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Compactness of Composition Operators on the Bloch Space in Classical Bounded Symmetric Domains

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1. Introduction

Let $\mathcal{D}$ be a bounded homogeneous domain in $\mathbb{C}^N$. The class of all holomorphic functions with domain $\mathcal{D}$ will be denoted by $H(\mathcal{D})$. Let ϕ be a holomorphic self-map of $\mathcal{D}$. For $f \in H(\mathcal{D})$, we denote the composition $f \circ \phi$ by $C_\phi f$ and call C_ϕ the composition operator induced by ϕ.

Let $K(z, z)$ be the Bergman kernel function of $\mathcal{D}$. The Bergman metric $H_z(u, u)$ in $\mathcal{D}$ is defined by

$$H_z(u, u) = \frac{1}{2} \sum_{l,k=1}^{N} \frac{\partial^2 \log K(z, z)}{\partial z_l \partial \bar{z}_k} u_l \bar{u}_k,$$

where $z \in \mathcal{D}$ and $u = (u_1, \dots, u_N) \in \mathbb{C}^N$.

Following Timoney [T], we say that $f \in H(\mathcal{D})$ is in the Bloch space $\beta(\mathcal{D})$ if

$$\|f\|_{\beta(\mathcal{D})} = \sup_{z \in \mathcal{D}} Q_f(z) < \infty, \tag{1}$$

where

$$Q_f(z) = \sup\left\{ \frac{|\nabla f(z) u|}{H_z^{1/2}(u, u)} : u \in \mathbb{C}^N - \{0\} \right\}$$

and where $\nabla f(z) = \left(\frac{\partial f(z)}{\partial z_1}, \dots, \frac{\partial f(z)}{\partial z_N}\right)$ and $\nabla f(z) u = \sum_{l=1}^{N} \frac{\partial f(z)}{\partial z_l} u_l$.

Let D be the unit disk in $\mathbb{C}$. Madigan and Matheson [MM] proved that C_ϕ is always bounded on $\beta(D)$. They also gave the sufficient and necessary conditions for C_ϕ to be compact on $\beta(D)$.

More recently, Shi and Luo [SL] proved that C_ϕ is always bounded on $\beta(\mathcal{D})$ and gave a sufficient condition for C_ϕ to be compact on $\beta(\mathcal{D})$, where $\mathcal{D}$ is a bounded homogeneous domain in $\mathbb{C}^N$.

By using Cartan's list, all irreducible bounded symmetric domains are divided into six types. The first four types of irreducible domains are called the classical bounded symmetric domains. The other two types, called exceptional domains, consist of one domain each (a 16- and a 27-dimensional domain).

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In what follows, $\Omega \subset \mathbb{C}^N$ denotes a classical bounded symmetric domain R_A ($A = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$) (the first four types of irreducible bounded symmetric domains) and ϕ denotes a holomorphic self-map of Ω. If $U = (u_{kl})_{m\times n}$ is an $m \times n$ complex matrix, write $u = (u_{11}, \dots, u_{1n}, \dots, u_{m1}, \dots, u_{mn})$ as the corresponding vector of matrix U and $\bar{u}'$ as the conjugate transpose of u. Note that C is a positive constant, not necessarily the same at each occurrence.

In this paper, we will give a sufficient and necessary condition for the composition operator C_ϕ to be compact on $\beta(\Omega)$.

Let $A = (a_{jk})_{m\times n}$ and $B = (b_{lr})_{p\times q}$. The Kronecker product of A and B, defined by $A \times B = C = (c_{jlkr})$, is an $mp \times nq$ matrix, where $c_{jlkr} = a_{jk}b_{lr}$.

It is well known [H] that the classical bounded symmetric domains R_{I}, R_{II}, R_{III}, and R_{IV} can be expressed as follows:

$R_{\mathrm{I}}(m,n) = \{Z : Z$ is an $m \times n$ complex matrix, $I_m - Z\bar{Z}' > 0\}$, where I_m is the $m \times m$ identity matrix ($m \le n$);

$R_{\mathrm{II}}(p) = \{Z : Z$ is a $p \times p$ symmetric matrix $Z = Z'$, $I_p - Z\bar{Z} > 0\}$;

$R_{\mathrm{III}}(q) = \{Z : Z$ is a $q \times q$ antisymmetric matrix $Z = -Z'$, $I_q + Z\bar{Z} > 0\}$;

$R_{\mathrm{IV}}(N) = \{z : z = (z_1, \dots, z_N),\ 1 + |zz'|^2 - 2z\bar{z}' > 0,\ |zz'| < 1\}$.

Their respective Bergman metrics may be listed as follows (cf. [Lu]):

$$H_z^{\mathrm{I}}(u,u) = (m+n)u(I_m - Z\bar{Z}')^{-1} \times (I_n - \bar{Z}'Z)^{-1}\bar{u}', \tag{2}$$

where $Z \in R_{\mathrm{I}}(m,n)$ and U is an $m \times n$ complex matrix, u is the corresponding vector of U, and $\bar{u}'$ is the conjugate transpose of u;

$$H_z^{\mathrm{II}}(u,u) = (p+1)u(I_p - Z\bar{Z})^{-1} \times (I_p - \bar{Z}Z)^{-1}\bar{u}', \tag{3}$$

where $Z \in R_{\mathrm{II}}(p)$, U is a $p \times p$ symmetric complex matrix, and u is the corresponding vector of U;

$$H_z^{\mathrm{III}}(u,u) = 2(q-1)u(I_q + Z\bar{Z})^{-1} \times (I_q + \bar{Z}Z)^{-1}\bar{u}', \tag{4}$$

where $Z \in R_{\mathrm{III}}(q)$ and U is a $q \times q$ antisymmetric complex matrix with u the corresponding vector of U; and

$$\begin{aligned} H_z^{\mathrm{IV}}(u,u) = {} & \frac{2N}{(1+|zz'|^2 - 2z\bar{z}')^2}u \\ & \times \left[(1+|zz'|^2 - 2z\bar{z}')I_N - 2\begin{pmatrix} z \\ \bar{z} \end{pmatrix}' \begin{pmatrix} 1-2|z|^2 & \overline{zz'} \\ zz' & -1 \end{pmatrix} \overline{\begin{pmatrix} z \\ \bar{z} \end{pmatrix}}\right]\bar{u}', \end{aligned} \tag{5}$$

where $z \in R_{\mathrm{IV}}(N)$ and $u \in \mathbb{C}^N$.

Our main result is the following theorem.

THEOREM. *Let $\Omega \subset \mathbb{C}^N$ be a classical bounded symmetric domain R_A ($A = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$) and ϕ a holomorphic self-map of Ω. Then C_ϕ is compact on the Bloch space $\beta(\Omega)$ if and only if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\frac{H_{\phi(z)}(J\phi(z)u, J\phi(z)u)}{H_z(u,u)} < \varepsilon \tag{6}$$

for all $u \in \mathbb{C}^N - \{0\}$ whenever $\operatorname{dist}(\phi(z), \partial\Omega) < \delta$, where $H_z(u,u)$ is the Bergman metric of Ω.

REMARK 1. We have $R_{\mathrm{I}}(m, n) = B_n$ when $m = 1$, so the result holds when $\Omega = B_n$.

REMARK 2. When $m, n = 1$ and $R_{\mathrm{I}} = D$, the Bergman metric of the unit disk D is $H_z(u, u) = |u|^2/(1 - |z|^2)^2$ $(z \in D,\ u \in \mathbb{C})$. Hence

$$\frac{H_{\phi(z)}(\phi'(z)u, \phi'(z)u)}{H_z(u, u)} = \left\{\frac{1 - |z|^2}{1 - |\phi(z)|^2}\right\}^2 |\phi'(z)|^2,$$

where ϕ is a holomorphic self-map of D. Thus, by the preceding theorem we can also obtain Theorem 2 in [MM].

REMARK 3. For the two exceptional domains we conjecture that condition (6) is also necessary, but their Bergman metrics are very complex. We will not discuss them here.

2. Some Lemmas

In order to prove our theorem, we need the following lemmas.

LEMMA 1 [T, Thm. 2.12]. *Let $\mathcal{D} \subset \mathbb{C}^N$ be a bounded homogeneous domain. Then there exists a constant C, depending only on $\mathcal{D}$, such that $H_{\phi(z)}(J\phi(z)u, J\phi(z)u) \le CH_z(u, u)$ for each $z \in \mathcal{D}$ whenever ϕ holomorphically maps $\mathcal{D}$ into itself. Here $H_z(u, u)$ denotes the Bergman metric on $\mathcal{D}$, $J\phi(z) = \left(\frac{\partial \phi_l(z)}{\partial z_k}\right)_{1 \le l, k \le N}$ denotes the Jacobian matrix of ϕ, and $J\phi(z)u$ denotes a vector whose lth component is $(J\phi(z)u)_l = \sum_{k=1}^N \frac{\partial \phi_l(z)}{\partial z_k} u_k$, $l = 1, 2, \dots, N$.*

Using the Bergman distance in $\beta(\mathcal{D})$ and Montel's theorem, it is easy to prove (by the definition of compact operators) the following lemma, which is a characterization of compactness of C_ϕ expressed in terms of sequential convergence.

LEMMA 2 [SL, Lemma 3]. *Let $\mathcal{D}$ be a bounded homogeneous domain in $\mathbb{C}^N$. Then C_ϕ is compact on $\beta(\mathcal{D})$ if and only if, for any bounded sequence $\{f_k\}$ in $\beta(\mathcal{D})$ that converges to 0 uniformly on compact subsets of $\mathcal{D}$, we have $\|f_k \circ \phi\|_{\beta(\mathcal{D})} \to 0$ as $k \to \infty$.*

LEMMA 3 [SL, Thm. 3]. *Let $\phi\colon \mathcal{D} \to \mathcal{D}$ be a holomorphic self-map, where $\mathcal{D}$ is a bounded homogeneous domain in $\mathbb{C}^N$. Then C_ϕ is compact on $\beta(\mathcal{D})$ if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\frac{H_{\phi(z)}(J\phi(z)u, J\phi(z)u)}{H_z(u, u)} < \varepsilon$$

for all $u \in \mathbb{C}^N - \{0\}$ whenever $\operatorname{dist}(\phi(z), \partial\mathcal{D}) < \delta$.

Proof. For the reader's convenience, we give the proof of this lemma again. In order to prove that C_ϕ is compact on $\beta(\mathcal{D})$, by Lemma 2 it is enough to show that, if $\{f_k\}$ is a bounded sequence in $\beta(\mathcal{D})$ that converges to 0 uniformly on compact subsets of $\mathcal{D}$, then $\|f_k \circ \phi\|_{\beta(\mathcal{D})} \to 0$.

In fact, if we let $M = \sup_k \|f_k\|_{\beta(\mathcal{D})}$ then, for given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\frac{H_{\phi(z)}(J\phi(z)u, J\phi(z)u)}{H_z(u,u)} < \left(\frac{\varepsilon}{M}\right)^2 \tag{7}$$

for all $u \in \mathbb{C}^N - \{0\}$ whenever $\operatorname{dist}(\phi(z), \partial\mathcal{D}) < \delta$.

Using the chain rule yields

$$\nabla(f_k \circ \phi)(z) = \nabla(f_k)(\phi(z))J\phi(z).$$

If $u \in \mathbb{C}^N - \{0\}$ and $J\phi(z)u = 0$ then it follows from the equality just displayed that $\nabla(f_k \circ \phi)(z)u = 0$. If $u \in \mathbb{C}^N - \{0\}$ and $J\phi(z)u \neq 0$, then

$$\frac{\nabla(f_k \circ \phi)(z)u}{H_z^{1/2}(u,u)} = \frac{\nabla(f_k)(\phi(z))J\phi(z)u}{H_{\phi(z)}^{1/2}(J\phi(z)u, J\phi(z)u)} \times \frac{H_{\phi(z)}^{1/2}(J\phi(z)u, J\phi(z)u)}{H_z^{1/2}(u,u)}. \tag{8}$$

It follows from (8) that

$$\begin{aligned} Q_{f_k\circ\phi}(z) &= \sup\left\{\frac{|\nabla(f_k \circ \phi)(z)u|}{H_z^{1/2}(u,u)},\ u \in \mathbb{C}^N - \{0\}\right\} \\ &= \sup\left\{\frac{|\nabla(f_k \circ \phi)(z)u|}{H_z^{1/2}(u,u)},\ u \in \mathbb{C}^N - \{0\},\ J\phi(z)u \neq 0\right\} \\ &\leq Q_{f_k}(\phi(z)) \sup\left\{\left[\frac{H_{\phi(z)}(J\phi(z)u, J\phi(z)u)}{H_z(u,u)}\right]^{1/2},\ u \in \mathbb{C}^N - \{0\}\right\}. \end{aligned} \tag{9}$$

For any $z \in \mathcal{D}$, if $\operatorname{dist}(\phi(z), \partial\mathcal{D}) < \delta$ then, by (7) and (9), we have

$$Q_{f_k\circ\phi}(z) \leq \|f_k\|_{\beta(\mathcal{D})}\frac{\varepsilon}{M} \leq M\frac{\varepsilon}{M} = \varepsilon. \tag{10}$$

On the other hand, it is easy to see that

$$\inf\{H_w^{1/2}(u,u) : |u| = 1,\ \operatorname{dist}(w, \partial\mathcal{D}) \geq \delta\} = m > 0.$$

So if $\operatorname{dist}(w, \partial\mathcal{D}) \geq \delta$, then

$$\frac{|\nabla(f_k)(w)u|}{H_w^{1/2}(u,u)} \leq \frac{|\nabla(f_k)(w)||u|}{H_w^{1/2}(u,u)} = \frac{|\nabla(f_k)(w)|}{H_w^{1/2}(u/|u|, u/|u|)} \leq \frac{|\nabla(f_k)(w)|}{m}. \tag{11}$$

Our hypothesis is that $\{f_k\}$ converges to 0 uniformly on compact subsets of $\mathcal{D}$, and inequality (11) implies that $Q_{f_k}(w) \to 0$ uniformly for $\operatorname{dist}(w, \partial\mathcal{D}) \geq \delta$ as $k \to \infty$. Thus, from (9) and Lemma 1 it follows that, for large enough k,

$$Q_{f_k\circ\phi}(z) \leq CQ_{f_k}(\phi(z)) < \varepsilon \tag{12}$$

whenever $\operatorname{dist}(\phi(z), \partial\mathcal{D}) \geq \delta$.

Combining (10) and (12) shows that $\|f_k \circ \phi\|_{\beta(\mathcal{D})} = \sup_{z\in\mathcal{D}} Q_{f_k\circ\phi}(z) < \varepsilon$ as $k \to \infty$. This ends the proof. $\square$

LEMMA 4. *Let $\mathcal{D}$ be a bounded homogeneous domain of $\mathbb{C}^N$, and let $T(z,z)$ denote its metric matrix. If $T(0,0) = \lambda I_N$, where λ is a constant depending only on $\mathcal{D}$, then a holomorphic function f on $\mathcal{D}$ is in $\beta(\mathcal{D})$ if and only if*

$$\sup_{z\in\mathcal{D}}\left\{\nabla f(z)T^{-1}(z,z)\overline{\nabla f(z)}'\right\} < \infty. \tag{13}$$

If (13) *holds, then there exists a constant C depending only on $\mathcal{D}$ such that*

$$\|f\|_{\beta(\mathcal{D})} \le C\sup_{z\in\mathcal{D}}\left\{\nabla f(z)T^{-1}(z,z)\overline{\nabla f(z)}'\right\}.$$

Proof. For any $a \in \mathcal{D}$, let $\phi_a \in \mathrm{Aut}(\mathcal{D})$ with $\phi_a(a) = 0$. Then

$$T(a,a) = (J\phi_a)'(a)T(0,0)\overline{(J\phi_a)(a)} = \lambda(J\phi_a)'(a)\overline{(J\phi_a)(a)},$$

where $(J\phi_a)'$ is the transpose of $J\phi_a$. Denote $\phi_a^{-1} = \psi_a$; then $\psi(0) = a$, $J\psi_a = (J\phi_a)^{-1}$, and

$$T^{-1}(a,a) = \frac{1}{\lambda}\overline{(J\phi_a)(a)}^{-1}((J\phi_a)'(a))^{-1} = \frac{1}{\lambda}\overline{(J\psi_a)(0)}(J\psi_a)'(0).$$

Thus

$$\begin{aligned}\overline{\nabla f(a)}T^{-1}(a,a)(\nabla f(a))' &= \frac{1}{\lambda}\overline{\nabla f(a)(J\psi_a)(0)}\,(\nabla f(a)J\psi_a(0))' \\ &= \frac{1}{\lambda}|\nabla(f\circ\psi_a)(0)|^2.\end{aligned}$$

Now the desired result follows from Theorem 3.4(5) in [T]. □

LEMMA 5. *Let $G(z) = \sqrt{1-z} + \sqrt{1-\lambda z}$. If $0 < \lambda < 1$ and $|z| < 1$, then*

$$|G(z)| \ge \sqrt{2(1-|z|)}.$$

Proof. We write $z = x + iy$, $1 - z = d_1 e^{i\theta_1}$, and $1 - \lambda z = d_2 e^{i\theta_2}$, where $d_1 = |1-z|$, $\theta_1 = \arg(1-z)$, $d_2 = |1-\lambda z|$, and $\theta_2 = \arg(1-\lambda z)$.

It is clear that $\theta_1 = \arctan(y/(1-x))$, $\theta_2 = \arctan(y/(1-\lambda x))$, $1-x > 0$, and $1-\lambda x > 0$, so

$$-\pi/2 \le \theta_1 \le \pi/2, \qquad -\pi/2 \le \theta_2 \le \pi/2;$$

furthermore,

$$-\pi/2 \le (\theta_1-\theta_2)/2 \le \pi/2.$$

Then

$$\begin{aligned}G(z) &= \sqrt{1-z} + \sqrt{1-\lambda z} = \sqrt{d_1}e^{i(\theta_1/2)} + \sqrt{d_2}e^{i(\theta_2/2)} \\ &= \left(\sqrt{d_1}\cos(\theta_1/2) + \sqrt{d_2}\cos(\theta_2/2)\right) + i\left(\sqrt{d_1}\sin(\theta_1/2) + \sqrt{d_2}\sin(\theta_2/2)\right),\end{aligned}$$

$$\begin{aligned}|G(z)| &= \sqrt{d_1 + d_2 + 2\sqrt{d_1}\sqrt{d_2}\cos((\theta_1-\theta_2)/2)} \\ &\ge \sqrt{d_1+d_2} = \sqrt{|1-z| + |1-\lambda z|} \ge \sqrt{2(1-|z|)}.\end{aligned}$$

The proof is complete. □

LEMMA 6 [T, Prop. 4.5]. *Let $\mathcal{D}$ be a bounded homogeneous domain in $\mathbb{C}^N$. If f is a bounded holomorphic function in $\mathcal{D}$, then $f \in \beta(\mathcal{D})$ and there exists a constant C depending only on $\mathcal{D}$ such that*

$$\|f\|_{\beta(\mathcal{D})} \leq C \sup_{z \in \mathcal{D}} |f(z)|.$$

LEMMA 7. *Let $F(z) = (1 - z)/(1 - \lambda z)$. If $0 < \lambda < 1$ and $|z| \leq 1$, then*

$$|F(z)| < 2.$$

Proof. Since $0 < \lambda < 1$ and $|1 - \lambda z| \geq 1 - \lambda|z| \geq 1 - \lambda > 0$, it follows that

$$\begin{aligned} |F(z)| &= \left|\frac{1-z}{1-\lambda z}\right| = \left|\frac{(1-\lambda z) - (1-\lambda)z}{1-\lambda z}\right| \\ &= \left|1 - z\frac{1-\lambda}{1-\lambda z}\right| \leq 1 + (1-\lambda)\frac{1}{1-|\lambda z|} < 2. \end{aligned}$$

This proof is complete. □

It is well known that every $m \times n$ $(m \leq n)$ matrix A may be written as $A = U\left(\sum_{k=1}^{m} \lambda_k E_{kk}\right)V$, where U and V are $m \times m$ and $n \times n$ unitary matrices (respectively), $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$, E_{kk} is an $m \times n$ matrix, the element of the kth row and kth column is 1, and other elements are 0. Hence for every $P \in R_{\mathrm{I}}(m, n)$, $m \leq n$, there exist an $m \times m$ unitary matrix U and an $n \times n$ unitary matrix V such that $P = U\left(\sum_{k=1}^{m} \lambda_k E_{kk}\right)V$, where $1 \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0$.

In [Lu] the author proved the following lemma in Chinese; for the reader's convenience, we give the proof of this lemma again.

LEMMA 8. *Let*

$$P = U \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_m & 0 & \dots & 0 \end{pmatrix} V \in R_{\mathrm{I}}$$

and write

$$Q = U \begin{pmatrix} \dfrac{1}{\sqrt{1-\lambda_1^2}} & 0 & \dots & 0 \\ 0 & \dfrac{1}{\sqrt{1-\lambda_2^2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dfrac{1}{\sqrt{1-\lambda_m^2}} \end{pmatrix} \bar{U}',$$

$$R = \bar{V}' \begin{pmatrix} \frac{1}{\sqrt{1-\lambda_1^2}} & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{\sqrt{1-\lambda_2^2}} & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{1}{\sqrt{1-\lambda_m^2}} & 0 & \dots & 0 \\ & 0_{(n-m)\times m} & & & & I_{n-m} & \end{pmatrix} V,$$

where U and V are $m \times m$ and $n \times n$ unitary matrices (*respectively*) *and $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$. For $Z \in R_{\mathrm{I}}$, denote $\Phi_P^{(\mathrm{I})}(Z) = Q(P-Z)(I_n - \bar{P}'Z)^{-1}R^{-1}$. Then*

(I) $\Phi_P^{(\mathrm{I})} \in \mathrm{Aut}(R_{\mathrm{I}})$;
(II) $(\Phi_P^{(\mathrm{I})})^{-1} = \Phi_P^{(\mathrm{I})}$;
(III) $\Phi_P^{(\mathrm{I})}(0) = 0$ *and* $\Phi_P^{(\mathrm{I})}(P) = P$;
(IV) $d\Phi_P^{(\mathrm{I})}(Z)|_{Z=P} = -QdZR$ *and* $d\Phi_P^{(\mathrm{I})}(Z)|_{Z=0} = -Q^{-1}dZR^{-1}$;
(V) $\Phi_P^{(\mathrm{I})}(Z) = Q^{-1}(I_m - Z\bar{P}')^{-1}(P-Z)R$ *for* $Z \in R_{\mathrm{I}}$;
(VI) $(I_m - Z\bar{P}')Q\big(I_m - \Phi_P^{(\mathrm{I})}(Z)\overline{\Phi_P^{(\mathrm{I})}(Z)'}\big)\bar{Q}'(I_m - P\bar{Z}') = I_m - Z\bar{Z}'$ *for* $Z \in R_{\mathrm{I}}$.

Proof. (V) It is easy to demonstrate that $Q^2 - PR\bar{P}' = I_m$, $R^2 - \bar{P}'Q^2P = I_n$, $PR^2 = Q^2P$, and $\bar{P}'Q^2 = R^2\bar{P}'$. In order to show (V) we only need to prove $(P-Z)R^2(I_n - \bar{P}'Z) = (I_m - Z\bar{P}')Q^2(P-Z)$, that is, $PR^2 - PR^2\bar{P}'Z - ZR^2 + ZR^2\bar{P}'Z = Q^2P - Q^2Z - Z\bar{P}'Q^2P + Z\bar{P}'Q^2Z$, but this is an equality.

(VI)

$$\begin{aligned}
&(I_m - Z\bar{P}')Q\big(I_m - \Phi_P^{(\mathrm{I})}(Z)\overline{\Phi_P^{(\mathrm{I})}(Z)'}\big)\bar{Q}'(I_m - P\bar{Z}') \\
&\quad = (I_m - Z\bar{P}')Q^2(I_m - P\bar{P}') - (P-Z)R^2(\bar{P}' - \bar{Z}') \\
&\quad = Q^2 - Q^2P\bar{Z}' - Z\bar{P}'Q^2 + Z\bar{P}'Q^2P\bar{Z}' - PR^2\bar{P}' \\
&\qquad + PR^2\bar{Z}' + ZR^2\bar{P}' - ZR^2\bar{Z}' \\
&\quad = I_m - Z\bar{Z}'.
\end{aligned}$$

(III) It is clear that $\Phi_P^{(\mathrm{I})}(P) = 0$. Since $QP = PQ$, we have $\Phi_P^{(\mathrm{I})}(0) = f\, Q^{-1}PR = P$.

(IV)

$$\begin{aligned}
d\Phi_P^{(\mathrm{I})}(Z)|_{Z=P} &= -Q^{-1}(I_m - P\bar{P}')^{-1}dZR = -QdZR, \\
d\Phi_P^{(\mathrm{I})}(Z)|_{Z=0} &= \bar{Q}'dZ\bar{P}'P - Q^{-1}dZR \\
&= -\bar{Q}^{-1}dZ(I_n - \bar{P}'P)R = -Q^{-1}dZR^{-1}.
\end{aligned}$$

(I), (II) From (VI) we know that $\Phi_P^{(\mathrm{I})}$ is a holomorphic self-map of R_{I}, so $\Phi = \Phi_P^{(\mathrm{I})} \circ \Phi_P^{(\mathrm{I})}$ is also a holomorphic self-map of R_{I}. Because $\Phi(0) = 0$ and

$d\Phi(Z)|_{Z=0} = dZ$, by Schwarz's lemma we have $\Phi(Z) = Z$, that is, $(\Phi_P^{(\mathrm{I})})^{-1} = \Phi_P^{(\mathrm{I})}$ and $\Phi_P^{(\mathrm{I})} \in \mathrm{Aut}(R_{\mathrm{I}})$. The proof of Lemma 8 is complete. □

3. Compactness of C_ϕ on $\beta(R_{\mathrm{I}})$ and $\beta(R_{\mathrm{II}})$

Since R_A $(A = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV})$ are all bounded homogeneous domains, the sufficiency of condition (6) has been proved by Lemma 3. Hence we need only prove that condition (6) is necessary.

Suppose C_ϕ is compact on $\beta(R_{\mathrm{I}})$ and that condition (6) fails. Then there exist a sequence $\{Z^j\}$ in $R_{\mathrm{I}}(m,n)$ with $\phi(Z^j) \to \partial R_{\mathrm{I}}$ as $j \to \infty$, a $u^j \in \mathbb{C}^{mn} - \{0\}$, and an ε_0 such that

$$\frac{H^{\mathrm{I}}_{\phi(Z^j)}(J\phi(Z^j)u^j, J\phi(Z^j)u^j)}{H^{\mathrm{I}}_{Z^j}(u^j, u^j)} \geq \varepsilon_0 \tag{14}$$

for all $j = 1, 2, \ldots$.

Using (14), we will construct a sequence of functions $\{f_j\}$ satisfying the following three conditions:

(i) $\{f_j\}$ is a bounded sequence in $\beta(R_{\mathrm{I}})$;
(ii) $\{f_j\}$ tends to 0 uniformly on any compact subsets of R_{I},
(iii) $\|C_\phi f_j\|_{\beta(R_{\mathrm{I}})} \not\to 0$ as $j \to \infty$.

This sequence will contradict (by Lemma 2) the compactness of C_ϕ.

We construct the functions according to the following four parts, A–D.

Part A: To construct the sequence of $\{f_j\}$, we first assume that

$$\phi(Z^j) = r_j E_{11}, \quad j = 1, 2, \ldots,$$

where E_{kl} is an $m \times n$ matrix, the element of the kth row and lth column is 1, and other elements are 0. It is clear that $0 < r_j < 1$ and $r_j \to 1$ as $j \to \infty$.

Denote $J\phi(Z^j)u^j = w^j = (w^j_{11}, \ldots, w^j_{1n}, w^j_{21}, \ldots, w^j_{2n}, \ldots, w^j_{m1}, \ldots, w^j_{mn})$. Using formula (2), we have

$$\begin{aligned}
&H^{\mathrm{I}}_{\phi(Z^j)}(w^j, w^j)\\
&= H^{\mathrm{I}}_{r_jE_{11}}(w^j, w^j)\\
&= (m+n)w^j \begin{pmatrix} (1-r_j^2)^{-1} & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix} \times \begin{pmatrix} (1-r_j^2)^{-1} & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix} \overline{w^j}'\\
&= (m+n)\Bigg[\frac{|w^j_{11}|^2}{(1-r_j^2)^2} + \frac{1}{1-r_j^2}\Bigg(\sum_{l=2}^{n}|w^j_{1l}|^2 + \sum_{k=2}^{m}|w^j_{k1}|^2\Bigg)\\
&\qquad + \sum_{2\leq k\leq m, 2\leq l\leq n}|w^j_{kl}|^2\Bigg].
\end{aligned}$$

Denote

$$A_j^{\mathrm{I}} = \frac{|w_{11}^j|^2}{(1-r_j^2)^2},$$

$$B_j^{\mathrm{I}} = \frac{1}{1-r_j^2}\left(\sum_{l=2}^{n}|w_{1l}^j|^2 + \sum_{k=2}^{m}|w_{k1}^j|^2\right),$$

$$C_j^{\mathrm{I}} = \sum_{2\le k\le m, 2\le l\le n}|w_{kl}^j|^2;$$

then

$$H_{\phi(Z^j)}^{\mathrm{I}}(w^j, w^j) = (m+n)(A_j^{\mathrm{I}} + B_j^{\mathrm{I}} + C_j^{\mathrm{I}}). \tag{15}$$

We construct the functions according to three different cases as follows.

Case 1. If, for some j,

$$\max(B_j^{\mathrm{I}}, C_j^{\mathrm{I}}) \le A_j^{\mathrm{I}}, \tag{16}$$

then set

$$f_j(Z) = \log(1 - e^{-a(1-r_j)}z_{11}) - \log(1 - z_{11}), \tag{17}$$

where $Z = (z_{kl})$ with $1 \le k \le m$ and $1 \le l \le n$ and where a is any positive number.

Case 2. If, for some j,

$$\max(A_j^{\mathrm{I}}, C_j^{\mathrm{I}}) \le B_j^{\mathrm{I}}, \tag{18}$$

then set

$$f_j(Z) = \left(\sum_{l=2}^{n} e^{-i\theta_{1l}^j}z_{1l} + \sum_{k=2}^{m} e^{-i\theta_{k1}^j}z_{k1}\right)\left(\frac{1}{\sqrt{1-e^{-a(1-r_j)}z_{11}}} - \frac{1}{\sqrt{1-z_{11}}}\right), \tag{19}$$

where a is any positive number and where $\theta_{1l}^j = \arg w_{1l}^j$ and $\theta_{k1}^j = \arg w_{k1}^j$. If $w_{1l}^j = 0$ for some l or $w_{k1}^j = 0$ for some k, replace the corresponding term $e^{-i\theta_{1l}^j}z_{1l}$ or $e^{-i\theta_{k1}^j}z_{k1}$ by 0.

Case 3. If, for some j,

$$\max(A_j^{\mathrm{I}}, B_j^{\mathrm{I}}) \le C_j^{\mathrm{I}}, \tag{20}$$

then set

$$f_j(Z) = \left(\sum_{2\le k\le m, 2\le l\le n} e^{-i\theta_{kl}^j}z_{kl}\right)\sqrt{1-z_{11}}\left(\frac{1}{\sqrt{1-e^{-a(1-r_j)}z_{11}}} - \frac{1}{\sqrt{1-z_{11}}}\right), \tag{21}$$

where a is any positive number and $\theta_{kl}^j = \arg w_{kl}^j$ for $2 \le k \le m$ and $2 \le l \le n$. If $w_{kl}^j = 0$ for some k or l, replace the corresponding term $e^{-i\theta_{kl}^j}z_{kl}$ by 0.

Now we will prove that the functions defined by (17), (19), and (21) all satisfy conditions (i), (ii), and (iii).

For the functions defined by (17), it is easy to see that

$$\begin{aligned}\nabla f_j(Z) &= \left(\frac{\partial f_j}{\partial z_{11}}(Z), \ldots, \frac{\partial f_j}{\partial z_{1n}}(Z), \ldots, \frac{\partial f_j}{\partial z_{m1}}(Z), \ldots, \frac{\partial f_j}{\partial z_{mn}}(Z)\right) \\ &= \left(\frac{\partial f_j}{\partial z_{11}}(Z), \ldots, 0, \ldots, 0, \ldots, 0, \ldots, 0\right).\end{aligned}$$

From formula (2), it is easy to know that the metric matrix of $R_{\mathrm{I}}(m,n)$ is

$$T(Z,Z) = (m+n)(I_m - Z\bar{Z}')^{-1} \times (I_n - \bar{Z}'Z)^{-1},$$

so $T(0,0) = (m+n)I_{mn}$ and

$$T^{-1}(Z,Z) = (m+n)^{-1}(I_m - Z\bar{Z}') \times (I_n - \bar{Z}'Z).$$

Thus

$$\begin{aligned}
&\nabla f_j(Z)T^{-1}(Z,Z)\overline{\nabla f_j(Z)}' \\
&\quad = (m+n)^{-1}\left|\frac{\partial f_j}{\partial z_{11}}(Z)\right|^2\left(1-\sum_{l=1}^{n}|z_{1l}|^2\right)\left(1-\sum_{k=1}^{n}|z_{k1}|^2\right) \\
&\quad = (m+n)^{-1}\left(1-\sum_{l=1}^{n}|z_{1l}|^2\right)\left(1-\sum_{k=1}^{n}|z_{k1}|^2\right)\left|\frac{-e^{-a(1-r_j)}}{1-e^{-a(1-r_j)}z_{11}} + \frac{1}{1-z_{11}}\right|^2 \\
&\quad \le (m+n)^{-1}(1-|z_{11}|^2)^2\left(\frac{2}{1-|z_{11}|}\right)^2 \\
&\quad \le 4(m+n)^{-1}(1+|z_{11}|)^2 \le 16(m+n)^{-1}.
\end{aligned}$$

Lemma 4 now gives

$$\|f_j\|_{\beta(R_{\mathrm{I}})} \le 16C(m+n)^{-1}.$$

This proves that the functions (17) satisfy condition (i).

Let E be a compact subset of R_{I}; then there exists a $\rho \in (0,1)$ such that

$$|z_{11}| \le \rho \tag{22}$$

for any $Z = (z_{kl}) \in E$. We have

$$f_j(Z) = \log(1-e^{-a(1-r_j)}z_{11}) - \log(1-z_{11}) = \log\frac{1-e^{-a(1-r_j)}z_{11}}{1-z_{11}}.$$

Since

$$\begin{aligned}
\left|\frac{1-e^{-a(1-r_j)}z_{11}}{1-z_{11}} - 1\right| &= \left|\frac{1-e^{-a(1-r_j)}z_{11} - 1 + z_{11}}{1-z_{11}}\right| \\
&= \left|\frac{z_{11}}{1-z_{11}}\right||1-e^{-a(1-r_j)}| \le \frac{1}{1-|z_{11}|}|1-e^{-a(1-r_j)}| \\
&\le \frac{1}{1-\rho}(1-e^{-a(1-r_j)}),
\end{aligned}$$

it is clear that $1-e^{-a(1-r_j)} \to 0$ as $j \to \infty$. Therefore, $(1-e^{-a(1-r_j)}z_{11})/(1-z_{11})$ converges to 1 uniformly on a compact subset E; that is, $f_j(Z) = \log((1-e^{-a(1-r_j)}z_{11})/(1-z_{11}))$ converges to 0 uniformly on E as $j \to \infty$, so the functions (17) satisfy condition (ii).

Now we prove that $\|C_\phi f_j\|_{\beta(R_{\mathrm{I}})} \not\to 0$. In fact, by (15) and (16),

$$H^{\mathrm{I}}_{\phi(Z^j)}(w^j, w^j) = (m+n)(A^{\mathrm{I}}_j + B^{\mathrm{I}}_j + C^{\mathrm{I}}_j) \le 3(m+n)A^{\mathrm{I}}_j. \tag{23}$$

Combining (14) and (23) yields

$$
\begin{aligned}
&\|C_\phi f_j\|_{\beta(R_{\mathrm{I}})}\\
&\quad= \|f_j\circ\phi\|_{\beta(R_{\mathrm{I}})} \geq Q_{f_j\circ\phi}(Z^j)\\
&\quad\geq \frac{|\nabla(f_j\circ\phi)(Z^j)u^j|}{H_{Z^j}^{1/2}(u^j,u^j)} = \frac{|\nabla(f_j)(\phi(Z^j))J\phi(Z^j)u^j|}{H_{Z^j}^{1/2}(u^j,u^j)}\\
&\quad= \frac{|\nabla(f_j)(\phi(Z^j))J\phi(Z^j)u^j|}{H_{\phi(Z^j)}^{1/2}(J\phi(Z^j)u^j, J\phi(Z^j)u^j)}\left\{\frac{H_{\phi(Z^j)}(J\phi(Z^j)u^j, J\phi(Z^j)u^j)}{H_{Z^j}(u^j,u^j)}\right\}^{1/2}\\
&\quad\geq \sqrt{\varepsilon_0}\frac{|\nabla(f_j)(r_jE_{11})w^j|}{H_{r_jE_{11}}^{1/2}(w^j,w^j)} \geq \sqrt{\frac{\varepsilon_0}{3(m+n)}}\frac{\left|\frac{\partial f_j}{\partial z_{11}}(r_jE_{11})w_{11}^j\right|}{|w_{11}^j|/(1-r_j^2)}\\
&\quad= \sqrt{\frac{\varepsilon_0}{3(m+n)}}(1-r_j^2)\left|\frac{1}{1-r_j} - \frac{e^{-a(1-r_j)}}{1-e^{-a(1-r_j)}r_j}\right|\\
&\quad\geq \sqrt{\frac{\varepsilon_0}{3(m+n)}}\left(1-\frac{(1-r_j)e^{-a(1-r_j)}}{1-e^{-a(1-r_j)}r_j}\right),
\end{aligned}
$$

and

$$
\lim_{j\to\infty}\left[1-\frac{(1-r_j)e^{-a(1-r_j)}}{1-e^{-a(1-r_j)}r_j}\right] = \frac{a}{a+1} \neq 0.
$$

This proves that $\|C_\phi f_j\|_{\beta(R_{\mathrm{I}})} \not\to 0$ as $j\to\infty$.

For the functions defined by (19), we will prove that $\{f_j\}$ is bounded on R_{I}. In fact, for any $Z\in R_{\mathrm{I}}(m,n)$ we have

$$
I_m - Z\bar{Z}' = \left(\delta_{st} - \sum_{k=1}^{n} z_{sk}\bar{z}_{tk}\right)_{1\leq s,t\leq m} > 0,
$$

$$
I_n - Z'\bar{Z} = \left(\delta_{st} - \sum_{k=1}^{m} z_{ks}\bar{z}_{kt}\right)_{1\leq s,t\leq n} > 0.
$$

Hence

$$
\delta_{11} - \sum_{l=1}^{n} z_{1l}\bar{z}_{1l} = 1 - \sum_{l=1}^{n}|z_{1l}|^2 > 0, \tag{24}
$$

$$
\delta_{11} - \sum_{k=1}^{m} z_{k1}\bar{z}_{k1} = 1 - \sum_{k=1}^{m}|z_{k1}|^2 > 0. \tag{25}
$$

Now (24) and (25) imply

$$
\begin{aligned}
&|f_j(Z)|\\
&\quad= \left|\left(\sum_{l=2}^{n} e^{-i\theta_{1l}^j}z_{1l} + \sum_{k=2}^{m} e^{-i\theta_{k1}^j}z_{k1}\right)\left(\frac{1}{\sqrt{1-e^{-a(1-r_j)}z_{11}}} - \frac{1}{\sqrt{1-z_{11}}}\right)\right|\\
&\quad\leq \left(\sum_{l=2}^{n}|z_{1l}| + \sum_{k=2}^{m}|z_{k1}|\right)\left(\left|\frac{1}{\sqrt{1-e^{-a(1-r_j)}z_{11}}}\right| + \left|\frac{1}{\sqrt{1-z_{11}}}\right|\right) <
\end{aligned}
$$

$$
\begin{aligned}
&< \left(\sqrt{n}\left(\sum_{l=2}^{n}|z_{1l}|^2\right)^{1/2} + \sqrt{m}\left(\sum_{k=2}^{m}|z_{k1}|^2\right)^{1/2}\right)\left(\frac{1}{\sqrt{1-|z_{11}|}} + \frac{1}{\sqrt{1-|z_{11}|}}\right)\\
&< \left(\sqrt{n}+\sqrt{m}\right)\sqrt{1-|z_{11}|^2}\left(\frac{1}{\sqrt{1-|z_{11}|}} + \frac{1}{\sqrt{1-|z_{11}|}}\right)\\
&\le 4\left(\sqrt{m}+\sqrt{n}\right). \qquad (26)
\end{aligned}
$$

By Lemma 6, (26) means that $f_j \in \beta(R_{\mathrm{I}})$ and $\{f_j\}$ is bounded on $\beta(R_{\mathrm{I}})$.

For the compact subset E of R_{I}, by (19) we have

$$
\begin{aligned}
f_j(Z) &= \left(\sum_{l=2}^{n} e^{-i\theta_{1l}^{j}} z_{1l} + \sum_{k=2}^{m} e^{-i\theta_{k1}^{j}} z_{k1}\right)\frac{\sqrt{1-z_{11}} - \sqrt{1-e^{-a(1-r_j)}z_{11}}}{\sqrt{1-e^{-a(1-r_j)}z_{11}}}\\
&= \left(\sum_{l=2}^{n} e^{-i\theta_{1l}^{j}} z_{1l} + \sum_{k=2}^{m} e^{-i\theta_{k1}^{j}} z_{k1}\right)\\
&\qquad\times \frac{(e^{-a(1-r_j)}-1)z_{11}}{\sqrt{(1-e^{-a(1-r_j)}z_{11})(1-z_{11})}\left(\sqrt{1-z_{11}}+\sqrt{1-e^{-a(1-r_j)}z_{11}}\right)}.
\end{aligned}
$$

Since $|z_{11}| < 1$ and since $0 < \lambda = e^{-a(1-r_j)} < 1$ we have $\sqrt{|1-e^{-a(1-r_j)}z_{11}|} \ge \sqrt{1-|z_{11}|}$; from Lemma 5, it then follows that

$$
\left|\sqrt{1-z_{11}}+\sqrt{1-e^{-a(1-r_j)}z_{11}}\right| \ge \sqrt{2(1-|z_{11}|)}.
$$

By (22), $1-|z_{11}| \ge 1-\rho > 0$; thus

$$
|f_j(Z)| \le C\frac{1-e^{-a(1-r_j)}}{(1-|z_{11}|)\sqrt{2(1-|z_{11}|)}} \le \frac{C}{\sqrt{2(1-\rho)}(1-\rho)}(1-e^{-a(1-r_j)}).
$$

It is clear that $\lim_{j\to\infty}(1-e^{-a(1-r_j)}) = 0$, so $\{f_j\}$ converges to 0 uniformly on compact subsets E of R_{I}; that is, the functions (19) satisfy condition (ii).

Now we prove that $\|C_\phi f_j\|_{\beta(R_{\mathrm{I}})} \not\to 0$. In fact, by (15) and (18),

$$
H^{\mathrm{I}}_{\phi(Z^j)}(w^j, w^j) = (m+n)(A_j^{\mathrm{I}} + B_j^{\mathrm{I}} + C_j^{\mathrm{I}}) \le 3(m+n)B_j^{\mathrm{I}}. \qquad (27)
$$

Combining (14) and (27) yields

$$
\begin{aligned}
&\|C_\phi f_j\|_{\beta(R_{\mathrm{I}})}\\
&\quad = \|f_j\circ\phi\|_{\beta(R_{\mathrm{I}})} \ge Q_{f_j\circ\phi}(Z^j)\\
&\quad \ge \frac{|\nabla(f_j\circ\phi)(Z^j)u^j|}{H_{Z^j}^{1/2}(u,u)} = \frac{|\nabla(f_j)(\phi(Z^j))J\phi(Z^j)u^j|}{H_{Z^j}^{1/2}(u,u)}\\
&\quad = \frac{|\nabla(f_j)(\phi(Z^j))J\phi(Z^j)u^j|}{H_{\phi(Z^j)}^{1/2}(J\phi(Z^j)u^j, J\phi(Z^j)u^j)}\left\{\frac{H_{\phi(Z^j)}(J\phi(Z^j)u^j, J\phi(Z^j)u^j)}{H_{Z^j}(u^j,u^j)}\right\}^{1/2}
\end{aligned}
$$

$$\geq \sqrt{\varepsilon_0}\frac{|\nabla(f_j)(r_jE_{11})w^j|}{H^{1/2}_{r_jE_{11}}(w^j,w^j)} \geq \sqrt{\frac{\varepsilon_0}{3(m+n)}}\frac{\left|\sum\limits_{1\leq k\leq m,1\leq l\leq n}\frac{\partial f_j}{\partial z_{kl}}(r_jE_{11})w^j_{kl}\right|}{\left(\frac{1}{1-r_j^2}\left(\sum\limits_{l=2}^{n}|w^j_{1l}|^2+\sum\limits_{k=2}^{m}|w^j_{k1}|^2\right)\right)^{1/2}}$$

$$= \sqrt{\frac{\varepsilon_0}{3(m+n)}}\frac{\sqrt{1-r_j^2}\left|\frac{1}{\sqrt{1-e^{-a(1-r_j)}r_j}}-\frac{1}{\sqrt{1-r_j}}\right|\left(\sum\limits_{l=2}^{n}|w^j_{1l}|+\sum\limits_{k=2}^{m}|w^j_{k1}|\right)}{\sum\limits_{l=2}^{n}|w^j_{1l}|^2+\sum\limits_{k=2}^{m}|w^j_{k1}|^2}$$

$$\geq \sqrt{\frac{\varepsilon_0}{3(m+n)}}\left|\sqrt{\frac{1-r_j}{1-e^{-a(1-r_j)}r_j}}-1\right| = \sqrt{\frac{\varepsilon_0}{3(m+n)}}\left(1-\sqrt{\frac{1-r_j}{1-e^{-a(1-r_j)}r_j}}\right),$$

and

$$\lim_{j\to\infty}\frac{1-r_j}{1-e^{-a(1-r_j)}r_j} = \lim_{r\to 1}\frac{1-r}{1-e^{-a(1-r)}r} = \frac{1}{a+1}.$$

Thus $\|C_\phi f_j\|_{\beta(R_{\mathrm{I}})} \not\to 0$ as $j\to\infty$, which means that the functions (19) satisfy condition (iii).

For the functions defined by (21), we will prove that $\{f_j\}$ is bounded on R_{I}. In fact, for any $z\in R_{\mathrm{I}}$, from Lemma 7 we have

$$\begin{aligned}|f_j(Z)| &= \left|\left(\sum_{2\leq k\leq m,2\leq l\leq n} e^{-i\theta^j_{kl}}z_{kl}\right)\sqrt{1-z_{11}}\left(\frac{1}{\sqrt{1-e^{-a(1-r_j)}z_{11}}}-\frac{1}{\sqrt{1-z_{11}}}\right)\right|\\ &\leq C\sqrt{|1-z_{11}|}\left(\sqrt{\left|\frac{1-z_{11}}{1-e^{-a(1-r_j)}z_{11}}\right|}+1\right)\\ &\leq C\big(\sqrt{2}+1\big)\leq C. \end{aligned}\tag{28}$$

By Lemma 6, (28) means that $f_j\in\beta(R_{\mathrm{I}})$ and $\{f_j\}$ is bounded on $\beta(R_{\mathrm{I}})$.

For the compact subset E of R_{I}, by (21) we have

$$\begin{aligned}&f_j(Z)\\ &\quad= \left(\sum_{2\leq k\leq m,2\leq l\leq n} e^{-i\theta^j_{kl}}z_{kl}\right)\sqrt{1-z_{11}}\left(\frac{1}{\sqrt{1-e^{-a(1-r_j)}z_{11}}}-\frac{1}{\sqrt{1-z_{11}}}\right)\\ &\quad= \left(\sum_{2\leq k\leq m,2\leq l\leq n} e^{-i\theta^j_{kl}}z_{kl}\right)\frac{(e^{-a(1-r_j)}-1)z_{11}}{\sqrt{1-e^{-a(1-r_j)}z_{11}}\big(\sqrt{1-z_{11}}+\sqrt{1-e^{-a(1-r_j)}z_{11}}\big)}.\end{aligned}$$

Since $|z_{11}|<1$ and since $0<\lambda=e^{-a(1-r_j)}<1$, we have $\sqrt{|1-e^{-a(1-r_j)}z_{11}|}\geq\sqrt{1-|z_{11}|}$, and from Lemma 5 it follows that

$$\left|\sqrt{1-z_{11}}+\sqrt{1-e^{-a(1-r_j)}z_{11}}\right|\geq\sqrt{2(1-|z_{11}|)}.$$

By (22), $1-|z_{11}|\geq 1-\rho>0$; thus

$$|f_j(Z)| \le C\frac{1-e^{-a(1-r_j)}}{\sqrt{1-|z_{11}|}\sqrt{2(1-|z_{11}|)}} \le \frac{C}{\sqrt{2}(1-\rho)}(1-e^{-a(1-r_j)}).$$

It is clear that $\lim_{j\to\infty}(1-e^{-a(1-r_j)})=0$, so $\{f_j\}$ converges to 0 uniformly on E and therefore the functions defined by (19) satisfy condition (ii).

Now we prove that $\|C_\phi f_j\|_{\beta(R_{\mathrm{I}})} \not\to 0$. In fact, by (15) and (20),

$$H^{\mathrm{I}}_{\phi(Z^j)}(w^j,w^j) = (m+n)(A^{\mathrm{I}}_j + B^{\mathrm{I}}_j + C^{\mathrm{I}}_j) \le 3(m+n)C^{\mathrm{I}}_j. \tag{29}$$

Combining (14) and (29) yields

$$\begin{aligned}
&\|C_\phi f_j\|_{\beta(R_{\mathrm{I}})}\\
&\quad = \|f_j\circ\phi\|_{\beta(R_{\mathrm{I}})} \ge Q_{f_j\circ\phi}(Z^j)\\
&\quad \ge \frac{|\nabla(f_j\circ\phi)(Z^j)u^j|}{H^{1/2}_{Z^j}(u,u)} = \frac{|\nabla(f_j)(\phi(Z^j))J\phi(Z^j)u^j|}{H^{1/2}_{Z^j}(u,u)}\\
&\quad = \frac{|\nabla(f_j)(\phi(Z^j))J\phi(Z^j)u^j|}{H^{1/2}_{\phi(Z^j)}(J\phi(Z^j)u^j, J\phi(Z^j)u^j)}\left\{\frac{H_{\phi(Z^j)}(J\phi(Z^j)u^j, J\phi(Z^j)u^j)}{H_{Z^j}(u^j,u^j)}\right\}^{1/2}\\
&\quad \ge \sqrt{\varepsilon_0}\frac{|\nabla(f_j)(r_jE_{11})w^j|}{H^{1/2}_{r_jE_{11}}(w^j,w^j)} \ge \sqrt{\frac{\varepsilon_0}{3(m+n)}}\frac{\left|\displaystyle\sum_{1\le k\le m,1\le l\le n}\frac{\partial f_j}{\partial z_{kl}}(r_jE_{11})w^j_{kl}\right|}{\left(\displaystyle\sum_{2\le k\le m,2\le l\le n}|w^j_{kl}|^2\right)^{1/2}}\\
&\quad = \sqrt{\frac{\varepsilon_0}{3(m+n)}}\frac{\sqrt{1-r_j}\left|\dfrac{1}{\sqrt{1-e^{-a(1-r_j)}r_j}} - \dfrac{1}{\sqrt{1-r_j}}\right|\displaystyle\sum_{2\le k\le m,2\le l\le n}|w^j_{kl}|}{\left(\displaystyle\sum_{2\le k\le m,2\le l\le n}|w^j_{kl}|^2\right)^{1/2}}\\
&\quad \ge \sqrt{\frac{\varepsilon_0}{3(m+n)}}\left|\sqrt{\frac{1-r_j}{1-e^{-a(1-r_j)}r_j}}-1\right|\\
&\quad = \sqrt{\frac{\varepsilon_0}{3(m+n)}}\left(1-\sqrt{\frac{1-r_j}{1-e^{-a(1-r_j)}r_j}}\right),
\end{aligned}$$

and

$$\lim_{j\to\infty}\frac{1-r_j}{1-e^{-a(1-r_j)}r_j} = \lim_{r\to 1}\frac{1-r}{1-e^{-a(1-r)}r} = \frac{1}{a+1}.$$

Consequently, $\|C_\phi f_j\|_{\beta(R_{\mathrm{I}})} \not\to 0$ as $j\to\infty$; that is, the functions defined by (21) satisfy condition (iii).

Part B: We assume that

$$\phi(Z^j) = r^{(1)}_j E_{11} + r^{(2)}_j E_{22},$$

where $1 > r^{(1)}_j \ge r^{(2)}_j \ge 0$. Since $\phi(Z^j)\to\partial R_{\mathrm{I}}$, we may assume that $r^{(1)}_j \to 1$ and $r^{(2)}_j \to \lambda_0$, where $\lambda_0 \le 1$.

If $\lambda_0 = 1$ then, using the same methods as in Part A, we can construct a sequence of functions $\{f_j(Z)\}$ satisfying conditions (i)–(iii).

If $\lambda_0 < 1$ then, by Lemma 8, there exist $\Phi^{\mathrm{I}}_{r_j^{(1)}E_{11}+r_j^{(2)}E_{22}} \in R_{\mathrm{I}}$ and $\Phi^{\mathrm{I}}_{r_j^{(1)}E_{11}} \in R_{\mathrm{I}}$ such that $\Phi^{\mathrm{I}}_{r_j^{(1)}E_{11}+r_j^{(2)}E_{22}}(r_j^{(1)}E_{11}+r_j^{(2)}E_{22}) = 0$ and $\Phi^{\mathrm{I}}_{r_j^{(1)}E_{11}}(r_j^{(1)}E_{11}) = 0$ ($j = 1, 2, \ldots$). If we denote $\Psi^{(j)}(Z) = \left(\Phi^{\mathrm{I}}_{r_j^{(1)}E_{11}}\right)^{-1} \circ \Phi^{\mathrm{I}}_{r_j^{(1)}E_{11}+r_j^{(2)}E_{22}}$, then $\Psi^{(j)} \in R_{\mathrm{I}}$ and $\Psi^{(j)}(\phi(Z^j)) = \Psi^{(j)}(r_j^{(1)}E_{11}+r_j^{(2)}E_{22}) = r_j^{(1)}E_{11} = r_jE_{11}$, where $r_j = r_j^{(1)}$.

Set $g_j = f_j \circ \Psi^{(j)}$, where $\{f_j\}$ are the functions obtained in Part A. Since $\Psi^{(j)}(Z) \in \mathrm{Aut}(R_{\mathrm{I}})$, it is clear that

$$\begin{aligned} H^{\mathrm{I}}_{\phi(Z^j)}(w^j, w^j) &= H^{\mathrm{I}}_{\Psi^j\circ\phi(Z^j)}(J\Psi^{(j)}(\phi(Z^j))w^j, J\Psi^{(j)}(\phi(Z^j))w^j) \\ &= H^{\mathrm{I}}_{r_jE_{11}}(v^j, v^j), \end{aligned} \tag{30}$$

where $w^j = J\phi(Z^j)u^j$ and $v^j = J\Psi^{(j)}(\phi(Z^j))w^j$. It follows from (30) that

$$\begin{aligned} \frac{|\nabla(g_j)(\phi(Z^j))w^j|}{H^{1/2}_{\phi(Z^j)}(w^j, w^j)} &= \frac{|\nabla(f_j)(r_jE_{11})J\Psi^{(j)}(\phi(Z^j))w^j)|}{H^{1/2}_{r_jE_{11}}(J\Psi^{(j)}(\phi(Z^j))w^j, J\Psi^{(j)}(\phi(Z^j))w^j)} \\ &= \frac{|\nabla(f_j)(r_jE_{11})v^j)|}{H^{1/2}_{r_jE_{11}}(v^j, v^j)} \end{aligned}$$

and

$$\begin{aligned} &\|C_\phi g_j\|_{\beta(R_{\mathrm{I}})} \\ &\quad= \|g_j \circ \phi\|_{\beta(R_{\mathrm{I}})} \geq Q_{g_j\circ\phi}(Z^j) \\ &\quad\geq \frac{|\nabla(g_j\circ\phi)(Z^j)u^j|}{H^{1/2}_{Z^j}(u^j, u^j)} = \frac{|\nabla(g_j)(\phi(Z^j))J\phi(Z^j)u^j|}{H^{1/2}_{Z^j}(u^j, u^j)} \\ &\quad= \frac{|\nabla(g_j)(\phi(Z^j))J\phi(Z^j)u^j|}{H^{1/2}_{\phi(Z^j)}(J\phi(Z^j)u^j, J\phi(Z^j)u^j)} \left\{\frac{H_{\phi(Z^j)}(J\phi(Z^j)u^j, J\phi(Z^j)u^j)}{H_{Z^j}(u^j, u^j)}\right\}^{1/2} \\ &\quad\geq \sqrt{\varepsilon_0}\,\frac{|\nabla(g_j)(\phi(Z^j))w^j|}{H^{1/2}_{\phi(Z^j)}(w^j, w^j)} = \sqrt{\varepsilon_0}\,\frac{|\nabla(f_j)(r_jE_{11})v^j)|}{H^{1/2}_{r_jE_{11}}(v^j, v^j)}. \end{aligned} \tag{31}$$

Now, the discussion in Part A shows that $\|C_\phi g_j\|_{\beta(R_{\mathrm{I}})} \not\to 0$ as $j \to \infty$; that is, $\{g_j\}$ satisfies condition (iii).

We prove that $\{g_j\}$ is a bounded sequence in $\beta(R_{\mathrm{I}})$. In fact, since $\Psi^{(j)}(Z) \in \mathrm{Aut}(R_{\mathrm{I}})$,

$$Q_{g_j}(Z) = Q_{f_j\circ\Psi^{(j)}}(Z) = Q_{f_j}(\Psi^{(j)}(Z))$$

and so $\|g_j\|_{\beta(R_{\mathrm{I}})} = \|f_j\|_{\beta(R_{\mathrm{I}})}$ is bounded.

Now we prove that $\{g_j\}$ tends to 0 uniformly on a compact subset E of R_{I}. If we write $\Psi^{(j)}(Z) = (\psi^{(j)}_{lk}(Z))_{1\leq l\leq m, 1\leq k\leq n}$ then, by the definition of $\Psi^{(j)}$ and Lemma 8, a direct calculation shows that

$$\psi_{11}^{(j)}(Z) = z_{11} + r_j^{(2)} \frac{z_{12} z_{21}}{1 - r_j^{(2)} z_{22}}. \tag{32}$$

It is easy to show that $\psi_{11}^{(j)}(Z)$ converges uniformly to $\psi_{11}(Z) = z_{11} + \lambda_0 \frac{z_{12} z_{21}}{1-\lambda_0 z_{22}}$ on $R_{\mathrm{I}}(m, n)$.

Since $\lambda_0 < 1$ and $\lambda_0 E_{11} + \lambda_0 E_{22} \in R_{\mathrm{I}}$, there similarly exists $\Psi(Z) \in \mathrm{Aut}(R_{\mathrm{I}})$ such that $\Psi(\lambda_0 E_{11} + \lambda_0 E_{22}) = \lambda_0 E_{11}$, and the first component of $\Psi(Z)$ is $\psi_{11}(Z)$. It is clear that $\psi_{11}(Z)$ is holomorphic on R_{I}. Let $M_1 = \sup_{Z \in E} |\psi_{11}(Z)| = |\psi_{11}(Z_0)|$ for $Z_0 \in E$. From $\Psi(Z) \in \mathrm{Aut}(R_{\mathrm{I}})$ we know $M_1 = |\psi_{11}(Z_0)| < 1$, so we may choose $M_0 > 0$ with $M_1 < M_0 < 1$. Thus, for j large enough, $|\psi_{11}^{(j)}(Z_0)| < M_0$ and from this it follows that

$$1 - |\psi_{11}^{(j)}(Z)| > 1 - M_0 > 0;$$

by the definition of $f_j(Z)$, it is easy to see that $g_j(Z) = f_j \circ \Psi^{(j)}(Z)$ tends to 0 uniformly on E.

Hence $g_j(Z)$ satisfies conditions (i)–(iii), and this contradicts the compactness of C_ϕ by Lemma 2.

Part C: Assume that

$$\phi(Z^j) = \sum_{k=1}^{m} r_j^{(k)} E_{kk}, \quad 1 > r_j^{(1)} \geq r_j^{(2)} \geq \cdots \geq r_j^{(m)} \geq 0.$$

Just as in Part B, we can construct a sequence of functions $f_j(z)$ that satisfies conditions (i)–(iii).

Part D: In the general situation we have $\phi(Z^j) \in R_{\mathrm{I}}(m, n)$, so there exist an $m \times m$ unitary matrix P_j and an $n \times n$ unitary matrix Q_j such that

$$P_j(\phi(Z^j))Q_j = \sum_{k=1}^{m} r_j^{(k)} E_{kk}.$$

We may assume that $P_j \to P$ and $Q_j \to Q$ as $j \to \infty$ (let $P_j = (p_j^{kl})$ and $P = (p^{kl})$; $P_j \to P$ means that $p_j^{kl} \to p^{kl}$ as $j \to \infty$ for any $1 \geq k \geq m$ and $1 \geq l \geq n$). Let $\psi^j(Z) = P_j Z Q_j$ and $\Psi(Z) = PZQ$ for $Z \in R_{\mathrm{I}}(m, n)$. It is easy to show that P is an $m \times m$ unitary matrix, Q is an $n \times n$ unitary matrix, and $\psi^{(j)}(Z)$ converges uniformly to $\psi(Z)$ on R_{I}.

Let $g_j(Z) = f_j(\psi^{(j)}(Z))$, where $\{f_j\}$ are the functions obtained in Part C. From the same disscussion as that of Part B, we know that $g_j(Z)$ satisfies conditions (i) and (iii). For the compact subset $E \subset R_{\mathrm{I}}$, it is easy to see that $\psi(E)$ is also a compact subset of R_{I}, so we can choose an open subset D_1 of R_{I} such that $\psi(E) \subset D_1 \subset \overline{D_1} \subset R_{\mathrm{I}}$. Since $\psi^{(j)}(Z)$ converges uniformly to $\psi(Z)$ on R_{I}, it follows that $\psi^j(E) \subset D_1$ as $j \to \infty$. Since $f_j(Z)$ tends to 0 uniformly on $\overline{D_1}$, we know $g_j(Z) = f_j(\psi^{(j)}(Z))$ tends to 0 uniformly on $E \subset R_{\mathrm{I}}$; that is, g_j satisfies condition (iii).

The last claim follows from the previous discussion. This completes the proof for $\beta(R_{\mathrm{I}})$. For $\beta(R_{\mathrm{II}})$, the proof is the same as that for $\beta(R_{\mathrm{I}})$; we omit the details.

4. Compactness of C_ϕ on $\beta(R_{\mathrm{III}})$

As in the case of $\beta(R_{\mathrm{I}})$, we need only prove that condition (6) is necessary. Suppose C_ϕ is compact on $\beta(R_{\mathrm{III}})$ and that condition (6) fails. Then there exist a sequence $\{Z^j\}$ in $R_{\mathrm{III}}(q)$ with $\phi(Z^j) \to \partial R_{\mathrm{III}}$ as $j \to \infty$, a sequence of $q \times q$ antisymmetric complex matrices $U^j \neq 0$, and an ε_0 such that

$$\frac{H_{\phi(Z^j)}(J\phi(Z^j)u^j, J\phi(Z^j)u^j)}{H_{Z^j}(u^j, u^j)} \geq \varepsilon_0 \tag{33}$$

for all $j = 1, 2, \ldots$, where u^j is the vector corresponding to U^j.

Using (33), we will construct a sequence of functions $\{f_j\}$ satisfying the following three conditions:

(i) $\{f_j\}$ is a bounded sequence in $\beta(R_{\mathrm{III}})$;
(ii) $\{f_j\}$ tends to 0 uniformly on compact subsets of R_{III};
(iii) $\|C_\phi f_j\|_{\beta(R_{\mathrm{III}})} \not\to 0$ as $j \to \infty$.

This sequence will contradict the compactness of C_ϕ, by Lemma 2.

To construct the sequence of $\{f_j\}$, we first assume that

$$\phi(Z^j) = r_j(E_{12} - E_{21}), \quad j = 1, 2, \ldots,$$

where E_{lk} is a $q \times q$ matrix whose element in the lth row and kth column is 1 and whose other elements are 0. It is clear that $0 < r_j < 1$ and $r_j \to 1$ as $j \to \infty$.

Denote

$$J\phi(Z^j)u^j = w^j = (w^j_{11}, \ldots, w^j_{1q}, \ldots, w^j_{q1}, \ldots, w^j_{qq}),$$

where W^j is the matrix corresponding to the vector w^j. Using formula (4), we have

$$\begin{aligned}
&H^{\mathrm{III}}_{\phi(Z^j)}(w^j, w^j) \\
&\quad = H^{\mathrm{III}}_{r_j(E_{12}-E_{21})}(w^j, w^j) \\
&\quad = 2(q-1)w^j \begin{pmatrix} (1-r_j^2)^{-1} & 0 & 0 & \ldots & 0 \\ 0 & (1-r_j^2)^{-1} & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}
\end{aligned}$$

$$\times \begin{pmatrix} (1-r_j^2)^{-1} & 0 & 0 & \dots & 0 \\ 0 & (1-r_j^2)^{-1} & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \overline{w^j}'$$

$$\begin{aligned} &= 2(q-1)\Bigg[\frac{1}{(1-r_j^2)^2}(|w_{11}^j|^2 + |w_{12}^j|^2 + |w_{21}^j|^2 + |w_{22}^j|^2) \\ &\qquad + \frac{2}{1-r_j^2}\Bigg(\sum_{k=3}^{q}(|w_{1k}^j|^2 + |w_{2k}^j|^2) + \sum_{l=3}^{q}(|w_{l1}^j|^2 + |w_{l2}^j|^2)\Bigg) \\ &\qquad + \sum_{3\le k,l\le q}|w_{kl}^j|^2\Bigg]. \end{aligned} \tag{34}$$

Since ϕ holomorphically maps R_{III} into itself, it follows that $\phi(Z) \in R_{\mathrm{III}}$ for each $Z \in R_{\mathrm{III}}$; namely, $\phi(Z)$ is a $q \times q$ antisymmetric matrix. If we write $\phi(Z) = (\phi_{lk}(Z))_{1\le l,k\le q}$, then $w^j = J\phi(Z^j)u^j$ gives

$$w_{lk}^j = \sum_{s,t=1}^{q} \frac{\partial \phi_{lk}(Z^j)}{\partial z_{st}} u_{st}^j,$$

and $\phi_{lk}(Z) = -\phi_{kl}(Z)$ implies $w_{lk}^j = -w_{kl}^j$ for $l, k = 1, 2, \dots, q$, so W^j is also an antisymmetric matrix. Thus (34) becomes

$$\begin{aligned} H_{\phi(Z^j)}^{\mathrm{III}}(w^j, w^j) &= 2(q-1)\Bigg[\frac{1}{(1-r_j^2)^2}|w_{12}^j|^2 + \frac{2}{1-r_j^2}\sum_{k=3}^{q}(|w_{2k}^j|^2 + |w_{1k}^j|^2) \\ &\qquad + \sum_{3\le k,l\le q}|w_{kl}^j|^2\Bigg]. \end{aligned} \tag{35}$$

Denote

$$A_j^{\mathrm{III}} = \frac{1}{(1-r_j^2)^2}|w_{12}^j|^2,$$

$$B_j^{\mathrm{III}} = \frac{2}{1-r_j^2}\sum_{k=3}^{q}(|w_{2k}^j|^2 + |w_{1k}^j|^2),$$

$$C_j^{\mathrm{III}} = \sum_{3\le k,l\le q}|w_{kl}^j|^2;$$

then

$$H_{\phi(Z^j)}^{\mathrm{III}}(w^j, w^j) = 2(q-1)(A_j^{\mathrm{III}} + B_j^{\mathrm{III}} + C_j^{\mathrm{III}}). \tag{36}$$

We construct the functions according to three different cases.

Case 1. If, for some j,

$$\max(B_j^{\mathrm{III}}, C_j^{\mathrm{III}}) \le A_j^{\mathrm{III}}, \tag{37}$$

then set

$$f_j(Z) = 2\log(1 - e^{-a(1-r_j)} z_{12}) - \log(1 - z_{12}), \tag{38}$$

where $Z \in R_{\mathrm{III}}$ and a is any positive number.

Case 2. If, for some j,

$$\max(A_j^{\mathrm{III}}, C_j^{\mathrm{III}}) \le B_j^{\mathrm{III}}, \tag{39}$$

then set

$$f_j(Z) = 2\sum_{k=3}^{q} (e^{-i\theta_{1k}^j} z_{1k} + e^{-i\theta_{2k}^j} z_{2k}) \left(\frac{1}{\sqrt{1 - e^{-a(1-r_j)} z_{12}}} - \frac{1}{\sqrt{1 - z_{12}}} \right), \tag{40}$$

where $Z \in R_{\mathrm{III}}$ and a is any positive number and where $\theta_{1k}^j = \arg w_{1k}^j$ and $\theta_{2k}^j = \arg w_{2k}^j$. If $w_{1k}^j = 0$ or $w_{2k}^j = 0$ for some k, then replace the corresponding term $e^{-i\theta_{1k}^j} z_{1k}$ or $e^{-i\theta_{2k}^j} z_{2k}$ by 0.

Case 3. If, for some j,

$$\max(A_j^{\mathrm{III}}, B_j^{\mathrm{III}}) \le C_j^{\mathrm{III}}, \tag{41}$$

then set

$$f_j(Z) = \left(\sum_{3 \le k, l \le q} e^{-i\theta_{kl}^j} z_{kl} \right) \sqrt{1 - z_{12}} \left(\frac{1}{\sqrt{1 - e^{-a(1-r_j)} z_{12}}} - \frac{1}{\sqrt{1 - z_{12}}} \right), \tag{42}$$

where $Z \in R_{\mathrm{III}}$, a is any positive number, and $\theta_{kl}^j = \arg w_{kl}^j$ for $2 \le k$ and $l \le q$. If $w_{kl}^j = 0$ for some k or l, replace the corresponding term $e^{-i\theta_{kl}^j} z_{kl}$ by 0.

The proofs that functions defined by (38), (40), and (42) satisfy conditions (i)–(iii) are similar to that for $\beta(R_{\mathrm{I}})$, so we omit the details here.

A simple exercise shows that, for any $Z \in R_{\mathrm{III}}(q)$, there exists a $q \times q$ unitary matrix U such that

$$Z = \begin{cases} U'(\lambda_1(E_{12} - E_{21}) + \cdots + \lambda_v(E_{2v-1,2v} - E_{2v,2v-1}))U, & q = 2v, \\ U'(\lambda_1(E_{12} - E_{21}) + \cdots + \lambda_v(E_{2v-1,2v} - E_{2v,2v-1}) + 0)U, & q = 2v+1, \end{cases}$$

where $1 \ge \lambda_1 \ge \cdots \ge \lambda_v \ge 0$. So, using constructions similar to those in Parts B, C, and D of Section 3 for $\beta(R_{\mathrm{I}})$, it is not hard to complete the proof for $\beta(R_{\mathrm{III}})$. We omit the details.

5. Compactness of C_ϕ on $\beta(R_{\mathrm{IV}})$

In order to study the compactness of composition operators C_ϕ on $\beta(R_{\mathrm{IV}})$, we first introduce a new domain $\mathbb{R}_{\mathrm{IV}}$ as follows:

$$\mathbb{R}_{\mathrm{IV}} = \left\{ \zeta \in \mathbb{C}^N : \zeta_1 = z_1 + iz_2,\ \zeta_2 = z_1 - iz_2,\ \zeta_k = \sqrt{2} z_k,\ 3 \le k \le N,\ z \in R_{\mathrm{IV}} \right\}.$$

If we define the biholomorphic map $\psi = (\psi_1, \ldots, \psi_N) : \mathbb{C}^N \to \mathbb{C}^N$ by

$$\psi_1(z) = z_1 + iz_2, \quad \psi_2(z) = z_1 - iz_2, \quad \psi_k(z) = \sqrt{2}z_k \ (k = 3, 4, \dots, N),$$

then $\mathbb{R}_{\mathrm{IV}} = \psi(R_{\mathrm{IV}})$; that is, $\mathbb{R}_{\mathrm{IV}}$ is equivalent biholomorphically to R_{IV}.

By the definition of R_{IV}, we can write $\mathbb{R}_{\mathrm{IV}}$ as

$$\begin{aligned}\mathbb{R}_{\mathrm{IV}} = \Big\{\zeta = (\zeta_1, \dots, \zeta_N) \in \mathbb{C}^N : 1 + \left|\zeta_1\zeta_2 + \tfrac{1}{2}\zeta_3^2 + \dots + \tfrac{1}{2}\zeta_N^2\right|^2 - \zeta\bar{\zeta}' > 0, \\ 1 - \left|\zeta_1\zeta_2 + \tfrac{1}{2}\zeta_3^2 + \dots + \tfrac{1}{2}\zeta_N^2\right|^2 > 0\Big\}.\end{aligned} \tag{43}$$

For $\mathbb{R}_{\mathrm{IV}}$, we have the following proposition.

PROPOSITION 1. *If $\zeta \in \mathbb{R}_{\mathrm{IV}}$, then $|\zeta_1| < 1$ and $|\zeta_2| < 1$.*

Proof. If $|\zeta_1| = 1$, then $\zeta \in \mathbb{R}_{\mathrm{IV}}$ shows that $|\zeta|^2 < 2$, so

$$|\zeta_2|^2 + |\zeta_3|^2 + \dots + |\zeta_N|^2 < 1; \tag{44}$$

moreover,

$$\begin{aligned}1 &+ \left|\zeta_1\zeta_2 + \tfrac{1}{2}\zeta_3^2 + \dots + \tfrac{1}{2}\zeta_N^2\right|^2 - \zeta\bar{\zeta}' \\ &= |\zeta_1|^2|\zeta_2|^2 + \tfrac{1}{2}\overline{\zeta_1\zeta_2}(\zeta_3^2 + \dots + \zeta_N^2) + \tfrac{1}{2}\zeta_1\zeta_2\overline{(\zeta_3^2 + \dots + \zeta_N^2)} \\ &\quad + \tfrac{1}{4}|\zeta_3^2 + \dots + \zeta_N^2|^2 - |\zeta_2|^2 - |\zeta_3|^2 - \dots - |\zeta_N|^2 \\ &= \tfrac{1}{2}\overline{\zeta_1\zeta_2}(\zeta_3^2 + \dots + \zeta_N^2) + \tfrac{1}{2}\zeta_1\zeta_2\overline{(\zeta_3^2 + \dots + \zeta_N^2)} \\ &\quad + \tfrac{1}{4}|\zeta_3^2 + \dots + \zeta_N^2|^2 - (|\zeta_3|^2 + \dots + |\zeta_N|^2) \\ &\le |\zeta_2|(|\zeta_3|^2 + \dots + |\zeta_N|^2) + \tfrac{1}{4}(|\zeta_3|^2 + \dots + |\zeta_N|^2)^2 - (|\zeta_3|^2 + \dots + |\zeta_N|^2) \\ &< \Big[\sqrt{1 - (|\zeta_3|^2 + \dots + |\zeta_N|^2)} \\ &\quad + \tfrac{1}{4}(|\zeta_3|^2 + \dots + |\zeta_N|^2) - 1\Big](|\zeta_3|^2 + \dots + |\zeta_N|^2).\end{aligned} \tag{45}$$

Let $\lambda = |\zeta_3|^2 + \dots + |\zeta_N|^2$. It follows from (44) that $0 \le \lambda < 1$. Let $g(\lambda) = \sqrt{1-\lambda} + \frac{1}{4}\lambda - 1$; then, since $0 \le \lambda < 1$,

$$\frac{dg(\lambda)}{d\lambda} = \frac{1}{4}\left(1 - \frac{2}{\sqrt{1-\lambda}}\right) < 0$$

and so $g(\lambda)$ is a nonincreasing function. Since $g(0) = 0$ we know $g(\lambda) \le 0$; hence, by (45) we have that

$$1 + \left|\zeta_1\zeta_2 + \tfrac{1}{2}\zeta_3^2 + \dots + \tfrac{1}{2}\zeta_N^2\right|^2 - \zeta\bar{\zeta}' \le 0.$$

This contradicts the first equality of (43), so $|\zeta_1| \ne 1$.

If $|\zeta_1| > 1$ then it follows, since $\zeta, 0 \in \mathbb{R}_{\mathrm{IV}} = \psi(R_{\mathrm{IV}})$ and $\mathbb{R}_{\mathrm{IV}}$ is a domain, that in $\mathbb{R}_{\mathrm{IV}}$ there exists a continuous curve $\zeta(t) = (\zeta_1(t), \dots, \zeta_N(t))$, $0 \le t \le 1$, such that $\zeta(0) = 0$ and $\zeta(1) = \zeta$. Given $\zeta_1(0) = 0$, $\zeta_1(1) = \zeta_1$, and $|\zeta_1| > 1$, there should exist a $\zeta(t_0) \in \psi(R_N)$ such that $\zeta_1(t_0) = 1$, but by our previous discussion this is impossible. So $|\zeta_1| < 1$ and, for the same reason as before, $|\zeta_2| < 1$. This ends the proof. □

PROPOSITION 2. *Let $\mathbb{H}^{\mathrm{IV}}_{\zeta}(w, w)$ be the Bergman metric of $\mathbb{R}_{\mathrm{IV}}$. Then*

$$\mathbb{H}^{\mathrm{IV}}_{2re_2}(w, w) = N\left(|w_1|^2 + \frac{1}{(1-4r^2)^2}|w_2|^2 + \frac{1}{1-4r^2}\sum_{k=3}^{N}|w_k|^2\right),$$

where $0 \le r < 1$, $e_2 = (0, 1, 0, \ldots, 0)$, *and* $w \in \mathbb{C}^N$.

Proof. Let $e_1 = (1, 0, \ldots, 0)$. By the definition of ψ it is clear that, if $z = r(e_1 + ie_2) \in R_{\mathrm{IV}}$, then $2re_2 = \psi(r(e_1 + ie_2)) \in \mathbb{R}_{\mathrm{IV}}$ and

$$J\psi(z) = \begin{bmatrix} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} & 0 \\ 0 & \sqrt{2} I^{N-2} \end{bmatrix},$$

$$J\psi^{-1}(\zeta) = \begin{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2i} & -\frac{1}{2i} \end{bmatrix} & 0 \\ 0 & \frac{1}{\sqrt{2}} I^{N-2} \end{bmatrix}.$$

Let $u = J\psi^{-1}w$. Since $z = r(e_1 + ie_2) = r(1, i, 0, \ldots, 0)$, a simple calculation shows that

$$zz' = 0, \qquad 1 + |zz'|^2 - 2|z|^2 = 1 - 4r^2. \tag{46}$$

If we write $u = (u_1, u_2, \ldots, u_N)$, then

$$u_1 = \frac{w_1 + w_2}{2}, \quad u_2 = \frac{w_1 - w_2}{2i}, \quad u_k = \frac{1}{\sqrt{2}} w_k \ \ (3 \le k \le N); \tag{47}$$

$$\begin{aligned}
& u\begin{pmatrix} z \\ \bar{z} \end{pmatrix}' \begin{pmatrix} 1 - 2|z|^2 & \overline{zz'} \\ zz' & -1 \end{pmatrix} \overline{\begin{pmatrix} z \\ \bar{z} \end{pmatrix}} \bar{u}' \\
&= u(z', \bar{z}') \begin{pmatrix} 1 - 4r^2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \bar{z} \\ z \end{pmatrix} \bar{u}' \\
&= r(u_1, u_2, \ldots, u_N) \begin{pmatrix} 1 & 1 \\ i & -i \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 - 4r^2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -i & 0 & \ldots & 0 \\ 1 & i & 0 & \ldots & 0 \end{pmatrix} \begin{pmatrix} \bar{u}'_1 \\ \bar{u}'_2 \\ \vdots \\ \bar{u}'_N \end{pmatrix} \\
&= r(u_1 + iu_2, u_1 - iu_2) \begin{pmatrix} 1 - 4r^2 & 0 \\ 0 & -1 \end{pmatrix} r \begin{pmatrix} \overline{u_1 + iu_2} \\ \overline{u_1 - iu_2} \end{pmatrix} \\
&= r^2(|u_1 + iu_2|^2 - |u_1 - iu_2|^2 - 4r^2|u_1 + iu_2|^2).
\end{aligned} \tag{48}$$

Using (46), (47), and (48), formula (5) gives

$$
\begin{aligned}
&\mathbb{H}^{\mathrm{IV}}_{2re_2}(w,w)\\
&\quad= \mathbb{H}^{\mathrm{IV}}_{\psi(r(e_1+ie_2))}(w,w)\\
&\quad= H^{\mathrm{IV}}_{r(e_1+ie_2)}(J\psi^{-1}(z)w, J\psi^{-1}(z)w) = H^{\mathrm{IV}}_z(u,u)\\
&\quad= \frac{2N}{(1-4r^2)^2}\biggl((1-4r^2)\sum_{k=1}^{N}|u_k|^2\\
&\qquad\qquad - 2r^2(|u_1+iu_2|^2-|u_1-iu_2|^2-4r^2|u_1+iu_2|^2)\biggr)\\
&\quad= \frac{2N}{(1-4r^2)^2}\biggl((1-4r^2+8r^4)(|u_1|^2+|u_2|^2)\\
&\qquad\qquad + 8r^2(1-2r^2)\operatorname{Im}(\bar{u}_1u_2)+(1-4r^2)\sum_{k=3}^{N}|u_k|^2\biggr)\\
&\quad= \frac{2N}{(1-4r^2)^2}\biggl((1-4r^2+8r^4)\biggl(\left|\frac{w_1+w_2}{2}\right|^2+\left|\frac{w_1-w_2}{2i}\right|^2\biggr)\\
&\qquad\qquad + 8r^2(1-2r^2)\operatorname{Im}\biggl(\overline{\frac{w_1+w_2}{2}}\,\frac{w_1-w_2}{2i}\biggr)\\
&\qquad\qquad + (1-4r^2)\sum_{k=3}^{N}\left|\frac{w_k}{2}\right|^2\biggr)\\
&\quad= N\biggl(|w_1|^2+\frac{1}{(1-4r^2)^2}|w_2|^2+\frac{1}{1-4r^2}\sum_{k=3}^{N}|w_k|^2\biggr).
\end{aligned}
$$

This completes the proof. □

Using Lemma 2, it is not hard to prove the following proposition. We omit the details.

PROPOSITION 3. *C_ϕ is compact on $\beta(R_{\mathrm{IV}})$ if and only if $C_\Phi = C_{\psi\circ\phi\circ\psi^{-1}}$ is compact on $\mathbb{R}_{\mathrm{IV}} = \psi(R_{\mathrm{IV}})$, where $\Phi = \psi\circ\phi\circ\psi^{-1}$.*

PROPOSITION 4. *Let $u \in \mathbb{C}^N$, $w = J\psi(\zeta)u$, $z \in R_{\mathrm{IV}}$, $\zeta = \psi(z)$, and $\Phi = \psi\circ\phi\circ\psi^{-1}$. Then*

$$
\frac{\mathbb{H}^{\mathrm{IV}}_{\Phi(\zeta)}(J\Phi(\zeta)w, J\Phi(\zeta)w)}{\mathbb{H}^{\mathrm{IV}}_{\zeta}(w,w)} = \frac{H^{\mathrm{IV}}_{\phi(z)}(J\phi(z)u, J\phi(z)u)}{H^{\mathrm{IV}}_z(u,u)}.
$$

Proof. It is clear that

$$
\begin{aligned}
&\mathbb{H}^{\mathrm{IV}}_{\Phi(\zeta)}(J\Phi(\zeta)w, J\Phi(\zeta)w)\\
&\quad= \mathbb{H}^{\mathrm{IV}}_{\psi\circ\phi\circ\psi^{-1}(\zeta)}\bigl(J(\psi\circ\phi\circ\psi^{-1})(\zeta)w, J(\psi\circ\phi\circ\psi^{-1})(\zeta)w\bigr)\\
&\quad= \mathbb{H}^{\mathrm{IV}}_{\psi(\phi(z))}\bigl(J\psi(\phi(z))J\phi(z)J\psi^{-1}(\zeta)w, J\psi(\phi(z))J\phi(z)J\psi^{-1}(\zeta)w\bigr)
\end{aligned}
$$

$$= H^{\mathrm{IV}}_{\phi(z)}(J\phi(z)J\psi^{-1}(\zeta)w, J\phi(z)J\psi^{-1}(\zeta)w)$$
$$= H^{\mathrm{IV}}_{\phi(z)}(J\phi(z)u, J\phi(z)u),$$

and

$$\mathbb{H}^{\mathrm{IV}}_{\zeta}(w, w) = \mathbb{H}^{\mathrm{IV}}_{\psi(z)}(w, w) = H^{\mathrm{IV}}_{z}(J\psi^{-1}(\zeta)w, J\psi^{-1}(\zeta)w) = H^{\mathrm{IV}}_{z}(u, u).$$

The desired result follows. □

Since R_{IV} is equivalent holomorphically to $\mathbb{R}_{\mathrm{IV}} = \psi(R_{\mathrm{IV}})$ (by Propositions 3 and 4), we may discuss the domain $\mathbb{R}_{\mathrm{IV}}(N)$ instead of $R_{\mathrm{IV}}(N)$. As with $\beta(R_{\mathrm{I}})$, we need only prove that condition (6) is necessary.

Assume for now that condition (6) fails. By Proposition 4, there would then exist a sequence $\{\zeta^j\} \in \mathbb{R}_{\mathrm{IV}}$ with $\Phi(\zeta^j) \to \partial\mathbb{R}_{\mathrm{IV}}$ as $j \to \infty$, a $w^j \in \mathbb{C}^N - \{0\}$, and an ε_0 such that

$$\frac{\mathbb{H}^{\mathrm{IV}}_{\Phi(\zeta^j)}(J\Phi(\zeta^j)w^j, J\phi(\zeta^j)w^j)}{\mathbb{H}^{\mathrm{IV}}_{\zeta^j}(w^j, w^j)} \geq \varepsilon_0 \tag{49}$$

for all $j = 1, 2, \ldots$.

Using (49), we will construct a sequence of functions $\{f_j\}$ satisfying the following three conditions:

(i) $\{f_j\}$ is a bounded sequence in $\beta(\mathbb{R}_{\mathrm{IV}})$;
(ii) $\{f_j\}$ tends to 0 uniformly on compact subsets of $\mathbb{R}_{\mathrm{IV}}$;
(iii) $\|C_\phi f_j\|_{\beta(\mathbb{R}_{\mathrm{IV}})} \not\to 0$ as $j \to \infty$.

This sequence will contradict the compactness of C_ϕ, by Lemma 2.

To construct the sequence of $\{f_j\}$, we first assume that

$$\Phi(\zeta^j) = 2r_j e_2, \quad j = 1, 2, \ldots, \tag{50}$$

where $e_k = (0, \ldots, 1, \ldots, 0)$, the kth coordinate is 1, and the other coordinates are 0 (for some fixed k, $1 \leq k \leq N$).

It is clear that $0 < r_j < \frac{1}{2}$ and $r_j \to \frac{1}{2}$ as $j \to \infty$. Denote $J\Phi(\zeta^j)w^j = v^j$.

Proposition 2 shows that

$$\begin{aligned}
&\mathbb{H}^{\mathrm{IV}}_{\Phi(\zeta^j)}(J\Phi(\zeta^j)w^j, J\Phi(\zeta^j)w^j)\\
&\quad = \mathbb{H}^{\mathrm{IV}}_{2re_2}(v^j, v^j)\\
&\quad = N\left(|v_1^j|^2 + \frac{1}{(1-4r^2)^2}|v_2^j|^2 + \frac{1}{1-4r^2}\sum_{k=3}^{N}|v_k^j|^2\right).
\end{aligned}$$

Denote

$$A_j^{\mathrm{IV}} = \frac{1}{1-4r^2}\sum_{k=3}^{N}|v_k^j|^2,$$

$$B_j^{\mathrm{IV}} = |v_1^j|^2,$$

$$C_j^{\mathrm{IV}} = \frac{1}{(1-4r^2)^2}|v_2^j|^2;$$

then

$$\mathbb{H}^{\mathrm{IV}}_{\Phi(\zeta^j)}(v^j, v^j) = N(A_j^{\mathrm{IV}} + B_j^{\mathrm{IV}} + C_j^{\mathrm{IV}}). \tag{51}$$

We construct the functions according to three different cases.

Case 1. If, for some j,

$$\max(B_j^{\mathrm{IV}}, C_j^{\mathrm{IV}}) \le A_j^{\mathrm{IV}}, \tag{52}$$

then set

$$f_j(\zeta) = \log(1 - e^{-a(1-2r_j)}\zeta_2) - \log(1 - \zeta_2), \tag{53}$$

where $\zeta \in \mathbb{R}_{\mathrm{IV}}$ and a is any positive number.

Case 2. If, for some j,

$$\max(A_j^{\mathrm{IV}}, C_j^{\mathrm{IV}}) \le B_j^{\mathrm{IV}}, \tag{54}$$

then set

$$f_j(\zeta) = \zeta_1\sqrt{1-\zeta_2}\left(\frac{1}{\sqrt{1 - e^{-a(1-2r_j)}\zeta_2}} - \frac{1}{\sqrt{1-\zeta_2}}\right), \tag{55}$$

where $\zeta \in \mathbb{R}_{\mathrm{IV}}$ and a is any positive number.

Case 3. If, for some j,

$$\max(A_j^{\mathrm{IV}}, B_j^{\mathrm{IV}}) \le C_j^{\mathrm{IV}}, \tag{56}$$

then set

$$f_j(\zeta) = \left(\sum_{k=3}^{N} e^{-i\theta_k^j}\frac{\zeta_k}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{1 - e^{-a(1-2r_j)}\zeta_2}} - \frac{1}{\sqrt{1-\zeta_2}}\right), \tag{57}$$

where a is any positive number and $\theta_k^j = \arg W_k^j$ for $k \ge 2$. If $W_k^j = 0$ for some k, then replace the corresponding term $e^{-i\theta_k^j}\zeta_k$ by 0.

The proofs that the functions defined by (53), (55), and (57) satisfy conditions (i)–(iii) are similar to the proof for $\beta(R_{\mathrm{I}})$; we omit the details.

A simple exercise shows that, for any $\zeta \in \mathbb{R}_{\mathrm{IV}}$, there exists an $N \times N$ unitary matrix U such that

$$\zeta = (\lambda, \mu, 0, \dots, 0)U,$$

where $0 \le \lambda \le \mu < 1$. So, using the same methods as in Parts B and D of Section 3 for $\beta(R_{\mathrm{I}})$, it is not hard to complete the proof for $\beta(\mathbb{R}_{\mathrm{IV}})$ or $\beta(R_{\mathrm{IV}})$. We omit the details.

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