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On the L^p Boundedness of Marcinkiewicz Integrals

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1. Introduction and Results

Let $n \geq 2$ and let S^{n-1} be the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $d\sigma$. Let $b(\cdot) \in L^\infty(\mathbb{R}_+)$ and let Ω be a homogeneous function of degree zero on $\mathbb{R}^n$ (which is then naturally identified with a function on S^{n-1}) satisfying $\Omega \in L^1(S^{n-1})$ and

$$\int_{S^{n-1}} \Omega(y)\, d\sigma(y) = 0. \tag{1.1}$$

For a suitable mapping $\Phi \colon \mathbb{R}^n \to \mathbb{R}^d$, we define the Marcinkiewicz integral operator $\mu_{\Phi,\Omega,b}$ on $\mathbb{R}^d$ by

$$\mu_{\Phi,\Omega,b}(f)(x) = \left(\int_0^\infty |F_{\Phi,t}(x)|^2 \frac{dt}{t^3} \right)^{1/2}, \tag{1.2}$$

where

$$F_{\Phi,t}(x) = \int_{|y| \leq t} \frac{\Omega(y)}{|y|^{n-1}} b(|y|) f(x - \Phi(y))\, dy. \tag{1.3}$$

If $n = d$, $\Phi(y) = (y_1, y_2, \dots, y_n)$, and $b \equiv 1$, then we shall simply denote the operator $\mu_{\Phi,\Omega,b}$ by μ_Ω.

The main purpose of this paper is to study the L^p boundedness of the operators $\mu_{\Phi,\Omega,b}$. The operator μ_Ω was introduced by Stein [S1]. He proved that if Ω satisfies a Lip_α $(0 < \alpha \leq 1)$ condition on S^{n-1}, then μ_Ω is of type (p, p) for $1 < p \leq 2$ and of weak type $(1, 1)$. Subsequently Benedek, Calderón, and Panzone [BCP] showed that if Ω is continuously differentiable on S^{n-1} then μ_Ω is of type (p, p) for $1 < p < \infty$. In a more recent paper [DFP] we obtained the L^p boundedness of μ_Ω under the substantially weaker assumption that $\Omega \in H^1(S^{n-1})$. In fact, it was proved in [DFP] that the operator $\mu_{\mathbf{1},\Omega,b}$ is bounded on $L^p(\mathbb{R}^n)$ provided that $\Omega \in H^1(S^{n-1})$ and $b(\cdot) \in L^\infty(\mathbb{R}_+)$. Here $\mathbf{1}$ represents the identity mapping from $\mathbb{R}^n$ to itself and $H^1(S^{n-1})$ denotes the Hardy space on the unit sphere that contains $L \log^+ L(S^{n-1})$ as a proper subspace (see Section 3 for its definition).

In this paper we shall establish the L^p boundedness of $\mu_{\Phi,\Omega,b}$ for several classes of mapping Φ with rough kernels Ω, mirroring recent developments in the theory

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of singular integrals. A sample of our results is the following statement concerning polynomial mappings.

THEOREM 1. *Let $\mathcal{P} = (P_1, \dots, P_d)$, where P_j is a real-valued polynomial on $\mathbb{R}^n$ for $1 \le j \le d$. Suppose that $b(r) \in L^\infty(\mathbb{R}_+)$ and that $\Omega \in H^1(S^{n-1})$ and satisfies* (1.1). *Then, for $1 < p < \infty$, there exists a constant $C_p > 0$ such that*

$$\|\mu_{\mathcal{P},\Omega,b}(f)\|_{L^p(\mathbb{R}^d)} \le C_p \|f\|_{L^p(\mathbb{R}^d)} \tag{1.4}$$

for every $f \in L^p(\mathbb{R}^d)$. The constant C_p may depend on n, d, and $\deg(P_j)$ $(1 \le j \le d)$, *but it is independent of the coefficients of P_j.*

Similar results for C^∞ mappings of finite type and homogeneous mappings will be described in Section 4. The proof of Theorem 1, to be presented in Sections 2–3, can be easily adapted to treat other classes of mappings. Finally, we include in Section 5 a brief discussion on the Marcinkiewicz integral operators related to area integrals and the Littlewood–Paley g_λ^* functions.

2. Main Lemma

For a family of measures $\tau = \{\tau_{k,t} \mid k \in \mathbb{N}, t \in \mathbb{R}\}$ on $\mathbb{R}^d$, we define the operators Δ_τ and τ_k^* by

$$\Delta_\tau(f)(x) = \sum_{k=1}^{\infty} \left(\int_{\mathbb{R}} |(\tau_{k,t} * f)(x)|^2 \, dt \right)^{1/2} \tag{2.1}$$

and

$$\tau_k^*(f)(x) = \sup_{t \in \mathbb{R}} (|\tau_{k,t}| * |f|)(x) \tag{2.2}$$

for $k \in \mathbb{N}$.

LEMMA 2.1. *Let $m \in \mathbb{N}$ and $L\colon \mathbb{R}^d \to \mathbb{R}^m$ be a linear transformation. Suppose that there are constants C_0, C_p $(1 < p < \infty)$, $\alpha, \beta > 0$, and $\gamma \ne 0$ such that the following hold for $k \in \mathbb{N}$, $t \in \mathbb{R}$, $\xi \in \mathbb{R}^d$, and $p \in (1, \infty)$:*

$$\|\tau_{k,t}\| \le C_0 2^{-k}; \tag{2.3}$$

$$|\hat{\tau}_{k,t}(\xi)| \le C_0 2^{-k} \min\{(2^{(t-k)\gamma}|L\xi|)^{\alpha}, (2^{(t-k)\gamma}|L\xi|)^{-\beta}\}; \tag{2.4}$$

$$\|\tau_k^*(f)\|_{L^p(\mathbb{R}^d)} \le C_p 2^{-k} \|f\|_{L^p(\mathbb{R}^d)}. \tag{2.5}$$

Then, for $1 < p < \infty$, there exists a constant $A_p > 0$ such that

$$\|\Delta_\tau(f)\|_{L^p(\mathbb{R}^d)} \le A_p \|f\|_{L^p(\mathbb{R}^d)} \tag{2.6}$$

for all $f \in L^p(\mathbb{R}^d)$. The constant A_p may depend on C_0, C_p, α, β, γ, n, d, and m, but it is independent of the linear transformation L.

Proof. Clearly we may assume that $|\gamma|\alpha, |\gamma|\beta < 1$. We shall begin with the special case in which $m \le d$ and $L\xi = \pi_m^d(\xi) = (\xi_1, \dots, \xi_m)$ for $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$. Choose a C^∞ function $\psi\colon \mathbb{R} \to [0, 1]$ such that $\operatorname{supp}(\psi) \subset [1/4, 4]$ and

$$\int_0^\infty \frac{\psi(r)}{r}\, dr = 2.$$

Define the Schwartz functions $\Psi, \Psi_t : \mathbb{R}^m \to \mathbb{C}$ by

$$\hat{\Psi}(\xi_1, \ldots, \xi_m) = \psi(\xi_1^2 + \cdots + \xi_m^2)$$

and $\Psi_t(z) = t^{-m}\Psi(z/t)$ for $t > 0$ and $z \in \mathbb{R}^m$. If we let δ_{d-m} represent the Dirac delta on $\mathbb{R}^{d-m}$, then

$$f(x) = \int_0^\infty (\Psi_t \otimes \delta_{d-m}) * f(x)\, \frac{dt}{t} = (\gamma \ln 2) \int_{\mathbb{R}} (\Psi_{2^{\gamma s}} \otimes \delta_{d-m}) * f(x)\, ds. \tag{2.7}$$

Define the g-function $g(f)$ by

$$g(f)(x) = \left(\int_{\mathbb{R}} |(\Psi_{2^{\gamma s}} \otimes \delta_{d-m}) * f(x)|^2\, ds \right)^{1/2}. \tag{2.8}$$

A fact that will be used a little later is the L^p boundedness of the operator $f \to g(f)$, which follows from

$$\int_{\mathbb{R}^m} \Psi_t(z)\, dz = \psi(0) = 0$$

and the Littlewood–Paley theory. By (2.7) and the Minkowski inequality,

$$\begin{aligned}
\Delta_\tau(f)(x) &= (|\gamma| \ln 2) \sum_{k=1}^\infty \left(\int_{\mathbb{R}} \left| \int_{\mathbb{R}} (\Psi_{2^{\gamma(s+t)}} \otimes \delta_{d-m}) * \tau_{k,t} * f(x)\, ds \right|^2 dt \right)^{1/2} \\
&\le (|\gamma| \ln 2) \sum_{k=1}^\infty \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |(\Psi_{2^{\gamma(s+t)}} \otimes \delta_{d-m}) * \tau_{k,t} * f(x)|^2\, dt \right)^{1/2} ds \\
&= (|\gamma| \ln 2) \int_{\mathbb{R}} H_s(f)(x)\, ds,
\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}
H_s(f)(x) &= \sum_{k=1}^\infty \left(\int_{\mathbb{R}} |(\Psi_{2^{\gamma(s+t)}} \otimes \delta_{d-m}) * \tau_{k,t} * f(x)|^2\, dt \right)^{1/2} \\
&:= \sum_{k=1}^\infty H_{s,k}(f)(x).
\end{aligned} \tag{2.10}$$

Thus the L^p boundedness of Δ_τ would follow if we can prove that, for $1 < p < \infty$,

$$\|H_s(f)\|_{L^p(\mathbb{R}^d)} \le C_p 2^{-\theta(p)|s|} \|f\|_{L^p(\mathbb{R}^d)} \tag{2.11}$$

holds for some $C_p, \theta(p) > 0$.

We shall first verify (2.11) for $p = 2$, which can be done by a simple application of Plancherel's theorem. We shall consider the case $\gamma > 0$ only, because the case $\gamma < 0$ can be dealt with in similar fashion. By (2.10),

$$\|H_{s,k}(f)\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}} \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 |\psi(|2^{\gamma(s+t)}\xi'|^2)\hat{\tau}_{k,t}(\xi)|^2\, d\xi\, dt,$$

where $\xi = (\xi_1, \dots, \xi_d)$ and $\xi' = (\xi_1, \dots, \xi_m)$. When $s \geq 0$, by (2.4) we have

$$\begin{aligned}\int_{\mathbb{R}} &|\psi(|2^{\gamma(s+t)}\xi'|^2)\hat{\tau}_{k,t}(\xi)|^2\,dt \\ &\leq C_0^2 2^{-2k} \int_{(2^{\gamma s+1}|\xi'|)^{-1} \leq 2^{\gamma t} \leq 2(2^{\gamma s}|\xi'|)^{-1}} (2^{\gamma(t-k)}|\xi'|)^{2\alpha}\,dt \\ &\leq C 2^{-2[\gamma\alpha s+(1+\gamma\alpha)k]}. \qquad (2.12)\end{aligned}$$

Similarly for $s < 0$, also by using (2.4), we have

$$\int_{\mathbb{R}} |\psi(|2^{\gamma(s+t)}\xi'|^2)\hat{\tau}_{k,t}(\xi)|^2\,dt \leq C 2^{2[\gamma\beta s-(1-\gamma\beta)k]}. \qquad (2.13)$$

Thus, there exists a $\theta > 0$ such that

$$\|H_{s,k}(f)\|_{L^2(\mathbb{R}^d)} \leq C 2^{-\theta(|s|+k)}\|f\|_{L^2(\mathbb{R}^d)}, \qquad (2.14)$$

which implies that (2.11) holds for $p = 2$. Next we shall prove that, for every $p_0 \in (1, \infty)$,

$$\|H_s(f)\|_{L^{p_0}(\mathbb{R}^d)} \leq C\|f\|_{L^{p_0}(\mathbb{R}^d)}. \qquad (2.15)$$

First let us consider the case $1 < p_0 < 2$. Let $G_u(x) = (\Psi_{2^{\gamma u}} \otimes \delta_{d-m}) * f(x)$. Then, for $k \in \mathbb{N}$, by (2.3) we have

$$\left\| \int_{\mathbb{R}} |\tau_{k,t} * G_{s+t}(\cdot)|\,dt \right\|_{L^1(\mathbb{R}^d)} \leq C 2^{-k} \left\| \int_{\mathbb{R}} |G_t(\cdot)|\,dt \right\|_{L^1(\mathbb{R}^d)},$$

where C is independent of s. On the other hand, by (2.5),

$$\left\| \sup_{t\in\mathbb{R}} |\tau_{k,t} * G_{s+t}(\cdot)| \right\|_{L^q(\mathbb{R}^d)} \leq \left\| \tau_k^*\Big(\sup_{t\in\mathbb{R}} |G_t(\cdot)|\Big) \right\|_{L^q(\mathbb{R}^d)} \leq C 2^{-k} \left\| \sup_{t\in\mathbb{R}} |G_t(\cdot)| \right\|_{L^q(\mathbb{R}^d)}$$

for $1 < q < \infty$, where C again is independent of s. The estimates we have given here show that the linear mapping $T\colon G_t(x) \to \tau_{k,t} * G_{s+t}(x)$ is bounded on $L^1(L^1(\mathbb{R}), \mathbb{R}^d)$ and $L^q(L^\infty(\mathbb{R}), \mathbb{R}^d)$, respectively. Thus, if q satisfies $1/q = 2/p_0 - 1$, then we conclude by using interpolation that the mapping T is also bounded on $L^{p_0}(L^2(\mathbb{R}), \mathbb{R}^d)$. More precisely,

$$\left\| \left(\int_{\mathbb{R}} |\tau_{k,t} * G_{s+t}(\cdot)|^2\,dt \right)^{1/2} \right\|_{L^{p_0}(\mathbb{R}^d)} \leq C_{p_0} 2^{-k} \left\| \left(\int_{\mathbb{R}} |G_t(\cdot)|^2\,dt \right)^{1/2} \right\|_{L^{p_0}(\mathbb{R}^d)}.$$

From this and the L^{p_0} boundedness of the g-function, we have

$$\|H_{s,k}(f)\|_{L^{p_0}(\mathbb{R}^d)} \leq C 2^{-k}\|f\|_{L^{p_0}(\mathbb{R}^d)} \quad \text{for } 1 < p_0 < 2 \text{ and } k \in \mathbb{N}.$$

As for the case $2 < p_0 < \infty$, it will be performed by imitating an argument in [DR]. Let $k \in \mathbb{N}$ and $q = (p_0/2)'$. There exists a $w = w_k \in L^q(\mathbb{R}^d)$ such that $\|w\|_q = 1$ and

$$\begin{aligned}
&\|H_{s,k}(f)\|_{L^{p_0}(\mathbb{R}^d)}\\
&\quad = \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}} |(\Psi_{2^{\gamma(s+t)}} \otimes \delta_{d-m}) * \tau_{k,t} * f(x)|^2 \, dt \, w(x)\, dx \right)^{1/2}\\
&\quad \le \Bigg[\Big(\sup_t \|\tau_{k,t}\| \Big) \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(\Psi_{2^{\gamma(s+t)}} \otimes \delta_{d-m}) * f(x-y)|^2 w(x)\\
&\qquad\qquad dx \, d\tau_{k,t}(y)\, dt \Bigg]^{1/2}\\
&\quad \le C \left(2^{-k} \int_{\mathbb{R}} \int_{\mathbb{R}^d} |(\Psi_{2^{\gamma(s+t)}} \otimes \delta_{d-m}) * f(u)|^2 \tau_k^*(\tilde{w})(-u)\, du\, dt \right)^{1/2}\\
&\qquad\qquad (\text{with } \tilde{w}(x) = w(-x))\\
&\quad \le C[2^{-k} \|g(f)\|^2_{L^{p_0}(\mathbb{R}^d)} \|\tau_k^*(\tilde{w})\|_{L^q(\mathbb{R}^d)}]^{1/2} \le C 2^{-k} \|f\|_{L^{p_0}(\mathbb{R}^d)},
\end{aligned}$$

where we used the boundedness of the g-function and assumption (2.5) again. Summing over $k \in \mathbb{N}$, we obtain (2.15). By (2.14), (2.15), and applying the Riesz–Thorin interpolation theorem for sublinear operators [CZ], we obtain (2.11) for $1 < p < \infty$. This concludes the proof of (2.6) in the special case $L = \pi_m^d$.

The general case can be resolved by using a technique developed in [FP2]. Suppose that $\tau = \{\tau_{k,t} \mid k \in \mathbb{N},\, t \in \mathbb{R}\}$ satisfies (2.3)–(2.5) with a given linear transformation $L\colon \mathbb{R}^d \to \mathbb{R}^N$. Let $m = \operatorname{rank}(L) \le d$. Then there are nonsingular linear transformations $G_m\colon \mathbb{R}^m \to \mathbb{R}^m$ and $G_d\colon \mathbb{R}^d \to \mathbb{R}^d$ such that

$$|G_m \pi_m^d G_d \xi| \le |L\xi| \le N |G_m \pi_m^d G_d \xi|. \tag{2.16}$$

Define $\nu_{k,t}$ by

$$\int_{\mathbb{R}^d} f(x)\, d\nu_{k,t}(x) = \int_{\mathbb{R}^d} f(U^t x)\, d\tau_{k,t}(x), \tag{2.17}$$

where $U = G_d^{-1} \circ (G_m^{-1} \otimes \mathrm{id}_{\mathbb{R}^{d-m}})$. Then, by (2.3)–(2.5) and (2.16), $\nu = \{\nu_{k,t} \mid k \in \mathbb{N},\, t \in \mathbb{R}\}$ satisfies (2.3)–(2.5) with L replaced by π_m^d. Thus

$$\|\Delta_\nu(f)\|_{L^p(\mathbb{R}^d)} \le A_p \|f\|_{L^p(\mathbb{R}^d)} \tag{2.18}$$

holds for $1 < p < \infty$. Finally, (2.6) follows from (2.18) after making a trivial change of variables. Lemma 2.1 is proved. □

3. Proof of Theorem 1

Recall that

$$H^1(S^{n-1}) = \{f \in \mathcal{S}'(S^{n-1}) : \|P^+ f\|_{L^1(S^{n-1})} < \infty\}$$

and $\|f\|_{H^1(S^{n-1})} = \|P^+ f\|_{L^1(S^{n-1})}$, where

$$P^+ f(z) = \sup_{0 \le r < 1} \left| \int_{S^{n-1}} \frac{(1-r^2) f(y)}{|rz - y|^n}\, d\sigma(y) \right|.$$

A useful property of $H^1(S^{n-1})$ is its atomic decomposition. A function $a(\cdot)$ on S^{n-1} is a (regular) H^1 atom if it satisfies the following:

$$\operatorname{supp}(a) \subset S^{n-1} \cap \{y \in \mathbb{R}^n : |y - \zeta| < \rho\}$$
$$\text{for some } \zeta \in S^{n-1} \text{ and } \rho \in (0, 2]; \tag{3.1}$$

$$\int_{S^{n-1}} a(y)\, d\sigma(y) = 0; \tag{3.2}$$

$$\|a\|_\infty \le \rho^{-(n-1)}. \tag{3.3}$$

LEMMA 3.1 [Co; CTW]. *If* $\Omega \in H^1(S^{n-1})$ *and satisfies* (3.1)–(3.3), *then there exist* $\{c_j\} \subset \mathbb{C}$ *and* H^1 *atoms* $\{a_j\}$ *such that*

$$\Omega = \sum_j c_j a_j$$

and $\|\Omega\|_{H^1(S^{n-1})} \approx \sum_j |c_j|$.

Proof of Theorem 1. In light of Lemma 3.1, it suffices to prove that

$$\|\mu_{\mathcal{P},\Omega,b}(f)\|_{L^p(\mathbb{R}^d)} \le C_p \|f\|_{L^p(\mathbb{R}^d)}$$

holds when Ω is a H^1 atom satisfying (3.1)–(3.3).

Define the family of measures $\sigma = \{\sigma_{k,t} \mid k \in \mathbb{N},\ t \in \mathbb{R}\}$ on $\mathbb{R}^d$ by

$$\int_{\mathbb{R}^d} f(x)\, d\sigma_{k,t}(x) = \frac{\sqrt{\ln 2}}{2^t} \int_{2^{t-k} \le |y| \le 2^{t-k+1}} f(\mathcal{P}(y)) \frac{\Omega(y)}{|y|^{n-1}} b(|y|)\, dy. \tag{3.4}$$

Then

$$\mu_{\mathcal{P},\Omega,b}(f) \le \Delta_\sigma(f). \tag{3.5}$$

By the arguments in [FP2, Sec. 7; see esp. (7.36)], there are families of measures

$$\tau^{(1)} = \{\tau^{(1)}_{k,t} \mid k \in \mathbb{N},\ t \in \mathbb{R}\}, \dots, \tau^{(M)} = \{\tau^{(M)}_{k,t} \mid k \in \mathbb{N},\ t \in \mathbb{R}\},$$

each of which satisfies (2.3)–(2.5), such that

$$\sigma_{k,t} = \sum_{j=1}^{M} \tau^{(j)}_{k,t} \tag{3.6}$$

for $k \in \mathbb{N}$ and $t \in \mathbb{R}$. It then follows from Lemma 2.1 and Minkowski inequality that

$$\|\mu_{\mathcal{P},\Omega,b}(f)\|_{L^p(\mathbb{R}^d)} \le \sum_{j=1}^{M} \|\Delta_{\tau^{(j)}}(f)\|_{L^p(\mathbb{R}^d)} \le C_p \|f\|_{L^p(\mathbb{R}^d)}$$

for $f \in L^p(\mathbb{R}^d)$ and $1 < p < \infty$.

4. Additional Results on Marcinkiewicz Integrals

Mappings of Finite Type

Let $B(0, r)$ denote the ball centered at the origin in $\mathbb{R}^n$ with radius r. For a suitable function Ω and a mapping $\Phi \colon B(0, r) \to \mathbb{R}^d$, we define the Marcinkiewicz integral operator $\mu_{\Phi,\Omega}$ by

$$\mu_{\Phi,\Omega}(f)(x) = \left(\int_0^r \left| \int_{|y|\le t} f(x - \Phi(y)) \frac{\Omega(y)}{|y|^{n-1}} \, dy \right|^2 \frac{dt}{t^3} \right)^{1/2}. \tag{4.1}$$

(If Φ is a mapping from $\mathbb{R}^n$ into $\mathbb{R}^d$, then $r = \infty$.)

A C^∞ mapping $\Phi \colon B(0,1) \to \mathbb{R}^d$ is said to be of *finite type* at the origin if, for each unit vector $\eta \in \mathbb{R}^d$, there is a multi-index α (with $|\alpha| \ge 1$) such that

$$\partial_y^\alpha [\Phi(y) \cdot \eta]|_{y=0} \neq 0. \tag{4.2}$$

We have the following result concerning the Marcinkiewicz integrals associated to mappings of finite type.

Theorem 2. *Let $\Phi \colon B(0,1) \to \mathbb{R}^d$ be a C^∞ mapping that is of finite type at the origin. If $\Omega \in L^q(S^{n-1})$ for some $q > 1$ and satisfies* (1.1), *then $\mu_{\Phi,\Omega}$ is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$.*

A proof of Theorem 2 can be obtained by imitating that of Theorem 1. The only difference is that, instead of using the arguments in [FP2, Sec. 7], one uses the arguments in [FGP1] in conjunction with Lemma 2.1.

Homogeneous Mappings

For $\Gamma = (\gamma_1, \dots, \gamma_d) \in \mathbb{R}^d$ and $t > 0$, let Γ_t denote the dilation on $\mathbb{R}^d$ given by

$$\Gamma_t(x_1, \dots, x_d) = (t^{\gamma_1} x_1, \dots, t^{\gamma_d} x_d). \tag{4.3}$$

A mapping $\Phi \colon \mathbb{R}^n \to \mathbb{R}^d$ is said to be homogeneous of degree Γ if

$$\Phi(ty) = \Gamma_t(\Phi(y)) \tag{4.4}$$

holds for $y \in \mathbb{R}^n \backslash \{0\}$ and $t > 0$.

By employing the methods in [FGP2; Ch] and Lemma 2.1, we obtain the following.

Theorem 3. *Let $\Phi \colon \mathbb{R}^n \to \mathbb{R}^d$ be homogeneous of degree $\Gamma = (\gamma_1, \dots, \gamma_d)$, with $\gamma_j \neq 0$ for $1 \le j \le d$. If $\Phi|_{S^{n-1}}$ is real-analytic and if $\Omega \in H^1(S^{n-1})$ and satisfies* (1.1), *then $\mu_{\Phi,\Omega}$ is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$.*

Surface of Revolution

Next we consider the Marcinkiewicz integrals associated to surfaces of revolution. Let ϕ be a real-valued function on $[0, \infty)$, and let

$$\Phi(y) = (y, \phi(|y|)) \tag{4.5}$$

for $y \in \mathbb{R}^n$. Let $\mathcal{M}_\phi$ denote the following maximal operator on $\mathbb{R}^2$:

$$(\mathcal{M}_\phi) g(u, v) = \sup_{k \in \mathbb{Z}} 2^{-k} \int_{2^k}^{2^{k+1}} |g(u - t, v - \phi(t))| \, dt.$$

Theorem 4. *Let $\Phi \colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ be given as in* (4.5). *If $\Omega \in H^1(S^{n-1})$ satisfies* (1.1), *then $\mu_{\Phi,\Omega}$ is bounded on $L^2(\mathbb{R}^{n+1})$. If, in addition, ϕ is convex and increasing and if $\mathcal{M}_\phi$ is bounded on $L^p(\mathbb{R}^2)$ for $1 < p < \infty$, then $\mu_{\Phi,\Omega}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $1 < p < \infty$.*

Proof. By Plancherel's theorem, the L^2 boundedness of $\mu_{\Phi,\Omega}$ is equivalent to

$$J(\xi,\eta)=\int_0^\infty |m_t(\xi,\eta)|^2\,\frac{dt}{t^3}\le C, \tag{4.6}$$

uniformly in $\xi\in\mathbb{R}^n$ and $\eta\in\mathbb{R}$, where

$$m_t(\xi,\eta)=\int_{|y|\le t} e^{-i(\xi\cdot y+\eta\phi(|y|))}\,\frac{\Omega(y)}{|y|^{n-1}}\,dy. \tag{4.7}$$

If we let

$$I_\Omega(\xi)=\int_{S^{n-1}} e^{i\xi\cdot y}\Omega(y)\,d\sigma(y), \tag{4.8}$$

then it follows from [FP1, Thm. A] that

$$\sup_{\xi\in\mathbb{R}^n}\left(\int_0^\infty |I_\Omega(s\xi)|^2\,\frac{ds}{s}\right)^{1/2}\le C\|\Omega\|_{H^1}<\infty. \tag{4.9}$$

Thus, by (4.6)–(4.9), Hölder's inequality, and Fubini's theorem,

$$\sup_{(\xi,\eta)\in\mathbb{R}^{n+1}} J(\xi,\eta)\le \sup_{\xi\in\mathbb{R}^n}\int_0^\infty\int_0^t |I_\Omega(s\xi)|^2\frac{1}{t^2}\,ds\,dt\le C\|\Omega\|_{H^1}^2,$$

which proves the first part of Theorem 4. The second part follows from the arguments in [LPY] and Lemma 2.1. □

5. Area Integrals and g_λ^* Functions

We shall end the paper with a theorem on the Marcinkiewicz integral operators related to the area integral and the Littlewood–Paley g_λ^* function. We shall consider polynomial mappings only, but it is clear that similar results can be obtained for other classes of mappings.

Let $\mathcal{P}$ be a polynomial mapping from $\mathbb{R}^n$ into $\mathbb{R}^d$, and let $F_{\mathcal{P},t}$ be given as in (1.3). We define the operators $\tilde{\mu}_{\mathcal{P}}=\tilde{\mu}_{\mathcal{P},\Omega,b}$ and $\mu^*_{\mathcal{P},\lambda}=\mu^*_{\mathcal{P},\Omega,b,\lambda}$ by

$$\tilde{\mu}_{\mathcal{P}}(f)(x)=\left(\int_{\Gamma(x)}|F_{\mathcal{P},t}(u)|^2\frac{1}{t^{d+3}}\,du\,dt\right)^{1/2}, \tag{5.1}$$

where $\Gamma(x)=\{(u,t)\in\mathbb{R}^{d+1}_+ : |x-u|<t\}$, and

$$\mu^*_{\mathcal{P},\lambda}(f)(x)=\left(\iint_{\mathbb{R}^{d+1}_+}\left(\frac{t}{t+|x-u|}\right)^{d\lambda}|F_{\mathcal{P},t}(u)|^2\frac{1}{t^{d+3}}\,du\,dt\right)^{1/2} \tag{5.2}$$

for $\lambda>1$.

Our results can be stated as follows.

THEOREM 5. *Suppose that $b(r)\in L^\infty(\mathbb{R}_+)$ and that $\Omega\in H^1(S^{n-1})$ and satisfies* (1.1). *Then, for $2\le p<\infty$, there exists a constant $C_p>0$ such that*

$$\|\mu^*_{\mathcal{P},\lambda}(f)\|_{L^p(\mathbb{R}^d)}\le C_p\|f\|_{L^p(\mathbb{R}^d)} \tag{5.3}$$

and

$$\|\tilde{\mu}_{\mathcal{P}}(f)\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)} \tag{5.4}$$

for every $f \in L^p(\mathbb{R}^d)$. The constant C_p may depend on n, d, and $\deg(P_j)$ $(1 \leq j \leq d)$, but it is independent of the coefficients of P_j.

The proof of Theorem 5 is based on the following lemma.

LEMMA 5.1. *Let $\lambda > 1$. Then, for any nonnegative function g, we have*

$$\int_{\mathbb{R}^d} (\mu^*_{\mathcal{P},\lambda}(f)(x))^2 g(x)\,dx \leq C_\lambda \int_{\mathbb{R}^d} (\mu_{\mathcal{P},\Omega,b}(f)(x))^2 (Mg)(x)\,dx, \tag{5.5}$$

where M denotes the usual Hardy–Littlewood maximal operator on $\mathbb{R}^d$.

Proof. By definition, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} (\mu^*_{\mathcal{P},\lambda}(f)(x))^2 g(x)\,dx \\
&\quad = \int_{\mathbb{R}^d} \iint_{\mathbb{R}^{d+1}_+} \left(\frac{t}{t+|x-u|}\right)^{d\lambda} |F_{\mathcal{P},t}(u)|^2 \frac{1}{t^{d+3}}\,du\,dt\,g(x)\,dx \\
&\quad \leq \int_{\mathbb{R}^d} \int_0^\infty |F_{\mathcal{P},t}(u)|^2 \left[\sup_{t>0} \int_{\mathbb{R}^d} \left(\frac{t}{t+|x-u|}\right)^{d\lambda} g(x) \frac{1}{t^d}\,dx\right] \frac{1}{t^3}\,dt\,du \\
&\quad \leq C_\lambda \int_{\mathbb{R}^d} (\mu_{\mathcal{P},\Omega,b}(f)(u))^2 (Mg)(u)\,du
\end{aligned}$$

for $\lambda > 1$. Lemma 5.1 is proved. □

Proof of Theorem 5. When $p = 2$, one can obtain (5.3) by simply taking $g \equiv 1$ in (5.5) and then invoking Theorem 1. For $2 < p < \infty$, we let $q = (p/2)'$. Then, by Lemma 5.1, Theorem 1, and Hölder's inequality,

$$\begin{aligned}
\|\mu^*_{\mathcal{P},\lambda}(f)\|^2_{L^p(\mathbb{R}^d)} &= \sup_{\|g\|_q=1} \left| \int_{\mathbb{R}^d} (\mu^*_{\mathcal{P},\lambda}(f)(x))^2 g(x)\,dx \right| \\
&\leq C_\lambda \sup_{\|g\|_q=1} \int_{\mathbb{R}^d} (\mu_{\mathcal{P},\Omega,b}(f)(x))^2 Mg(x)\,dx \\
&\leq C_\lambda \Big(\sup_{\|g\|_q=1} \|Mg\|_q\Big) \|\mu_{\mathcal{P},\Omega,b}(f)\|^2_{L^p(\mathbb{R}^d)} \leq C_{\lambda,p} \|f\|^2_{L^p(\mathbb{R}^d)}.
\end{aligned}$$

Thus (5.3) holds for $2 \leq p < \infty$. Inequality (5.4) follows from (5.3) and the observation that $\tilde{\mu}_{\mathcal{P}}(f)(x) \leq C_\lambda[\mu^*_{\mathcal{P},\lambda}(f)(x)]$. Theorem 5 is proved. □

References

[BCP] A. Benedek, A. Calderón, and R. Panzones, *Convolution operators on Banach space valued functions,* Proc. Nat. Acad. Sci. U.S.A. 48 (1962), 356–365.

[CZ] A. Calderón and A. Zygmund, *A note on the interpolation of sublinear operators,* Amer. J. Math. 78 (1956), 282–288.

[Ch] L. Cheng, *Singular integrals related to homogeneous mappings,* Michigan Math. J. 47 (2000), 407–416.

[Co] L. Colzani, *Hardy spaces on spheres,* Ph.D. thesis, Washington University, St. Louis, MO, 1982.

[CTW] L. Colzani, M. Taibleson, and G. Weiss, *Maximal estimates for Cesàro and Riesz means on sphere,* Indiana Univ. Math. J. 33 (1984), 873–889.

[DFP] Y. Ding, D. Fan, and Y. Pan, *L^p-boundedness of Marcinkiewicz integrals with Hardy space function kernels,* Acta Math. Sinica (English Ser.) 16 (2000), 593–600.

[DR] J. Duoandikoetxea and J. L. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates,* Invent. Math. 84 (1986), 541–561.

[FGP1] D. Fan, K. Guo, and Y. Pan, *Singular integrals along submanifolds of finite type,* Michigan Math. J. 45 (1998), 135–142.

[FGP2] ———, *L^p estimates for singular integrals associated to homogeneous surfaces,* preprint.

[FP1] D. Fan and Y. Pan, *Oscillatory integrals and atoms on the unit sphere,* Manuscripta Math. 89 (1996), 179–192.

[FP2] ———, *Singular integral operator with rough kernel supported by subvarieties,* Amer. J. Math. 119 (1997) 799–839.

[Fe] R. Fefferman, *Multiparameter Fourier analysis,* Beijing lectures in harmonic analysis (Beijing, 1984), pp. 47–130, Ann. of Math. Stud., 112, Princeton Univ. Press, Princeton, NJ, 1986.

[LPY] S. Lu, Y. Pan, and D. Yang, *Rough singular integrals associated to surfaces of revolution,* Proc. Amer. Math. Soc. 129 (2001), 2931–2940.

[S1] E. Stein, *On the functions of Littlewood–Paley, Lusin and Marcinkiewicz,* Trans. Amer. Math. Soc. 88 (1958), 430–466.

[S2] ———, *Harmonic analysis: Real variable methods, orthogonality and oscillatory integrals,* Princeton Univ. Press, Princeton, NJ, 1993.

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