HERMITIAN MANIFOLDS WITH ZERO CURVATURE

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1. INTRODUCTION

In this note we consider the problem of determining those complex-analytic manifolds with a Hermitian metric whose curvature vanishes everywhere. It is easy to see that the identical vanishing of the curvature implies that there exists in a neighborhood of each point a field of n independent (in fact, orthonormal) parallel analytic vectors, where n is the dimension of the manifold. If the manifold is simply connected, such a field may then be defined over the entire manifold, and the manifold is therefore parallelisable (a complex-analytic manifold of complex dimension n is said to be parallelisable if there exist n analytic vector fields defined over it which are independent at each point). On the other hand, if a complex-analytic manifold is parallelisable, then it has a Hermitian metric with curvature zero. Hence, for a complex-analytic manifold, the existence of such a metric is a somewhat weaker property than parallelisability. H. C. Wang [6] has shown that a compact, complex-analytic, parallelisable manifold has a complex Lie group as its universal covering space. Here this is generalized to the corresponding theorem for the case of vanishing curvature.

We use the notation of [4], except that we denote the conjugate of a complex number by a bar, and that the * on indices is replaced by a bar. Thus Greek indices range from 1 to 2n, unbarred Latin indices from 1 to n, and barred Latin indices from n + 1 to 2n. In local coordinates $z^1, \cdots, z^n$, and relative to the natural (affine) frames, the metric tensor is denoted by $g_{i\bar{j}}\, dz^i d\bar{z}^j$, and the components of the connection $C_{\beta\gamma}^{\alpha}$ are given by

$$C_{jk}^{i} = g^{i\bar{l}} \frac{\partial g_{j\bar{l}}}{\partial z^k}, \qquad C_{\bar{j}\bar{k}}^{\bar{i}} = \overline{C}_{jk}^{i},$$

all other components being zero. The torsion tensor is simply the skew-symmetric part of the connection, that is, $A_{\beta\gamma}^{\alpha} = C_{\beta\gamma}^{\alpha} - C_{\gamma\beta}^{\alpha}$. Its vanishing is the condition that the metric be Kählerian. Covariant derivatives are given by the usual formula

$$X_{\beta_1\cdots\beta_s|\gamma}^{\alpha_1\cdots\alpha_r} = \frac{\partial X_{\beta_1\cdots\beta_s}^{\alpha_1\cdots\alpha_r}}{\partial z^\gamma} + \sum_{t=1}^{r} X_{\beta_1\cdots\beta_s}^{\alpha_1\cdots\sigma\cdots\alpha_r} C_{\sigma\gamma}^{\alpha_t} - \sum_{t=1}^{s} X_{\beta_1\cdots\sigma\cdots\beta_s}^{\alpha_1\cdots\alpha_r} C_{\beta_t\gamma}^{\sigma}.$$

There are natural decompositions of a tensor into a sum of pure tensors of special types, those of a given type having components which vanish except for a particular pattern of Latin indices, for example, for all except the unbarred Latin indices. It is easy to see from the definition of covariant derivatives that if a tensor is pure and has only unbarred indices (example: $X_{j_1\cdots j_s}^{i_1\cdots i_s}$), then the components are analytic functions of the local coordinates in each coordinate system if and only if

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$X^{i_1\cdots i_r}_{j_1\cdots j_s|\bar{k}} \equiv 0$. We conclude by noting that the curvature tensor is defined by

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} + g^{p\bar{q}} \frac{\partial g^{p\bar{j}}}{\partial \bar{z}^l} \frac{\partial g^{i\bar{q}}}{\partial z^k}.$$

This corrects a misprint in [4, p. 524, formula (24)].

2. BOCHNER'S LEMMA

Bochner's Lemma will be needed in the sequel; it may be stated as follows:

LEMMA. *If* Φ *is a real-valued function on a compact Hermitian manifold, and if* $L(\Phi) = g^{i\bar{j}}\Phi_{i\bar{j}}$, *the subscripts indicating covariant differentiation, then* $L(\Phi) \geq 0$ *everywhere implies that* $\Phi \equiv$ constant *and* $L(\Phi) \equiv 0$.

This is a slight extension of the lemma proved by Yano and Bochner [7, p. 30], and it may be proved as follows. Locally, we have $\Phi_{k\bar{l}} = \frac{\partial^2 \Phi}{\partial z^k \partial \bar{z}^l}$. Since Φ is real-valued, $\bar{\Phi}_{k\bar{l}} = \Phi_{\bar{k}l}$ and hence $L(\Phi) = g^{k\bar{l}}\frac{\partial^2 \Phi}{\partial z^k \partial \bar{z}^l}$ is real-valued. If the right-hand side is translated into an expression in real variables by means of the relations

$$z^k = x^k + ix^{\bar{k}}, \qquad \frac{\partial}{\partial z^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} - i\frac{\partial}{\partial x^{\bar{k}}}\right), \qquad \frac{\partial}{\partial \bar{z}^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} + i\frac{\partial}{\partial x^{\bar{k}}}\right),$$

$$g^{k\bar{l}} = a^{kl} + ib^{kl},$$

then we get $L(\Phi) = h^{\alpha\beta}\frac{\partial^2 \Phi}{\partial x^\alpha \partial x^\beta}$, where $h^{\alpha\beta}$ is a positive definite symmetric matrix defined by

$$h^{ij} = a^{ij}, \qquad h^{i\bar{j}} = b^{ij}, \qquad h^{\bar{i}j} = -b^{ij}, \qquad h^{\bar{i}\bar{j}} = a^{ij}.$$

Then E. Hopf's Theorem as stated in [7, Chapter II] applies, and exactly as in that case we may deduce the lemma from the maximum principle. We remark that this version of the lemma for Hermitian manifolds is an improvement over that given in [4, Section 14], since it does not involve the torsion term.

3. HERMITIAN MANIFOLDS WITH ZERO CURVATURE

In this section, Theorems 1 and 2 are proved essentially by the methods of Auslander and Markus [3]. As in [4], we introduce the bundle of orthonormal frames relative to the Hermitian metric, the fibre over a point p of M being all n-tples $e_1, \cdots, e_n$ of mutually orthogonal unit vectors at p. Then there exist independent Pfaffian forms ω^i, $\omega^i{}_j$, with $\omega^i{}_j + \bar{\omega}^j{}_i = 0$, defined intrinsically in the bundle, and such that if $pe_1 \cdots e_n$ is a frame, then $dp = \omega^i e_i$ and $de_i = \omega^j{}_i e_j$. These forms satisfy the equations of structure:

$$d\omega^i - \omega^k \wedge \omega^i{}_k = A^i{}_{jk}\,\omega^j \wedge \omega^k,$$

$$d\omega^j{}_i - \omega^k{}_i \wedge \omega^j{}_k = R_{i\bar{j}k\bar{l}}\,\omega^k \wedge \omega^l,$$

where $A^i{}_{jk}$ and $R_{i\bar{j}k\bar{l}}$, evaluated at a given frame $pe_1 \cdots e_n$, are the components of the torsion and curvature tensor at the point p relative to the frame $e_1 \cdots e_n$. In this case $R_{i\bar{j}k\bar{l}} = 0$, and hence the system of equations $\omega^j{}_i = 0$ is completely integrable (see [5], p. 58). Therefore, in a suitably chosen coordinate neighborhood N(p) of each point p of M, it is possible to introduce a field of orthonormal frames $qe_1 \cdots e_n$ which are parallel (since $de_i = \omega_i{}^j e_j = 0$), and which are uniquely determined by the choice of a frame at one point, say p. The vectors e_i are pure vectors, and, since their covariant derivatives vanish, they are analytic. Thus in N(p) we have introduced a field of parallel, orthonormal, analytic frames, assuming only that the curvature vanishes. On the other hand, given any field of orthonormal frames which are parallel, then, restricted to these frames, the left-hand side of the second equation of structure reduces to zero; hence the curvature relative to these frames is zero. But then it must vanish relative to all frames. This gives

THEOREM 1. *The curvature of a Hermitian manifold vanishes if and only if each point has a neighborhood over which it is possible to choose a field of analytic vectors* $e_1, \cdots, e_n$ *such that the vectors are orthonormal at each point, and such that each* e_i *is a parallel vector field in the sense that its covariant derivative is identically zero.*

Let these neighborhoods be called admissible. Parallel displacement of a frame at p along any path in an admissible neighborhood N(p) is independent of the path, since $\omega^i{}_j = 0$ has a unique solution throughout N(p) coinciding with the given frame at p. Similarly, given any two points p_0 and p_1 of M and a path C joining them, there is a neighborhood $N(C) = \bigcup_{q \in C} N(q)$ such that displacement of a frame from p_0 to p_1 is the same along any path from p_0 to p_1 in N(C). Let C_t $(0 \leq t \leq 1)$ be a homotopy class of curves joining p_0 to p_1. From what has been said, it follows that the subset S composed of those points of $0 \leq s \leq 1$ for which parallel displacement of a frame from p_0 to p_1 along C_s is the same as along C_0, is an open subset of $0 \leq t \leq 1$. It is also closed and not empty, and hence it is the entire interval. This gives

THEOREM 2. *If* M *is a Hermitian manifold of zero curvature, then parallel displacement of a vector or frame depends only on the homotopy class of the path. If the manifold is simply connected, it is parallelisable by parallel, orthonormal frames.*

THEOREM 3. *A complex-analytic manifold which is parallelisable has a natural metric which is Hermitian, which has curvature zero, and relative to which the given field of frames is parallel and orthonormal.*

To prove this, let $v_1, \cdots, v_n$ be the n independent analytic vector fields assumed to exist on the manifold. Let $\theta_1, \cdots, \theta_n$ be Pfaffian forms dual to $v_1, \cdots, v_n$. Define $ds^2 = \Sigma_{i=1}^n \theta^i \bar{\theta}^i$; then it is clear that $v_1, \cdots, v_n$ are orthonormal. In local coordinates, if $\theta^i = a^i{}_j dz^j$, we have then $g_{i\bar{j}} = \Sigma_{k=1}^n a^k{}_i \bar{a}^k{}_j$. By direct computation, and with the use of the expressions for $C^i{}_{jk}$ and for $R_{i\bar{j}k\bar{l}}$ in Section 1, it is easy to verify that $dv_i = 0$ and that $R_{i\bar{j}k\bar{l}} = 0$.

Example. As a simple example, we consider $\theta^1 = z^1 z^2 dz^1$, $\theta^2 = z^1 z^2 dz^2$, v^1 and v^2 being dual to these forms, which are defined on the 2-dimensional complex plane minus the origin. To compute components of the torsion tensor relative to v^1, v^2, we write

$$d\theta^1 = \frac{-1}{z^1(z^2)^2}\,\theta^1 \wedge \theta^2, \qquad d\theta^2 = \frac{1}{(z^1)^2 z^2}\,\theta^1 \wedge \theta^2.$$

Clearly, $A^i{}_{jk} = 0$, except that

$$A^1{}_{12} = \frac{-1}{z^1(z^2)^2} = -A^1{}_{21} \quad \text{and} \quad A^2{}_{12} = \frac{1}{(z^1)^2 z^2} = -A^2{}_{21}.$$

We note that these components are not constant, although v_1 and v_2 form parallel frames. Therefore not all the covariant derivatives of the torsion are zero. It will be shown below that this cannot happen in the compact case.

4. THE COMPACT CASE

We now consider a compact Hermitian manifold with zero curvature. The Bianchi identities for the curvature and torsion of such a manifold are (see [4, p. 525])

$$R_{i\bar{j}k\bar{l}} - R_{k\bar{j}i\bar{l}} = 2A^j{}_{ik|\bar{l}},$$

$$A^i{}_{hk}A^h{}_{lj} + A^i{}_{hl}A^h{}_{jk} + A^i{}_{hj}A^h{}_{kl} = \frac{1}{2}(A^i{}_{jl|k} + A^i{}_{lk|j} + A^i{}_{kj|l}).$$

In our case $A^j{}_{ik|\bar{l}} = 0$, that is, the components of the torsion tensor are complex-analytic. However, $A^j{}_{ik|l}$ may not be zero, as is shown by the example above; that is, the $A^j{}_{ik}$ may not be constants relative to a local field of parallel frames. We shall show, however, that in the compact case this follows. Let $\Phi = \Sigma_{i,j,k}\bar{A}^i{}_{jk}A^i{}_{jk}$. Since $A^i{}_{jk|\bar{l}} = 0$, we get for $L(\Phi)$ the expression

$$L(\Phi) = \sum_h \Phi_{h\bar{h}} = \sum_{i,j,k,h} \bar{A}^i{}_{jk}A^i{}_{jk|h\bar{h}} + \bar{A}^i{}_{jk|h}A^i{}_{jk|h}.$$

By the interchange formulas in [4, Section 13], we get

$$A^i{}_{jk|h\bar{h}} = A^i{}_{jk|\bar{h}h} = 0,$$

and hence

$$L(\Phi) = \sum_{i,j,k,h} \bar{A}^i{}_{jk|h}A^i{}_{jk|h} \geq 0.$$

It follows from Bochner's Lemma that $A^i{}_{jk|h} \equiv 0$. From the second Bianchi identity it follows that the $A^i{}_{jk}$ satisfy the Jacobi identities. If, in an admissible neighborhood of each point, we restrict to parallel orthonormal frames, then the equations of structure become

$$d\omega^i = A^i{}_{jk}\,\omega^j \wedge \omega^k, \qquad A^i{}_{jk} + A^i{}_{kj} = 0,$$

and we have $\omega^i{}_j = 0$. Since relative to these frames the $A^i{}_{jk}$ are constants and satisfy the Jacobi identity, the ω^i are the left invariant forms of a local Lie group. Let $\tilde{M}$ be the universal covering space of M; then, since it is simply connected and complete (being a covering space of a compact, and therefore complete, space), it is a complex Lie group with the images $\tilde{\omega}^i$ of ω^i as left invariant forms. Then, by standard arguments on covering spaces, we get

THEOREM 4. *If* M *is a compact Hermitian manifold with curvature zero, then its universal covering space* $\tilde{M}$ *is a complex Lie group, and* M *is the factor space* $\tilde{M}/D$ *of* $\tilde{M}$ *by a discontinuous group* D *of covering transformations each of which is an isometry without fixed points mapping* $\tilde{M}$ *onto* $\tilde{M}$.

COROLLARY. *A compact Hermitian manifold with zero curvature cannot be simply connected.*

The corollary is immediate, since the only compact complex Lie groups are the tori, and these are not simply connected.

We conclude with two remarks:

(i) H. C. Wang characterized all compact complex parallelisable manifolds: they are coset spaces of a complex Lie group by a discrete subgroup. Thus, in the event that M is complex parallelisable by parallel, orthonormal frames, D is actually a subgroup of the Lie group M. That in general it need not be such a subgroup, even in the case where the Hermitian metric is a Kähler metric, is shown by several examples of compact Kähler manifolds of zero curvature, in a recent paper by L. Auslander [2]. In these cases, since the homogeneous holonomy group is not the identity, the manifold is not parallelisable by parallel, orthonormal frames.

(ii) M. F. Atiyah [1] has recently considered complex-analytic fibre bundles with a complex-analytic connection. For this reason it is perhaps of interest to point out that, in the bundle of all complex-analytic frames (structure group GL(n, c)) over M, the connection determined by the Hermitian metric is complex-analytic if and only if the curvature vanishes. This follows from formula (20) of [4], namely

$$R_i{}^j{}_{k\bar{l}} = -\frac{\partial C^j{}_{ik}}{\partial \bar{z}^l}\,.$$

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