INDEPENDENT PERFECT SETS IN GROUPS

Walter Rudin

INTRODUCTION. We shall consider locally compact abelian groups (written additively) in which every neighborhood of the identity 0 contains elements of infinite order; for brevity, we shall call such a group an I-group.

Hewitt has recently proved [1] that the convolution algebra of all regular, complex, bounded Borel measures on an I-group is not symmetric. This is an interesting extension of an earlier result of Šreĭder [3] concerning the measure algebra on the real line. The crux of Hewitt's extension is the construction, in every I-group, of a Cantor set (that is, a set homeomorphic to Cantor's ternary set) which is independent in the sense defined below. His construction depends on a fairly involved structure theorem and on the consideration of special cases (p-adic groups and complete direct sums of cyclic groups, in particular).

The present paper contains a much simpler construction of such sets. We use a modest amount of structure theory to reduce the problem to the case of a metric I-group, but in the metric case we simply imitate the usual construction of a Cantor set on the line as the intersection of a sequence of sets E_n which are unions of 2^n intervals.

DEFINITIONS. A subset E of an abelian group G is *independent* if the following is true: for every choice of distinct points $x_1, \cdots, x_j$ in E and of integers $n_1, \cdots, n_j$, not all 0, we have

$$n_1 x_1 + n_2 x_2 + \cdots + n_j x_j \neq 0.$$

By a *compact neighborhood* we shall mean the compact closure of a nonempty open set.

For $k = 1, 2, 3, \cdots$, G^k will denote the topological space which is the cartesian product of G with itself, taken k times; that is, $G^1 = G$, $G^k = G^{k-1} \times G$.

For any group-theoretic terms used, we refer to [2].

The main result of the paper is as follows:

THEOREM. *Every* I-*group contains an independent Cantor set.*

The proof will be in two steps:

STEP 1. *Every metric* I-*group contains an independent Cantor set.*

STEP 2. *Every* I-*group contains a closed subgroup which is a metric* I-*group.* (We use *metric* synonymously with *metrizable.*)

LEMMA 1. *Suppose* G *is an* I-*group,* $n_1, \cdots, n_k$ *are integers, not all equal to zero, and* E *is the set of all points* $(x_1, \cdots, x_k)$ *in* G^k *at which*

$$n_1 x_1 + n_2 x_2 + \cdots + n_k x_k \neq 0.$$

Then E *is a dense open subset of* G^k.

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Proof. Let f be the mapping of G^k into G defined by

$$f(x_1, \cdots, x_k) = n_1 x_1 + n_2 x_2 + \cdots + n_k x_k .$$

Since f is continuous, $f^{-1}(0)$ is closed, so that E is open.

Assume that $f^{-1}(0)$ contains a nonempty open set $V = V_1 \times V_2 \times \cdots \times V_k$. Fix j so that $n_j \neq 0$, and fix $x_i \in V_i$ for $i \neq j$. Put $x_0 = \Sigma_{i \neq j} n_i x_i$. Then $n_j x = -x_0$ for all $x \in V_j$. Let W be the neighborhood of 0 in G which consists of all elements x - y ($x \in V_j$, $y \in V_j$). For every $w \in W$ we have

$$n_j w = n_j x - n_j y = - x_0 + x_0 = 0 ,$$

which contradicts the assumption that G is an I-group.

Consequently, $f^{-1}(0)$ has empty interior, and the lemma is proved.

PROOF OF STEP 1. Let G be a metric I-group. Let P_1 be a compact neighborhood in G.

Suppose P_r has been constructed (r = 1, 2, 3, ⋯), such that

$$P_r = P_r^{(1)} \cup P_r^{(2)} \cup \cdots \cup P_r^{(s)} ,$$

where $s = 2^{r-1}$ and the sets $P_r^{(i)}$ are disjoint compact neighborhoods. Let Q be the set of all points $(x_1, \cdots, x_{2s}) \in G^{2s}$ such that the conditions

$$(*) \qquad |n_1| + \cdots + |n_{2s}| > 0, \qquad |n_i| \leq r \text{ for } i = 1, \cdots, 2s$$

imply $n_1 x_1 + \cdots + n_{2s} x_{2s} \neq 0$. Applying Lemma 1 a finite number of times, we see that Q is a dense open subset of G^{2s}. Thus the set

$$P_r^{(1)} \times P_r^{(1)} \times P_r^{(2)} \times P_r^{(2)} \times \cdots \times P_r^{(s)} \times P_r^{(s)}$$

contains an open set $V_1 \times V_2 \times \cdots \times V_{2s-1} \times V_{2s}$ which lies in Q.

Since G is dense in itself, there are disjoint compact neighborhoods

$$P_{r+1}^{(i)} \subset V_i \quad (i = 1, \cdots, 2s)$$

whose diameters are less that 1/r. Put

$$P_{r+1} = P_{r+1}^{(1)} \cup \cdots \cup P_{r+1}^{(2s)} .$$

Then $P_{r+1} \subset P_r$, and if $x_i \in P_{r+1}^{(i)}$ (i = 1, ⋯, 2s), the conditions (*) imply that $n_1 x_1 + \cdots + n_{2s} x_{2s} \neq 0$.

Consequently the following statement is true: if $x_1, \cdots, x_j$ are distinct points of P_{r+1}, if no two of these points lie in the same set $P_{r+1}^{(i)}$, if $|n_1| + \cdots + |n_j| > 0$ and $|n_i| \leq r$ for i = 1, ⋯, j, then $n_1 x_1 + \cdots + n_j x_j \neq 0$.

Define $P = \bigcap_{r=1}^{\infty} P_r$. Then P is evidently a Cantor set. Suppose $x_1, \cdots, x_j$ are distinct points of P, and $n_1, \cdots, n_j$ are integers, not all 0. Choose r so large that $|n_i| \leq r$ for i = 1, ⋯, j and that none of the sets $P_{r+1}^{(i)}$ contains two of the points $x_1, \cdots, x_j$. The above remark shows that $n_1 x_1 + \cdots + n_j x_j \neq 0$, so that P is an independent set.

This completes the proof of the theorem for metric I-groups.

We now insert a purely algebraic lemma:

LEMMA 2. *Let* S *be an abelian group which is not of bounded order. Then there is a homomorphism of* S *onto a countable group* T *which is not of bounded order.*

Proof. S contains a countable subgroup S_1 which is not of bounded order, and S_1 can be embedded in a countable divisible group T_1 [2; p. 12, Exercise 5]. Since T_1 is divisible, the identity mapping of S_1 into T_1 can be extended to a homomorphism h of S into T_1 [2; p. 11, Exercise 1]. Put $T = h(S)$. Since $S_1 = h(S_1) \subset T \subset T_1$, T is countable and not of bounded order.

LEMMA 3. *Let* K *be a compact abelian group which is not of bounded order. Then* K *is an* I-*group.*

Proof. For $n = 1, 2, 3, \cdots$, let E_n be the set of all $x \in K$ such that $nx = 0$. Assume that one of these sets E_n contains an open set V. Let W be the neighborhood of 0 which consists of all elements $x - y$ $(x \in V,\ y \in V)$. Then $nz = 0$ for all $z \in W$. The group H generated by W is compact and open, and K/H is finite (being compact and discrete). If K/H has p elements, it follows that $px \in H$ and $npx = 0$, for every $x \in K$; thus K is of bounded order.

This contradiction shows that none of the compact sets E_n contains an open subset of K. Hence the set of all elements of infinite order, which is the complement of $\bigcup_1^\infty E_n$, is a dense subset of K. The lemma follows.

PROOF OF STEP 2. Let G be an I-group and let G_1 be an open-closed subgroup of G, generated by a symmetric compact neighborhood of 0. It is well known [4; p. 110] that G_1 contains an open-closed subgroup isomorphic to the direct sum of R^n and K, where R^n denotes n-dimensional euclidean space (for some $n \geq 0$) and K is compact. If $n > 0$, R^n furnishes the desired metric I-group.

If $n = 0$, then K is an open subgroup of G and hence is an I-group. We now use the duality theory for compact and discrete abelian groups:

The character group S of K is not of bounded order. By Lemma 2, S can be mapped homomorphically onto a countable discrete group T. Let H be the character group of T. Then H is a compact subgroup of K, H is not of bounded order, and Lemma 3 implies that H is an I-group. On the other hand, the fact that T is countable implies that a countable family of continuous functions separates points on H, so that H has a countable base and is therefore metric. This completes the proof of the theorem.

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University of Rochester