviously, we may remove the terms of order $m+2$ and higher to see that

$$R_2\big(\omega(s)\cdot(f_j^{(m)}(s) + g_{j|(m+1)}(s) + \theta_{j|(m+1)}(s))\big) = R_2\big(\omega(s)\cdot(f_j^{(m)}(s))\big) + O(|s|^{m+2}).$$

From the previous lemma, $R_2\big(\omega(s)\cdot(f_j^{(m)}(s))\big) = R_2\big(\phi(h^{(m)}(s))\big) + O(|s|^{m+2})$ and so

$$R_2\big(\omega(s)\cdot(f_j^{(m)}(s) + g_{j|(m+1)}(s) + \theta_{j|(m+1)}(s))\big) = R_2\big(\phi(h^{(m)}(s))\big) + O(|s|^{m+2}).$$

A similar computation shows that

$$R_3\big(\omega(s)\cdot(f_j^{(m)}(s) + g_{j|(m+1)}(s) + \theta_{j|(m+1)}(s))\big) = O(|s|^{m+2})$$

(and the zero follows from Lemma 8.6). The integrability condition is thus

$$\bar\partial^{(1)}\omega(s)\cdot(f_j^{(m)}(s) + g_{j|(m+1)}(s) + \theta_{j|(m+1)}(s)) + R_2\big(\phi(h^{(m)}(s))\big) = O(|s|^{m+2}).$$

Because $\phi(t)$ takes its values in $\Gamma(M, E_1)$, we have $\pi_F\big(R_2\big(\phi(h^{(m)}(s))\big)\big) = 0$. Hence

$$\pi_F\big(\bar\partial^{(1)}\big(\omega(s)\cdot(f_j^{(m)}(s) + g_{j|(m+1)}(s) + \theta_{j|(m+1)}(s))\big)\big) = O(|s|^{m+2}).$$

This is equivalent to equation (8.12) and so proves Proposition 8.5. □

LEMMA 8.7.

$$(1 - \bar\partial_1 N \bar\partial_1^* L) R_2\big(\omega(s)\cdot(f_j^{(m)}(s) + g_{j|(m+1)}(s) + \theta_{j|(m+1)}(s))\big) = O(|s|^{m+2}).$$

Proof. We recall that

$$P\big(\omega(s)\cdot(f_j^{(m)}(s) + g_{j|(m+1)}(s) + \theta_{j|(m+1)}(s))\big) = O(|s|^{m+2}).$$

(The map defined on each U_j by $f_j^{(m)}(s) + g_{j|(m+1)}(s) + \theta_{j|(m+1)}(s)$ makes sense globally modulo $O(|s|^{m+2})$.) Thus

$$\begin{aligned}\bar\partial_1\omega(s)&\cdot\big(f_j^{(m)}(s) + g_{j|(m+1)}(s) + \theta_{j|(m+1)}(s))\big)\\ &+ R_2\big(\omega(s)\cdot\big(f_j^{(m)}(s) + g_{j|(m+1)}(s) + \theta_{j|(m+1)}(s))\big)\big) = O(|s|^{m+2}).\end{aligned}$$

We apply the operator $1 - \bar\partial_1 N \bar\partial_1^* L$ to this equality. By Proposition 8.5, the left-hand side is the image of an element of $\Gamma(M, E_1)$ under $\bar\partial_1 + R_2$, so this makes sense. The decomposition of Theorem 5.4 implies that $(1 - \bar\partial_1 N\bar\partial_1^* L)\bar\partial_1 = 0$, and from this Lemma 8.7 follows easily. □

PROPOSITION 8.8.

$$\bar\partial_1\big[\omega(s)\cdot(f_j^{(m)}(s) + g_{j|(m+1)}(s) + \theta_{j|(m+1)}(s)) - \phi(h^{(m)}(s))\big] = O(|s|^{m+2}).$$

Proof. The first term on the left-hand side satisfies

$$\begin{aligned}\bar\partial_1\omega(s)&\cdot(f_j^{(m)}(s) + g_{j|(m+1)}(s) + \theta_{j|(m+1)}(s))\\ &+ R_2\big(\omega(s)\cdot(f_j^{(m)}(s) + g_{j|(m+1)}(s))\big) = O(|s|^{m+2}),\end{aligned}$$

as we have seen in the proof of Proposition 8.5. By the construction of $\phi(t)$ (equation (7.6)), we have

$$\bar\partial_1\phi(h^{(m)}(s)) + \bar\partial_1 N\bar\partial_1^* L R_2\big(\phi(h^{(m)}(s))\big) = 0.$$

Taking the difference of the last two equations implies

$$\begin{aligned}&\bar\partial_1\big[\omega(s)\cdot(f_j^{(m)}(s) + g_{j|(m+1)}(s) + \theta_{j|(m+1)}(s)) - \phi(h^{(m)}(s))\big]\\ &\quad+ \bar\partial_1 N\bar\partial_1^* L\big[R_2\big(\omega(s)\cdot(f_j^{(m)}(s) + g_{j|(m+1)}(s) + \theta_{j|(m+1)}(s))\big)\\ &\qquad\qquad - R_2\big(\phi(h^{(m)}(s))\big)\big]\\ &\quad+ (1 - \bar\partial_1 N\bar\partial_1^* L)\big(R_2\big(\omega(s)\cdot(f_j^{(m)}(s) + g_{j|(m+1)}(s) + \theta_{j|(m+1)}(s))\big)\big)\\ &= O(|s|^{m+2}).\end{aligned}$$

The proposition then follows from Lemmas 8.6 and 8.7. □

Proof of Theorem 8.4. We wish to solve equation (8.11), which can be written as

$$\omega(s)\cdot(f_j^{(m)}(s)+g_{j|(m+1)}(s))-\phi(h^{(m)}(s)) \\ = -\bar\partial_{T'}\zeta_{(m+1)}(s)+h_{(m+1)}(s)+O(|s|^{m+2}). \quad (8.15)$$

We begin by solving

$$\omega(s)\cdot(f_j^{(m)}(s)+g_{j|(m+1)}(s)+\theta_{j|(m+1)}(s))-\phi(h^{(m)}(s)) \\ = -\bar\partial_{T'}\eta_{(m+1)}(s)+h_{(m+1)}(s)+O(|s|^{m+2}) \quad (8.16)$$

for $\eta_{(m+1)}$ and $h_{(m+1)}$. By Proposition 8.8, the left-hand side of this equation is in the kernel of $\bar\partial_1$ modulo $O(|s|^{m+2})$. The decomposition of Theorem 5.4 implies that $\bar\partial_{T'}\eta_{(m+1)}$ and $h_{(m+1)}$, defined as follows, satisfy equation (8.16):

$$\bar\partial_{T'}\eta_{(m+1)}(s) = -\kappa_{m+1}\big[DD^*N\big(\omega(s)\cdot(f_j^{(m)}(s)+g_{j|(m+1)}(s) \\ +\theta_{j|(m+1)}(s))-\phi(h^{(m)}(s))\big)\big];$$
$$h_{(m+1)} = \kappa_{m+1}\big[H\big(\omega(s)\cdot(f_j^{(m)}(s)+g_{j|(m+1)}(s) \\ +\theta_{j|(m+1)}(s))-\phi(h^{(m)}(s))\big)\big].$$

Since $DD^*=\bar\partial_0\rho\rho^*\bar\partial_0^*$ and since $\bar\partial_{T'}=\bar\partial_0$ for elements of $H_0\subset\Gamma(M,T')$, we may define $\eta_{(m+1)}$ locally by

$$\eta_{j|(m+1)}(s) = -\kappa_{m+1}\big[\rho\rho^*\bar\partial_0^*N\big(\omega(s)\cdot(f_j^{(m)}(s)+g_{j|(m+1)}(s) \\ +\theta_{j|(m+1)}(s))-\phi(h^{(m)}(s))\big)\big].$$

To solve equation (8.15) and thus equation (8.11), we simply set $\zeta_{(m+1)} = \theta_{(m+1)}+\eta_{(m+1)}$. This ζ and $h_{(m+1)}$ solve equation (8.11), so we have proved Theorem 8.4. □

Continuing our proof of Theorem 8.1, we turn to the proof of convergence of the formal series. This part of the proof uses the standard method of Kodaira and Spencer (see [A2; AM1]). We define a Sobolev $(0,l)$-norm on a power series by setting

$$\|f_j(s)\|_{0,l}=\sum_{|\alpha|=0}^{\infty}\|f_{j|\alpha}\|_{0,l}s^\alpha \quad\text{and}\quad \|h(s)\|_{0,l}=\sum_{|\alpha|=0}^{\infty}\|h_{|\alpha}\|_{0,l}s^\alpha.$$

Consider the power series

$$A(s)=\frac{b}{16c}\sum_{|\alpha|=1}^{\infty}\left(\frac{c^{|\alpha|}}{|\alpha|^2}\right)s^\alpha; \quad (8.17)$$

this series converges for any positive c. Moreover, for positive b we have $A(s)^2 \ll (b/c)A(s)$, where $\ll$ means every coefficient of the left-hand side is less than the corresponding coefficient of the right-hand side. This implies that $A(s)^k \ll (b/c)^{k-1}A(s)$ for all integers $k\ge 2$. By choosing suitable b and c (see [A2, pp. 842–846] or [AM1, Sec. 3(II), p. 832]) we wish to show, for any integer $l\ge 3$, that

$$\|f_j(s) - z_j(s)\|_{0,l} \ll A(s) \quad \text{and} \quad \|h(s)\|_{0,l} \ll A(s). \tag{8.18}$$

(The reason for subtracting $z_j(s)$ in (8.18) is because $A(s)$ has no s^0 term.) By the Sobolev embedding theorem, this would give us all the convergence and regularity claimed in Theorem 8.1.

Proof of the convergence (8.18) is done by induction on the partial sums. That is, we assume that

$$\|f_j^{(m)}(s) - z_j(s)\|_{0,l} \ll A(s) \quad \text{and} \quad \|h^{(m)}(s)\|_{0,l} \ll A(s); \tag{8.19}$$

we then establish the same inequality for $m+1$. The special properties of $A(s)$ are used here: we bound the $(m+1)$th-degree terms with lower-degree terms that we have previously bounded. If b and c are chosen properly then we can bound sums of powers of $A(s)$ by $A(s)$ itself.

The $h_{(m+1)}$ term is well behaved: for any l there is a constant C_l such that the harmonic projector H satisfies the estimate

$$\|Hf\|_{0,l} \le C_l \|f\|$$

for any $f \in \Gamma(M, E_1)$. However, we may have to correct ζ to ensure convergence because our construction of $\theta_{(m+1)}$ involved first derivatives of $f_j^{(m)}(s)$. Recall from Theorem 8.4 that $\zeta_{(m+1)}$ is a solution to equation (8.11), which can be viewed as a linear $\bar\partial_{T'}$ equation for the standard deformation complex (3.3). Because T' is a holomorphic vector bundle, by the results of [T] there is a Neumann operator $N_{T'}\colon \Gamma_2(M, T'\otimes({}^0T'')^*) \to \Gamma_2(M, T'\otimes({}^0T'')^*)$ satisfying $u = H_{T'}u + \Box_{T'}N_{T'}u$ for all u, where $\Box_{T'} = \bar\partial_{T'}\bar\partial_{T'}^* + \bar\partial_{T'}^*\bar\partial_{T'}$ and $H_{T'}$ is the projection onto $\ker \Box_{T'}$. Arguing as in [A2], we let

$$\zeta = -\kappa_{m+1}\big[\bar\partial_{T'}^* N_{T'}\big(\omega(s)\cdot(f_j^{(m)}(s) + g_{j|(m+1)} + \theta_{j|(m+1)}) - \phi(h^{(m)}(s))\big)\big]$$

and

$$f_j^{(m+1)}(s) = f_j^{(m)}(s) + g_{j|(m+1)} + \zeta_{j|(m+1)}.$$

It is true that there is a first derivative of $f_j^{(m)}$ in $\omega(s)\cdot(f_j^{(m)}(s) + g_{j|(m+1)} + \theta_{j|(m+1)}) - \phi(h^{(m)}(s))$, but only in the $\mathbf{C}\otimes H$ direction. In fact, we recall that $\omega(s)\cdot f_j^{(m)}(s)$ is defined on U_j by

$$(\bar X + \omega(s)\cdot f_j^{(m)}(s)(\bar X)) f_j^{(m)}(s) = 0, \quad \bar X \in {}^0T''.$$

(The CR structure defined on U_j by $\omega(s)\cdot f_j^{(m)}(s)$ makes sense globally, modulo $O(|s|^{m+1})$.) Thus

$$\omega(s)\cdot f_j^{(m)}(s) z_j(s) + \omega(s)\cdot f_j^{(m)}(s)(f_j^{(m)}(s) - z_j(s)) + \bar X f_j^{(m)}(s) = O(|s|^{m+2}).$$

By the inductive hypothesis we have

$$\omega(s)\cdot f_j^{(m)}(s) z_j(s) + \phi(h^{(m)}(s))(f_j^{(m)}(s) - z_j(s)) + \bar X f_j^{(m)}(s) = O(|s|^{m+2}),$$

as $\phi(h^{(m)}(s))$ takes its values in ${}^0T'$. Since the composition $\bar\partial_{T'}^* N_{T'}$ of the adjoint operator and the standard Neumann operator gains 1 in this direction, there is no problem in the convergence of our formal solution. This finishes the proof of Theorem 8.1. □

References

[A1] T. Akahori, *Intrinsic formula for Kuranishi's* $\bar{\partial}_b^{\varphi}$, Publ. Res. Inst. Math. Sci. 14 (1978), 615–641.

[A2] ———, *Complex analytic construction of the Kuranishi family on a normal strongly pseudoconvex manifold,* Publ. Res. Inst. Math. Sci. 14 (1978), 789–847.

[A3] ———, *The new estimate for the subbundles* E_j *and its application to the deformation of the boundaries of strongly pseudoconvex domains,* Invent. Math. 63 (1981), 311–334.

[A4] ———, *The new Neumann operator associated with deformations of strongly pseudoconvex domains and its application to deformation theory,* Invent. Math. 68 (1982), 317–352.

[A5] ———, *Complex analytic construction of the Kuranishi family on a normal strongly pseudoconvex manifold with real dimension 5,* Manuscripta Math. 63 (1989), 29–43.

[A6] ———, *A mixed Hodge structure on a CR manifold,* MSRI preprint 1996-026.

[AM1] T. Akahori and K. Miyajima, *Complex analytic construction of the Kuranishi family on a normal strongly pseudoconvex manifold II,* Publ. Res. Inst. Math. Sci. 16 (1980), 811–834.

[AM2] ———, *An analogy of Tian–Todorov theorem on deformations of CR-structures,* Compositio Math. 85 (1993), 57–85.

[AM3] ———, *A note on the analogue of the Bogomolov type theorem on deformations of CR-structures,* Canad. Math. Bull. 37 (1994), 8–12.

[BE] J. Bland and C. L. Epstein, *Embeddable CR-structures and deformations of pseudoconvex surfaces, part I: formal deformations,* J. Algebraic Geom. 5 (1996), 277–368.

[BM] R. Buchweitz and J. Millson, *CR-geometry and deformations of isolated singularities,* Mem. Amer. Math. Soc. 125 (1997).

[CL] J. H. Cheng and J. M. Lee, *A local slice theorem for 3-dimensional CR structures,* Amer. J. Math. 117 (1995), 1249–1298.

[D] I. F. Donin, *Complete families of deformations of germs of complex spaces,* Math. Sb. (N.S.) 89 (1972), 390–399.

[G] P. M. Garfield, *The Rumin complex on CR manifolds,* Ph.D. dissertation, University of Washington, December 2001.

[Gr] H. Grauert, *Über die Deformation isolierter Singularitäten analytischer Mengen,* Invent. Math. 15 (1972), 171–198.

[K] M. Kuranishi, *Application of* $\bar{\partial}_b$ *to deformation of isolated singularities,* Proc. Sympos. Pure Math., 30, pp. 97–106, Amer. Math. Soc., Providence, RI, 1977.

[M1] K. Miyajima, *Deformations of a complex manifold near a strongly pseudo-convex real hypersurface and a realization of Kuranishi family of strongly pseudo-convex CR structures,* Math. Z. 205 (1990), 593–602.

[M2] ———, *Deformations of strongly pseudo-convex CR structures and deformations of normal isolated singularities,* Complex analysis (Wuppertal, 1991), Aspects Math., E17, pp. 200–204, Vieweg, Braunschweig, 1991.

[M3] ———, *CR construction of the flat deformations of normal isolated singularities,* J. Algebraic Geom. 8 (1999), 403–470.

[R] M. Rumin, *Formes différentielles sur les variétés de contact,* J. Differential Geom. 39 (1994), 281–330.

[T] N. Tanaka, *A differential geometric study on strongly pseudo-convex manifolds,* Lectures in Mathematics, 9, Department of Mathematics, Kyoto Univ., Kinokuniya, Tokyo, 1975.

[Tj] G. N. Tjurina, *Locally semi-universal flat deformations of isolated singularities of complex spaces,* Izv. Akad. Nauk SSSR Ser. Mat. 33 (1969), 1026–1058.

[W] S. M. Webster, *Pseudo-Hermitian structures on a real hypersurface,* J. Differential Geom. 13 (1978), 25–41.

T. Akahori
Department of Mathematics
Himeji Institute of Technology
Hyogo, Shosha 2167
Japan

akahorit@sanynet.ne.jp

P. M. Garfield
Department of Mathematics
University of Toronto
Toronto, Ontario M5S 3G3
Canada

garfield@math.toronto.edu

J. M. Lee
Department of Mathematics
University of Washington
Seattle, WA 98195-4350

lee@math.washington.edu