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On the Analyticity of Smooth CR Mappings between Real-Analytic CR Manifolds

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Introduction

Let $M \subset \mathbb{C}^n$ $(n \geq 2)$ be a generic real-analytic CR submanifold, $M' \subset \mathbb{C}^{n'}$ a real-analytic subset, and $f : M \to M'$ a smooth (i.e., of class $\mathcal{C}^\infty$) CR mapping defined near the point $p \in M$. It is natural to ask the following question: Under what conditions is f real-analytic? (Of course, in this case, it extends holomorphically to a neighborhood of p.)

In the equidimensional case, many authors considered the situation when f is a CR diffeomorphism [3; 16; 18; 20; 23; 27]. The more general situation, when f is supposed to be of only finite multiplicity, was studied in [4; 17]. When M and M' have different dimensions, more recent results give also some sufficient conditions [12; 13; 19]. We would also like to mention related works on the regularity of continuous CR mappings [10; 11; 14; 28].

In this paper, generalizing the result of Coupet, Pinchuk, and Sukhov [12] to arbitrary codimension, we give a new sufficient condition for the analyticity of a smooth CR mapping $f : M \to M'$ between a generic real-analytic submanifold $M \subset \mathbb{C}^n$ and a real-analytic subset $M' \subset \mathbb{C}^{n'}$. We prove that, if M is minimal at $p \in M$ and if the characteristic variety of f at p is 0-dimensional, then f is real-analytic near p (see Theorem 1.2). Our result generalizes many situations previously considered by other authors:

(1) $M, M' \subset \mathbb{C}^n$ are hypersurfaces, M' is strictly pseudoconvex, and f is a CR diffeomorphism (Lewy [23] and Pinchuk [27]);
(2) $M, M' \subset \mathbb{C}^n$ are submanifolds, M is minimal, M' is essentially finite, and f is a CR diffeomorphism (Baouendi, Jacobowitz, and Trèves [3]);
(3) $M, M' \subset \mathbb{C}^n$ are hypersurfaces, M' is essentially finite, and f is of finite multiplicity (Diederich and Fornæss [17] and Baouendi and Rothschild [4]).

We point out that our main result applies to situations not listed here—in particular, when M and M' have different dimensions. Our main result seems to be new also in the equidimensional case, when $M, M' \subset \mathbb{C}^n$ are submanifolds of *higher codimension*. In this case, our sufficient condition can be seen as a generalization of the finite multiplicity condition of [4; 17].

In this paper, we introduce the notion of "characteristic variety" associated to the sets M and M' *and* to the mapping f; this is a generalization to higher codimension

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of the notion introduced in [12]. The characteristic variety is the complex-analytic subset of $\mathbb{C}^{n'}$ defined by the vanishing of the family of holomorphic functions obtained by applying the CR operators of M to the equations of M' complexified and pulled back by f (see Definition 1.1).

We explain now the idea of the proof of our main result. First of all, the assumption that the characteristic variety is 0-dimensional implies, in a classical way using the *fundamental relation* $f(M) \subset M'$, that each component function f_j satisfies a polynomial equation with coefficients that are quotients of smooth functions on M that are analytic with respect to z, $\bar{z}$ and the jet of $\overline{f(z)}$ (see Lemma 3.3). Proceeding by contradiction, we deduce that these coefficients are CR on M outside their singular locus.

Then we prove that, since M is assumed to be minimal at p, these coefficients extend meromorphically to a neighborhood of p (see Proposition 2.5). This is the main technical proposition of our paper; we believe that this result is of independent interest and can be useful in other close situations. The proof of this proposition is divided into two steps. In the first step, using a symmetry in relation to M and Rothstein's separate meromorphy theorem [29], we establish the meromorphic extension to a wedge $\mathcal{W}_p^s$, which is the symmetry of the wedge $\mathcal{W}_p$ given by Tumanov's extension theorem [31] at p. The second step is crucial. By a theorem of Ivashkovich [21], the envelope of meromorphy and the envelope of holomorphy of $\mathcal{W}_p^s$ coincide. Thus, it suffices to prove that a function h holomorphic in $\mathcal{W}_p^s$ extends to a full neighborhood of p. The idea is to extend h holomorphically to $\mathcal{W}_p$ by Tumanov's theorem and to conclude by the edge-of-the-wedge theorem. But the crucial problem (see Remark 2.10) is that the direction of wedge extendability of a CR function defined on a neighborhood U_p of p in M *depends on* U_p. In our situation, we actually need to control this direction of extendability, but the set U_p we deal with is defined as the edge of the wedge $\mathcal{W}_p^s$, which can be arbitrarily small. Our original method of dealing with this problem is to reason *at every point* $q \in M$. Gluing together the associated wedges $\mathcal{W}_q^s$, we obtain a "wedge attached to M" (see Section 2.5 or [25] for precise definition). Its edge is all M and thus the associated direction of extendability is fixed.

Finally, the last ingredient that we use is a theorem of Malgrange [24], which ensures that the graph of each f_j is real-analytic, since by the construction just described it is contained in a real-analytic subset of $M \times \mathbb{C}$ of the same real dimension as M (see Lemma 3.4).

The present paper is organized as follows. In Section 1, we give precise notation, definition, and statement of the results. Section 2 is devoted to the proof of the meromorphic extension property for the coefficients of the polynomial equations verified by the component functions f_j. In Section 3 we prove a "generalized" reflection principle, which is a more general statement than our main result. Finally, in Section 4, we study the relation between characteristic variety and essential finiteness and we give the proofs of the corollaries of our main result.

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1. Statement of the Results

1.1. Notation and Definition

Let $M \subset \mathbb{C}^n \simeq \mathbb{R}^{2n}$ $(n \geq 2)$ be a real-analytic submanifold defined in a neighborhood of the point $p \in M$ by the equations $r_k(z) = 0$ $(k = 1, \ldots, d)$, where the r_k are real-valued real-analytic functions satisfying $dr_1 \wedge \cdots \wedge dr_d \neq 0$ near p and where d is the codimension of M. Let $T_z M$ denote the real tangent space of M at $z \in M$ and let $T_z^c M := T_z M \cap iT_z M$ denote the complex tangent space. We assume that the submanifold M is *Cauchy–Riemann* (CR); that is, the complex dimension of $T_z^c M$ is a constant m, called the CR dimension of M. We write the defining equations of M in the usual form:

$$\rho_k(z, \bar{z}) = 0, \quad k = 1, \ldots, d,$$

where the ρ_k are holomorphic functions of $2n$ variables satisfying $\rho_k(z, \bar{z}) \in \mathbb{R}$, $k = 1, \ldots, d$. We assume that M is *generic,* that is, $\bar{\partial}\rho_1 \wedge \cdots \wedge \bar{\partial}\rho_d \neq 0$ near $(p, \bar{p})$ or (equivalently) $m = n - d$. The submanifold M is *minimal* at p (in the sense of [31]) if it contains no proper CR submanifold through p with CR dimension m. Recall that, since M is real-analytic, minimality is equivalent to *finite type* in the sense of Bloom and Graham [7]. By the holomorphic implicit function theorem, we may write the equations of M near p in the form

$$\overline{y_k} = \phi_k(\bar{x}, x, y), \quad k = 1, \ldots, d, \tag{1.1}$$

where

$$\mathbb{C}^n \ni z = (x, y) \in \mathbb{C}^m \times \mathbb{C}^d \tag{1.2}$$

is a system of local holomorphic coordinates near $p = (x_p, y_p)$ and where the $\phi_k(\xi, x, y)$ are holomorphic functions near $(\overline{x_p}, x_p, y_p)$ satisfying $\phi_k(\overline{x_p}, x, y) \equiv \phi_k(\xi, x_p, y) \equiv y_k$, $k = 1, \ldots, d$. The operators

$$L_j = \frac{\partial}{\partial \overline{x_j}} + \sum_{k=1}^{d} \frac{\partial \phi_k}{\partial \overline{x_j}}(\bar{x}, x, y)\frac{\partial}{\partial \overline{y_k}}, \quad j = 1, \ldots, m,$$

form a (commuting) basis of the CR operators on M with real-analytic coefficients. Recall that a $\mathcal{C}^1$ function ψ defined on M is called Cauchy–Riemann if $L_j \psi = 0$ on M for all $j = 1, \ldots, m$. A mapping is CR if all its component functions are CR.

Similarly to M, let $M' \subset \mathbb{C}^{n'} \simeq \mathbb{R}^{2n'}$ be a real-analytic subset defined in a neighborhood of the point $p' \in M'$ by the real-analytic equations $\rho'_k(z', \overline{z'}) = 0$, $k = 1, \ldots, d'$. Let $f : M \to M'$ be a smooth (i.e., of class $\mathcal{C}^\infty$) CR mapping defined in a neighborhood of p in M and such that $f(p) = p'$. For $k = 1, \ldots, d'$, $\alpha \in \mathbb{N}^m$, and fixed $z' \in \mathbb{C}^{n'}$, we may apply the composed operator $L^\alpha := L_1^{\alpha_1} \ldots L_m^{\alpha_m}$ to the smooth function $\rho'_k(z', \overline{f(\cdot)})$ defined on M as follows.

DEFINITION 1.1. The *characteristic variety* of f at p is the complex-analytic subset $\mathcal{V}_p(f) \subset \mathbb{C}^{n'}$ defined in a neighborhood of p' by the equations in z',

$$L^\alpha \rho'_k(z', \overline{f(\cdot)})|_p = 0 \quad \text{for all } k = 1, \ldots, d' \text{ and } \alpha \in \mathbb{N}^m.$$

Notice that $p' \in \mathcal{V}_p(f)$, since f is CR and $\rho'_k(f(z), \overline{f(z)}) = 0$ for all $k = 1, \ldots, d'$ and $z \in M$.

This notion of characteristic variety was first introduced in [12] for the case of M a hypersurface. It was then generalized to arbitrary codimension by the author in [15] for the algebraic case. The characteristic variety is related to "partial" analytic determinacy of f by its jet, that is, finite analytic determinacy of some of the component functions of f by the other ones and by the jet of f. In case the characteristic variety is 0-dimensional, we prove that f is finitely and analytically determined by its jet (see Lemma 3.3). This condition holds in many known situations (see [3; 4; 12; 17; 22; 23; 27]).

1.2. Results

The main theorem of our article generalizes the result of [12, Thm. 1] to arbitrary codimension.

THEOREM 1.2. *Let $f: M \to M'$ be a smooth CR mapping between a generic real-analytic submanifold $M \subset \mathbb{C}^n$ and a real-analytic subset $M' \subset \mathbb{C}^{n'}$, with $p \in M$, $p' \in M'$, and $f(p) = p'$. If M is minimal at p and if the dimension of $\mathcal{V}_p(f)$ at p' is zero, then f is real-analytic near p.*

The proof of this theorem is given in Sections 2 and 3.

In fact, we prove in Section 3 a "generalized" reflection principle (see Theorem 3.2), which is a more general statement than Theorem 1.2. This result shows that the *fundamental* condition $f(M) \subset M'$, equivalent to $\rho'_k(f(z), \overline{f(z)}) = 0$ for all $k = 1, \ldots, d'$ and $z \in M$, is not necessary. It is sufficient to assume that $f: M \to \mathbb{C}^{n'}$ satisfies a system of equations of the form $R_l(f(z), \overline{g(z)}) = 0$ for all $l = 1, \ldots, D$ and $z \in M$, where $g = (g_1, \ldots, g_{N'})$ are arbitrary smooth CR functions on M and where $R_1, \ldots, R_D$ are arbitrary holomorphic functions in $n' + N'$ variables.

If M' is a *smooth* generic real-analytic submanifold of $\mathbb{C}^{n'}$, then the *Segre variety* of M' associated to the point z' close to p' is the complex submanifold $Q'_{z'}$ defined in a neighborhood of p' by the equations $\rho'_k(\cdot, \overline{z'}) = 0$, $k = 1, \ldots, d'$. The submanifold M' is called *essentially finite* at p' if the complex-analytic set $A'_{p'} := \{z' : Q'_{z'} = Q'_{p'}\}$ is of dimension 0 at p' (see e.g. [3; 4; 17; 18]). The following result, due to [3], is a corollary of Theorem 1.2.

COROLLARY 1.3. *Let $f: M \to M'$ be a smooth CR diffeomorphism between generic real-analytic submanifolds $M, M' \subset \mathbb{C}^n$, with $p \in M$, $p' \in M'$ and $f(p) = p'$. If M is minimal at p and if M' is essentially finite at p'* (*or, equivalently, if M is essentially finite at p*), *then f is real-analytic near p.*

The first result in the diffeomorphic case was established by Lewy [23] and Pinchuk [27]. They proved the following *reflection principle*: Any local $\mathcal{C}^1$ CR diffeomorphism between strictly pseudoconvex real-analytic hypersurfaces is real-analytic. It is a consequence of Corollary 1.3 in the situation when f is smooth because, in codimension 1, strict pseudoconvexity implies both minimality and essential finiteness. Notice that, in this context, the complex-analytic subset defined by the first-order equations $L_j\rho'_k(z', \overline{f(\cdot)})|_p = 0$ for all $k = 1, \ldots, d'$ and $j = 1, \ldots, m$ is already 0-dimensional at p'.

The following statement is a corollary of Theorem 1.2 when $M, M' \subset \mathbb{C}^n$ are hypersurfaces and f is of finite multiplicity; it was proved in [4] and [17]. We refer the reader to [4] for a precise algebraic definition of *finite multiplicity*.

COROLLARY 1.4. *Let $f: M \to M'$ be a smooth CR mapping between real-analytic hypersurfaces $M, M' \subset \mathbb{C}^n$, with $p \in M$, $p' \in M'$ and $f(p) = p'$. If f is of finite multiplicity at p and if M' is essentially finite at p', then f is real-analytic near p.*

In the general situation defined in Section 1.1, we say that f is *K-nondegenerate* at p for some positive integer K if the complex vector space spanned by the gradients $(\partial/\partial z')L^\alpha\rho'_k(z', \overline{f(\cdot)})|_p$ at $z' = p'$, for $k = 1, \ldots, d'$ and $|\alpha| \le K$, is all $\mathbb{C}^{n'}$. The following statement, established in [22], is an easy corollary of Theorem 1.2, since in this situation the holomorphic implicit function theorem applies.

COROLLARY 1.5. *Let $f: M \to M'$ be a smooth CR mapping between a generic real-analytic submanifold $M \subset \mathbb{C}^n$ and a real-analytic subset $M' \subset \mathbb{C}^{n'}$, with $p \in M$, $p' \in M'$ and $f(p) = p'$. If M is minimal at p and if f is K-nondegenerate at p, then f is real-analytic near p.*

2. Meromorphic Extension

2.1. Preliminaries

We suppose that the submanifold $M \subset \mathbb{C}^n$ is given near p by (1.1) in the system of local holomorphic coordinates (1.2). It may also be defined near p by the equations

$$\operatorname{Im} y_k = \mathcal{G}_k(x, \operatorname{Re} y), \quad k = 1, \ldots, d, \tag{2.1}$$

where the $\mathcal{G}_k$ are real-valued real-analytic functions near $(x_p, \operatorname{Re} y_p)$. In the following, we will use the notation $\phi = (\phi_1, \ldots, \phi_d)$ and $\mathcal{G} = (\mathcal{G}_1, \ldots, \mathcal{G}_d)$.

DEFINITION 2.1. A *wedge* associated to the submanifold M at the point $q \in M$ is a domain of the form

$$\mathcal{W}(\mathcal{N}, C) := \{z \in \mathcal{N} : \operatorname{Im} y - \mathcal{G}(x, \operatorname{Re} y) \in C\}, \tag{2.2}$$

where $\mathcal{N}$ is a sufficiently small neighborhood of q in $\mathbb{C}^n$ and where C is a nonempty open convex cone in $\mathbb{R}^d$ (with vertex 0). The *edge* of $\mathcal{W}(\mathcal{N}, C)$ is the open subset $M \cap \mathcal{N}$ of M.

The following extension theorem is well known; we will need a precise statement of it.

THEOREM 2.2 (Tumanov [31]). *Let q be a point in M and let V be a neighborhood of q in M. If M is minimal at q, then there exist a neighborhood $\mathcal{N} = \mathcal{N}(q, V)$ of q in $\mathbb{C}^n$ and a nonempty open convex cone $C = C(q, V)$ in $\mathbb{R}^d$ such that every continuous CR function on V extends holomorphically to the wedge $\mathcal{W}(\mathcal{N}, C)$.*

In order to study extendability properties for some classes of functions defined on M (see Sections 2.2 and 2.4), it will be convenient to cut the complex affine space $\mathbb{C}^n$ into slices. For $a \in \mathbb{C}^m$ sufficiently close to x_p, let $E_a \subset \mathbb{C}^n$ denote the complex affine subspace $\{x = a\}$ of complex dimension d. The CR submanifold $M_a := M \cap E_a$ is real-analytic *totally real* (i.e., of CR dimension 0) and of maximal real dimension in E_a. If $\mathcal{W}$ is a wedge associated to M, then $\mathcal{W}_a := \mathcal{W} \cap E_a$ is a wedge associated to M_a in E_a.

The mapping $s\colon z \mapsto (x, \overline{\phi(\bar{x}, x, y)})$ defined near p is real-analytic in x and antiholomorphic in y. Moreover, s is a symmetry in relation to M, because M is invariant by s and s is an involution in a neighborhood of p. Indeed, $\phi(\bar{x}, x, \overline{\phi(\bar{x}, x, y)}) - \bar{y} \equiv 0$, since for fixed x this mapping is antiholomorphic and vanishes on the generic submanifold M_x of E_x.

For a wedge $\mathcal{W}$ associated to M, the *symmetric wedge* of $\mathcal{W}$ is $\mathcal{W}^s := s(\mathcal{W})$. It is not actually a wedge according to Definition 2.1, but it contains a real wedge of a cone possibly slightly smaller than $-C$. Notice that the relation $\mathcal{W} = s(\mathcal{W}^s)$ also holds, provided that $\mathcal{W}$ is small enough.

2.2. Definition and Basic Properties of the Ring of Functions $\mathcal{R}_p(M)$

Let $\mathcal{R}_p(M)$ be the ring of germs at p of functions defined on M of the form

$$h(z) = H(z, \bar{z}, \overline{g(z)}), \tag{2.3}$$

where $g = (g_1, \ldots, g_K)$ are germs at p of smooth CR functions on M and where H is a germ at $(p, \bar{p}, \overline{g(p)})$ of a holomorphic function in $\mathbb{C}^{2n+K}$. Notice that the CR operators L_j are derivations of the ring $\mathcal{R}_p(M)$. Let h be a representative of a germ of $\mathcal{R}_p(M)$ defined in some connected open neighborhood U of p in M. Assume that M is minimal at p and let $\mathcal{U} := \mathcal{N}(p, U)$, $\Gamma := C(p, U)$, and $\mathcal{W} := \mathcal{W}(\mathcal{U}, \Gamma)$ be (respectively) the neighborhood of p, the cone, and the wedge given by Tumanov's extension theorem. Let $U' := M \cap \mathcal{U} \subset U$ be the edge and let $\mathcal{W}^s$ be the symmetric wedge (cf. Section 2.1).

We can now state the following useful extension lemma.

LEMMA 2.3. *If M is minimal at p, then the function h extends as a real-analytic function $\tilde{h}$ in $\mathcal{W}$ (resp., $\tilde{h}^s$ in $\mathcal{W}^s$), smooth up to the edge U' and antiholomorphic (resp., holomorphic) with respect to y.*

Proof. First, we extend h to $\mathcal{W}$ using the holomorphic extension of CR functions; we then prove the extension to $\mathcal{W}^s$. This property may be seen as an analog of the Schwarz symmetry principle for wedges in $\mathbb{C}^n$ instead of half-domains in $\mathbb{C}$.

Step 1: Extension with Tumanov's theorem. The function h is given by (2.3) for $z \in U$. Without loss of generality, we may assume that $g(p) = 0$ and expand H as a power series in $\bar{g}$,

$$h(z) = \sum_{\nu \in \mathbb{N}^K} c_\nu(z, \bar{z}) \overline{g^\nu(z)},$$

where the coefficients c_ν are holomorphic functions near $(p, \bar{p})$. Since each g_j is a CR function on U, it admits a holomorphic extensions $\tilde{g}_j$ to $\mathcal{W}$ according to Tumanov's theorem. Then, the extension of h to $\mathcal{W}$,

$$\tilde{h}(z) = \sum_{\nu \in \mathbb{N}^K} c_\nu(s(z), \bar{z}) \overline{\tilde{g}^\nu(z)},$$

is clearly real-analytic and antiholomorphic with respect to y.

Step 2: Reflection principle. The extension of h to $\mathcal{W}^s$,

$$\tilde{h}^s(z) = \tilde{h}(s(z)), \tag{2.4}$$

is real-analytic and holomorphic with respect to y. □

The functions of $\mathcal{R}_p(M)$ are neither CR nor real-analytic. Nevertheless, they verify the following uniqueness principle.

Lemma 2.4. *Let h be as before and assume that M is minimal at p.*

(i) *If h vanishes on a nonempty open subset V of U', then $h \equiv 0$ on U'.*
(ii) *$\mathcal{R}_p(M)$ is an integral domain.*

Proof. (i) By Lemma 2.3, h has an extension $\tilde{h}$ to $\mathcal{W}$, real-analytic and antiholomorphic with respect to y. Let V' be the projection of V onto $\mathbb{C}^m_x$ by $\pi : (x, y) \mapsto x$. For all $a \in V'$, $\tilde{h}$ is antiholomorphic in $\mathcal{W}_a$ and vanishes on $V \cap E_a$, which is a nonempty open subset of M_a. Since M_a is a totally real submanifold of E_a of maximal dimension, the uniqueness theorem of Pinchuk [26] implies that $\tilde{h}|_{\mathcal{W}_a} \equiv 0$. Since M is a graph above $\mathbb{C}^m_x \times \mathbb{R}^d_{\operatorname{Re} y}$, it follows that V' is a nonempty open subset of $\mathbb{C}^m_x$. Therefore, if a moves in V' then $\mathcal{W}_a$ fills an open subset of $\mathcal{W}$. Hence, $\tilde{h}$ vanishes in an open subset of $\mathcal{W}$. Because $\tilde{h}$ is real-analytic, it vanishes identically in $\mathcal{W}$. By continuity up to the edge, $h|_{U'} \equiv 0$.

(ii) Let h_1 and h_2 be in $\mathcal{R}_p(M)$, and assume that $h_1 h_2 = 0$. If $h_1 \not\equiv 0$ near p, then there exists a nonempty open subset V of M, sufficiently close to p, such that h_2 vanishes on V. Thus, (i) applies and $h_2 \equiv 0$ near p. □

2.3. *Statement of the Meromorphic Extension Property*

Let $\hat{\mathcal{R}}_p(M)$ be the quotient field of the integral domain $\mathcal{R}_p(M)$ (cf. Lemma 2.4(ii)) and let $\mathcal{S}_p(M)$ be the subfield of $\hat{\mathcal{R}}_p(M)$ of CR functions. More precisely, the elements of $\mathcal{S}_p(M)$ are of the form $\psi = h_1/h_2$, where $h_1, h_2 \in \mathcal{R}_p(M)$, $h_2 \not\equiv 0$, and ψ is CR on $M \setminus \Sigma$ near p with $\Sigma := \{z \in M \text{ near } p : h_2(z) = 0\}$. By the foregoing uniqueness principle (cf. Lemma 2.4(i)), Σ is a closed subset of M near p *with empty interior.*

The main result of Section 2 is the following.

PROPOSITION 2.5. *If M is minimal at p, then every germ $\psi \in \mathcal{S}_p(M)$ extends meromorphically to a neighborhood of p in $\mathbb{C}^n$.*

The very technical proof of this proposition is given in Sections 2.4–2.6.

When ψ has no singularities at p, we have the following stronger result.

PROPOSITION 2.6. *If M is minimal at p, then every germ $\psi \in \mathcal{R}_p(M)$ that is CR on M near p extends holomorphically to a neighborhood of p in $\mathbb{C}^n$.*

The proof of this result is trivial (see Section 2.6) in comparison to that of Proposition 2.5. However, we actually need Proposition 2.5 because, in the proof of Lemma 3.4, we may divide by elements of $\mathcal{R}_p(M)$ and hence singularities at p may appear.

2.4. Edge-of-the-Wedge Theorem and Separate Meromorphy

PROPOSITION 2.7. *If M is minimal at p then, for every germ $\psi \in \mathcal{S}_p(M)$, there exits a wedge $\mathcal{W}^s$ at p such that ψ extends meromorphically to $\mathcal{W}^s$.*

Proof. The proof of this proposition is divided into three steps.

Step 1: Tumanov's extension theorem and reflection principle. Let h_1 and $h_2 \not\equiv 0$ be representatives of germs of $\mathcal{R}_p(M)$ defined in some connected open neighborhood of p in M. Up to shrinking M, we may assume that h_1 and h_2 are defined in all M and that M is minimal at every point $q \in M$, because minimality is an open property on real-analytic CR submanifolds. Let $\Sigma := \{z \in M : h_2(z) = 0\}$ and assume that the quotient $\psi := h_1/h_2$ is CR on $M \setminus \Sigma$, that is, $\psi \in \mathcal{S}_p(M)$. Let $U \Subset M$ be a relatively compact connected open neighborhood of p in M. As in Section 2.2, let $\mathcal{U}$, Γ, $\mathcal{W}$, U', and $\mathcal{W}^s$ be (respectively) the neighborhood of p, the cone, the wedge, the edge and the symmetric wedge associated to (p, U) by Tumanov's extension theorem.

By Lemma 2.3, h_j has an extension $\tilde{h}_j^s$ to $\mathcal{W}^s$ that is real-analytic and holomorphic with respect to y for $j = 1, 2$. Thus, $m = \tilde{h}_1^s/\tilde{h}_2^s$ is an extension of ψ to $\mathcal{W}^s$ that is meromorphic with respect to y.

Step 2: Edge-of-the-wedge theorem in each slice. We use the following notation. For $a \in \mathbb{C}^m$, $E_a := \{x = a\}$ denotes a slice of $\mathbb{C}^n$ as in Section 2.1; $\Delta^k(a, \rho)$ denotes the open polydisc of $\mathbb{C}^k$ of center a and radius $\rho > 0$ and, if $a = 0$, we write $\Delta^k_\rho := \Delta^k(0, \rho)$; $\mathcal{C}^\infty(\mathcal{D})$, $\mathcal{O}(\mathcal{D})$, and $\mathcal{M}(\mathcal{D})$ denote (respectively) the rings of smooth, holomorphic, and meromorphic functions in the domain $\mathcal{D} \subset \mathbb{C}^n$.

Let q be a point in $U' \setminus \Sigma$ and let V be a neighborhood of q in $U' \setminus \Sigma$. Since M is minimal at q (see step 1), Tumanov's extension theorem gives a neighborhood $\mathcal{V} := \mathcal{N}(q, V)$ of q, an open convex cone $\Lambda := C(q, V)$, and a wedge $\mathcal{W}^* := \mathcal{W}(\mathcal{V}, \Lambda)$ of edge $V' := M \cap \mathcal{V}$ such that every CR function on V extends holomorphically to $\mathcal{W}^*$. In particular, ψ extends holomorphically to $\mathcal{W}^*$; we also denote this extension by m.

In order to simplify the notation, we may assume that q is the origin 0. We may also assume that m has no singularities in the wedge $\mathcal{W}^{s\prime} := \mathcal{W}^s \cap \mathcal{V}$ (up to shrinking $\mathcal{V}$). Let $\Gamma^\sharp$ be an arbitrary large proper subcone of the convex hull of $-\Gamma \cup \Lambda$ and let $\mathcal{W}^\sharp$ be the wedge $\mathcal{W}(\mathcal{V}, \Gamma^\sharp)$. For every $a \in \Delta_\varepsilon^m$, $\varepsilon > 0$ sufficiently small, we use the following notation: $\mathcal{W}_a^{s\prime} := \mathcal{W}^{s\prime} \cap E_a$, $\mathcal{W}_a^* := \mathcal{W}^* \cap E_a$, $\mathcal{W}_a^\sharp := \mathcal{W}^\sharp \cap E_a$, and $V_a' := V' \cap E_a$.

LEMMA 2.8. *Let $h \in \mathcal{O}(\mathcal{W}^*)$ be such that, for every $a \in \Delta_\varepsilon^m$, $h_a := h|_{E_a} \in \mathcal{O}(\mathcal{W}_a^{s\prime} \cup \mathcal{W}_a^*) \cap \mathcal{C}^\infty(\mathcal{W}_a^{s\prime} \cup \mathcal{W}_a^* \cup V_a')$. Then h extends holomorphically to $\mathcal{W}^\sharp$ near* 0.

Proof. Let $a \in \Delta_\varepsilon^m$ and denote by $\eta_a := (0, \phi(a, 0))$ the point in M_a such that $\operatorname{Re} \eta_a = 0$. By the edge-of-the-wedge theorem of [1] (see also [6]), there exists a neighborhood $\mathcal{N}_a$ of η_a in $E_a \simeq \mathbb{C}^d$ such that h_a extends holomorphically to $\mathcal{W}_a^\sharp \cap \mathcal{N}_a$. We may assume that, for all $a \in \Delta_\varepsilon^m$, $\mathcal{N}_a \supset \Delta_\delta^d$ for some $\delta = \delta(\varepsilon)$. Thus, h is holomorphic in y in $\mathcal{W}^\sharp \cap (\Delta_\varepsilon^m \times \Delta_\delta^d)$ and holomorphic in all the variables in $\mathcal{W}^*$. By Hartogs's theorem, h is holomorphic in $\mathcal{W}^\sharp$ intersected with a neighborhood of 0. □

Applying Lemma 2.8 to the function m, we obtain that m is holomorphic in $\mathcal{W}^\sharp$ intersected with a neighborhood of 0. In particular, m is holomorphic in a nonempty domain $\Omega' \subset \mathcal{W}^s$.

Step 3: Propagation of meromorphy and separate meromorphy.

LEMMA 2.9. *Let $\Omega' \subset \Omega$ be nonempty domains in $\mathbb{C}^n$ and let h_1 and $h_2 \not\equiv 0$ be real-analytic functions in Ω. If $m := h_1/h_2$ is meromorphic in Ω', then m is meromorphic in all Ω.*

Proof. Up to shrinking Ω', we may assume without loss of generality that h_2 does not vanish in Ω'.

Case 1: $n = 1$, Ω' and Ω are discs. This case is treated in [13, Lemma 3.6]. Let c' be the center of Ω'. For $\zeta \in \Omega$, let γ denote the closed segment $[c', \zeta]$. Let $\tilde{h}_1$ (resp. $\tilde{h}_2$) be the holomorphic extension of $h_1|\gamma$ (resp. $h_2|\gamma$) to a neighborhood Γ of γ. We may assume that $\tilde{h}_2$ does not vanish in $\Omega' \cap \Gamma$ (up to shrinking Γ). Therefore, $\tilde{m} := \tilde{h}_1/\tilde{h}_2$ is holomorphic in $\Omega' \cap \Gamma$ and coincides with m on $\Omega' \cap \gamma$. By the uniqueness theorem, $\tilde{m} = m$ in $\Omega' \cap \Gamma$. Then the function $\tilde{h}_1 h_2 - \tilde{h}_2 h_1$, which is real-analytic in Γ, vanishes in $\Omega' \cap \Gamma$. Hence, it vanishes in all Γ and so $m|_\Gamma \equiv \tilde{m}$ is meromorphic. Using this argument for all $\zeta \in \Omega$, we show that m is meromorphic in all Ω.

Case 2: $n \geq 1$, Ω' and Ω are polydiscs. Assume that $\Omega' = \Delta^n(c', R')$ and $\Omega = \Delta^n(c, R)$. We prove inductively that the meromorphy of m propagates to each complex direction of $\mathbb{C}^n$. Proceeding by induction on $k = 0, \ldots, n$, we assume that m is meromorphic in $\Delta^k((c_1, \ldots, c_k), R) \times \Delta^{n-k}((c'_{k+1}, \ldots, c'_n), R')$ for some $k \in \{0, \ldots, n-1\}$. For each $\zeta \in \Delta^k((c_1, \ldots, c_k), R)$ and $\zeta' \in \Delta^{n-k-1}((c'_{k+2}, \ldots, c'_n), R')$

such that h_2 does not vanish identically in $\Delta' := \{\zeta\} \times \Delta^1(c'_{k+1}, R') \times \{\zeta'\}$, we apply case 1 to Δ' and $\Delta := \{\zeta\} \times \Delta^1(c_{k+1}, R) \times \{\zeta'\}$, which proves that $m|_\Delta$ is meromorphic. Consequently, by Rothstein's separate meromorphy theorem (see [29] or [30]), we obtain that m is meromorphic in $\Delta^{k+1}((c_1, \dots, c_{k+1}), R) \times \Delta^{n-k-1}((c'_{k+2}, \dots, c'_n), R')$.

Case 3: General case. Let c' be a point in Ω'. For each $\zeta \in \Omega$, let γ be a compact smooth simple curve linking c' and ζ and let $(\Delta_1^n, \dots, \Delta_r^n)$ be a finite cover of γ by polydiscs in Ω. Case 2 implies that the meromorphy of m propagates from Δ_ν^n to $\Delta_{\nu+1}^n$, and we obtain that m is meromorphic in a neighborhood of ζ for all $\zeta \in \Omega$. □

Lemma 2.9 applied to the function m and the domains $\Omega' \subset \mathcal{W}^s$ proves that m is meromorphic in all $\mathcal{W}^s$.

The proof of Proposition 2.7 is complete. □

REMARK 2.10. At this stage, we could easily conclude that m extends meromorphically near p if the direction of wedge extendability at p of CR functions on $U \subset M$ were independent of U. This condition is satisfied, for instance, if the submanifold M has finite type at p with all Hörmander's numbers being the same (see [8] and related results in [5]). Under this assumption, pushing M into $\mathcal{W}^s$ in the opposite direction of extendability shows that all holomorphic functions in $\mathcal{W}^s$ extend holomorphically near p, and a theorem of Ivashkovich [21] gives the conclusion of Proposition 2.5.

2.5. Meromorphic Extension to a Wedge Attached to M

We denote by $NM := T\mathbb{C}^n|_M/TM$ the normal bundle to M. Let q be a point in M, $n_q \in N_qM$ a normal vector to M at q, and $\mathcal{W}_q = \mathcal{W}(\mathcal{N}_q, C_q)$ a wedge at q. Identifying N_qM with $\mathbb{R}^d$, we may assume that $C_q \subset N_qM$. We say that $\mathcal{W}_q$ *has direction* n_q if $n_q \in C_q$. By definition, "$\mathcal{W}_q$ has direction $n_q = 0$" means that $\mathcal{W}_q$ is a full neighborhood of q in $\mathbb{C}^n$.

DEFINITION 2.11. Let Ω be a connected open subset of M. The domain ω is a *wedge attached to* Ω (see [25]) if there exists a smooth section $n: \Omega \mapsto N\Omega$ of the normal bundle such that, for every $q \in \Omega$, ω contains a wedge at q with direction $n(q)$.

This notion of attached wedge allows us to give the following global meromorphic extension result.

PROPOSITION 2.12. *Let $M \subset \mathbb{C}^n$ be a generic real-analytic submanifold that is minimal at every point $p \in M$. Let $\Sigma \subset M$ be a closed subset of empty interior and let ψ be a smooth CR function on $M \setminus \Sigma$. Assume that, for every point $p \in M$, there exist* (a) *a wedge $\mathcal{W}_p$ whose edge is a neighborhood U_p of p in M and* (b) *an extension $m_p \in \mathcal{M}(\mathcal{W}_p)$ of $\psi|_{U_p\setminus\Sigma}$. Then, for every connected open subset $\Omega \Subset M$, there exist* (a) *a wedge ω attached to Ω containing $\mathcal{W}_p$ for every $p \in \Omega$ and* (b) *an extension $m \in \mathcal{M}(\omega)$ of $\psi|_{\Omega\setminus\Sigma}$.*

For the proof of Proposition 2.12, we need some technical lemmas. The following lemma is a uniqueness principle with singularities on the edge.

LEMMA 2.13. *Let $\mathcal{W}$ be a wedge of edge U and let $m \in \mathcal{M}(\mathcal{W})$ be an extension of $\psi \in \mathcal{C}^\infty(U \setminus \Sigma)$. If $\psi \equiv 0$, then $m \equiv 0$.*

Proof. Let $p \in U \setminus \Sigma$. There exists a neighborhood $\mathcal{V}$ of p such that m is holomorphic in $\mathcal{W} \cap \mathcal{V}$. By Pinchuk's uniqueness principle [26], $m|_{\mathcal{V}} \equiv 0$. Then, by the uniqueness principle for holomorphic mappings between connected complex manifolds (here, $\mathcal{W}$ and $\mathbb{P}^1(\mathbb{C})$), $m \equiv 0$. □

The following lemma is an edge-of-the-wedge theorem with singularities on the edge.

LEMMA 2.14. *Let $M \subset \mathbb{C}^n$ be a generic real-analytic submanifold minimal at some point $p \in M$ and let U be a connected open neighborhood of p in M. Let $\Sigma \subset M$ be a closed subset of empty interior and let ψ be a smooth CR function on $M \setminus \Sigma$. Assume that there exist wedges $\mathcal{W}_j$ of edge U and of cones C_j and extensions $m_j \in \mathcal{M}(\mathcal{W}_j)$ of $\psi|_{U \setminus \Sigma}$ for $j = 1, 2$. Then there exist* (a) *a wedge $\mathcal{W}$ of edge $U' \subset U$ a neighborhood of p in M and of cone C almost containing the convex hull of $C_1 \cup C_2$ and* (b) *an extension $m \in \mathcal{M}(\mathcal{W})$ of $\psi|_{U \setminus \Sigma}$.*

REMARK. We will always use the following conventions.

(i) All our cones are presumed to be convex.
(ii) The phrase "a cone C *almost contains* a cone C'" means that C contains a proper subcone of C' with vertex 0. In practice, this subcone can be chosen as large as we wish, so this slight abuse of notation makes no difference in the following.

Proof of Lemma 2.14. Let h_1 be a holomorphic function in $\mathcal{W}_1$. Since M is minimal at p, there exists a wedge $\mathcal{W}'$ of edge $U' \subset U$ a neighborhood of p in M and of cone C such that every CR function in U extends holomorphically to $\mathcal{W}'$.

We may assume that the positive axis $\operatorname{Im} z_n$ is inside the cone C_1. For $d > 0$, let t^d be the translation along $\operatorname{Im} z_n$ with length d and let $U^d := t^d(U)$. Then $h_1|_{U^d}$ is CR and extends holomorphically to $\mathcal{W}'^d := t^d(\mathcal{W}')$. According to Aĭrapetyan's edge-of-the-wedge theorem [1], there exist a neighborhood $U_1' \subset U'$ of p in M and a cone C_1' almost containing the convex hull of $C_1 \cup C'$ such that h_1 extends holomorphically to the wedge $\mathcal{W}_1'^d$ of edge $U_1'^d := t_d(U_1')$ and of cone C_1'. Notice that $\mathcal{W}_1'^d = t_d(\mathcal{W}_1')$, where $\mathcal{W}_1'$ is the wedge of edge U_1' and of cone C_1'. Letting d tend to zero, we obtain that h_1 extends holomorphically to $\mathcal{W}_1'$. By a theorem of Ivashkovich [21], the envelope of holomorphy and the envelope of meromorphy of the open set $\mathcal{W}_1$ coincide. Consequently, m_1 extends meromorphically to $\mathcal{W}_1'$. Similarly, m_2 extends meromorphically to the wedge $\mathcal{W}_2'$ of edge $U_2' \subset U'$ a neighborhood of p in M and of cone C_2' almost containing the convex hull of $C_2 \cup C'$. We may assume that $U_1' = U_2' =: U_p'$. By the uniqueness principle (Lemma 2.13), the extensions of m_1 and m_2 coincide in $\mathcal{W}_1' \cap \mathcal{W}_2'$. Thus, we obtain a common extension $m \in \mathcal{M}(\mathcal{W}_1' \cup \mathcal{W}_2')$ of $\psi|_{U_p' \setminus \Sigma}$.

Finally, applying Ivashkovich's and Aĭrapetyan's theorems for $\mathcal{W}_1' \cup \mathcal{W}_2'$, we prove that m extends meromorphically to the wedge $\mathcal{W}''$ of edge $U'' \subset U_p'$ a neighborhood of p in M and of cone C'' almost containing the convex hull of $C_1' \cup C_2'$. Moreover, m admits $\psi|_{U''\setminus\Sigma}$ as smooth boundary value on $U'' \setminus \Sigma$. □

Following the notation of Proposition 2.12, we may assume that the U_p are traces on M of balls of $\mathbb{C}^n$; that is, $U_p = B(p, R_p) \cap M$ with $R_p > 0$. For $\varepsilon > 0$, we define the *ε-shrinking* of U_p to be $U_p^\varepsilon := B(p, R_p - \varepsilon) \cap M$. In the following, when a wedge ω is attached to some connected open subset Ω of M, we will always assume that Ω is a finite union of some U_p, that is, $\Omega = \bigcup_{j=1}^s U_{p_j}$. Thus, we may also define the *ε-shrinking* of Ω to be $\Omega^\varepsilon := \bigcup_{j=1}^s U_{p_j}^\varepsilon$.

REMARK 2.15. Let K be a compact subset of M and let $(U_{p_j})_{j=1,\dots,s}$ be an open cover of K. Then there exists an $\varepsilon > 0$ such that $(U_{p_j}^\varepsilon)_{j=1,\dots,s}$ is a cover of K, too.

The following lemma allows us to glue two attached wedges together.

LEMMA 2.16. *Let $M \subset \mathbb{C}^n$ be a generic real-analytic submanifold that is minimal at every point $p \in M$. Let $\Sigma \subset M$ be a closed subset of empty interior, and let ψ be a smooth CR function on $M \setminus \Sigma$. Let ω_j be a wedge attached to a connected open subset Ω_j of M, and let $m_j \in \mathcal{M}(\omega_j)$ be an extension of $\psi|_{\Omega_j\setminus\Sigma}$ for $j = 1, 2$. Assume that $\Omega_1 \cap \Omega_2 \neq \emptyset$. Then, for all sufficiently small $\varepsilon > 0$, there exist* (a) *a wedge ω^ε attached to $\Omega^\varepsilon := \Omega_1^\varepsilon \cup \Omega_2^\varepsilon$ that contains the restriction of ω_j to Ω_j^ε for $j = 1, 2$ and* (b) *an extension $m^\varepsilon \in \mathcal{M}(\omega^\varepsilon)$ of $\psi|_{\Omega^\varepsilon\setminus\Sigma}$.*

Proof. By Definition 2.11, for every $p \in \Omega_1 \cap \Omega_2$ and for $j = 1, 2$, there exists a wedge $\mathcal{W}_{p,j} \subset \omega_j$ of edge $U_{p,j}$, of cone $C_{p,j}$, and of direction $n_j(p)$, where n_j is the smooth section of $N\Omega_j$ associated to ω_j. By Lemma 2.14, there exist (a) a wedge $\mathcal{W}_p$ of edge $U_p' \subset U_{p,1} \cap U_{p,2}$ and of cone C_p almost containing the convex hull of $C_{p,1} \cup C_{p,2}$ and (b) a function $m_p \in \mathcal{M}(\mathcal{W}_p)$ extending $\psi|_{U_p'\setminus\Sigma}$.

Let $\varepsilon > 0$ and let ω_j^ε be the restriction of ω_j to Ω_j^ε for $j = 1, 2$. Let $(U_{p_1}', \dots, U_{p_s}')$ be a finite open cover of the adherence $\mathrm{Adh}(\Omega_1^\varepsilon \cap \Omega_2^\varepsilon) \Subset M$ of $\Omega_1^\varepsilon \cap \Omega_2^\varepsilon$. The domain $\omega^\varepsilon := \omega_1^\varepsilon \cup \omega_2^\varepsilon \cup \mathcal{W}_{p_1} \cup \cdots \cup \mathcal{W}_{p_s}$ is a wedge attached to $\Omega^\varepsilon := \Omega_1^\varepsilon \cup \Omega_2^\varepsilon$. Indeed, we build a smooth section of the normal bundle using a smooth partition of unity associated to the open cover of $\mathrm{Adh}(\Omega_1^\varepsilon \cup \Omega_2^\varepsilon) \Subset M$ by Ω_1 and Ω_2 and using the fact that C_{p_k} almost contains the convex hull of $C_{p_k,1} \cup C_{p_k,2}$. In view of Lemma 2.13, the functions m_j in ω_j $(j = 1, 2)$ and m_{p_k} in $\mathcal{W}_{p_k}$ $(k = 1, \dots, s)$ coincide on the intersections of these wedges. Hence, we obtain a meromorphic extension m^ε of $\psi|_{\Omega^\varepsilon\setminus\Sigma}$ to ω^ε. □

Proof of Proposition 2.12 (cont.). Let $(U_{p_1}, \dots, U_{p_s})$ be a finite open cover of $\mathrm{Adh}(\Omega)$ and let $\varepsilon > 0$ be such that $(U_{p_1}^\varepsilon, \dots, U_{p_s}^\varepsilon)$ is still a cover of $\mathrm{Adh}(\Omega)$ (cf. Remark 2.15). We prove by induction on $n \in \{1, \dots, s\}$ that there exist a wedge ω_n attached to $\Omega_n = U_{p_1}^{n\varepsilon/s} \cup \cdots \cup U_{p_n}^{n\varepsilon/s}$ containing the $\mathcal{W}_{p_k}$ $(k = 1, \dots, n)$ and an extension $m_n \in \mathcal{M}(\omega_n)$ of $\psi|_{\Omega_n\setminus\Sigma}$. For $n = 1$, the statement is clear with $\omega_1 = \mathcal{W}_{p_1}$ and $m_1 = m_{p_1}$. Assume that the statement is verified for some

$n \in \{1, \dots, s-1\}$. According to Lemma 2.16, for ε/s we may glue the wedges $\mathcal{W}_{p_{n+1}}$ and ω_n together. This gives the statement for $n+1$.

For $n = s$, we obtain a wedge ω_s attached to $\Omega_s \supset \Omega$ and an extension $m_s \in \mathcal{M}(\omega_s)$ of $\psi|_{\Omega_s \setminus \Sigma}$. Finally, we take ω to be the restriction of ω_s to Ω and $m := m_s|_\omega$. To refine the result, we glue ω with all the $\mathcal{W}_p$, $p \in \Omega$, and extend m to this larger domain (using Ivashkovich's and Aĭrapetyan's theorems only).

The proof of Proposition 2.12 is complete. □

2.6. *Deformation of the Submanifold M*

Proof of Proposition 2.5 (cont.). We follow the notation of Section 2.4; we assume that M is minimal at every point $q \in M$ and that $U \Subset M$ is a relatively compact connected open neighborhood of p in M. Therefore, for every point $q \in U$, Proposition 2.7 holds and there exist a wedge $\mathcal{W}_q^s$ of edge U_q and a meromorphic extension m_q of $\psi|_{U_q \setminus \Sigma}$ to $\mathcal{W}_q^s$.

By Proposition 2.12, we can glue the wedges $\mathcal{W}_q^s$ together and obtain a wedge ω^s attached to U such that $\psi|_{U \setminus \Sigma}$ extends to ω^s as a meromorphic function m^s and such that ω^s contains the wedge $\mathcal{W}_p^s$. With the help of a smooth partition of unity, as in Lemma 2.16, we may apply a small smooth deformation to U in the direction of n^s, the smooth section of the normal bundle to U associated to the wedge ω^s. We assume that this deformation depends smoothly on the parameter $d \geq 0$ and that the deformation is the identity for $d = 0$. We denote by $U^d \subset \omega^s$ the deformation of U.

The wedge $\mathcal{W}_p$ is obtained by analytic discs attached to U (see [31]). Therefore, there is still an analytic disc attached to U^d, making a wedge $\mathcal{W}_p^d$ that is a small smooth deformation of $\mathcal{W}_p$. In particular, $\mathcal{W}_p^d$ tends to $\mathcal{W}_p$ as d tends to zero. For sufficiently small $d > 0$, $\mathcal{W}_p^d$ is "almost symmetric" to $\mathcal{W}_p^s$ in the sense that the cones of $\mathcal{W}_p^d$ and $\mathcal{W}_p$ intersect. With possibly smaller $d > 0$, we may even assume that the cone of $\mathcal{W}_p^d$ contains the direction $-n^s(p)$ and hence that $p \in \mathcal{W}_p^d$. Thus, we obtain that the envelope of holomorphy of ω^s contains a neighborhood of p. By Ivashkovich's theorem [21], this proves that m^s extends meromorphically to this neighborhood of p.

The proof of Proposition 2.5 is complete. □

When ψ has no singularities at p, the proof of the (holomorphic) extendability property is trivial.

Proof of Proposition 2.6. Let ψ be a representative of a germ of $\mathcal{R}_p(M)$ defined in some connected open neighborhood U of p in M, and assume that ψ is CR on U. By Tumanov's theorem, ψ extends holomorphically to $\mathcal{W}$; by Lemma 2.3, it extends to $\mathcal{W}^s$ as a function holomorphic in y, where $\mathcal{W}$ is the wedge associated to (p, U) and $\mathcal{W}^s$ is its symmetric wedge. Now, by the *classical* edge-of-the-wedge theorem applied in each slice E_a and by Hartogs's theorem, we conclude that ψ extends holomorphically to a neighborhood of p in $\mathbb{C}^n$.

The proof of Proposition 2.6 is complete. □

3. Generalized Reflection Principle

3.1. Statement of the Generalized Reflection Principle and Application to the Mapping Problem

Let $M \subset \mathbb{C}^n$ be a generic real-analytic submanifold and let $p \in M$. Let $G = (G_1, \ldots, G_{N'})$ be smooth CR functions on M (near p) and let $R_l(Z, W)$, $l = 1, \ldots, D$, be holomorphic functions in a neighborhood of $(P, \overline{G(p)})$ in $\mathbb{C}^N \times \mathbb{C}^{N'}$. We consider the following system of equations in F:

$$R_l(F(z), \overline{G(z)}) = 0, \quad l = 1, \ldots, D, \; z \in M, \tag{S}$$

where $F\colon M \to \mathbb{C}^N$ is a smooth CR mapping defined near p and such that $F(p) = P$.

DEFINITION 3.1. The *characteristic variety* at p of the system of equations (S) is the complex-analytic subset $\mathcal{V}_p(S) \subset \mathbb{C}^N$ defined in a neighborhood of P by the equations in Z,

$$L^\alpha R_l(Z, \overline{G(\cdot)})|_p = 0 \quad \text{for all } \; l = 1, \ldots, D \text{ and } \alpha \in \mathbb{N}^m.$$

The following “generalized” reflection principle generalizes [12, Prop. 3] to arbitrary codimension.

THEOREM 3.2. *Let $F\colon M \to \mathbb{C}^N$ be a smooth CR mapping with $F(p) = P$ satisfying the system of equations (S). If M is minimal at p and if the dimension of $\mathcal{V}_p(S)$ at P is zero, then F is real-analytic near p.*

The proof of this theorem is given in Sections 3.2–3.3.

This theorem applies to the *mapping problem* defined in Section 1, and we obtain Theorem 1.2 as a special case of Theorem 3.2.

Proof of Theorem 1.2 (cont.). The *fundamental* condition $f(M) \subset M'$ is equivalent to the following equations:

$$\rho'_k(f(z), \overline{f(z)}) = 0, \quad k = 1, \ldots, d', \; z \in M.$$

This system of equations is equivalent to (S) with $F = G = f$ and $R_k = \rho'_k$ $(k = 1, \ldots, d')$. Clearly, $\mathcal{V}_p(f) = \mathcal{V}_p(S)$ and Theorem 3.2 applies.

The proof of Theorem 1.2 is complete. □

3.2. Algebraicity over the Ring $\mathcal{R}_p(M)$

LEMMA 3.3. *Let $F\colon M \to \mathbb{C}^N$ be a smooth CR mapping with $F(p) = P$ satisfying the system of equations (S). If the dimension of $\mathcal{V}_p(S)$ at P is zero, then each component function F_j $(j = 1, \ldots, N)$ is algebraic over the ring $\mathcal{R}_p(M)$.*

Proof. In the following, all our reasonings will be localized at p. Applying the operators $L^\alpha = L_1^{\alpha_1} \ldots L_m^{\alpha_m}$ to the system of equations (S), we obtain

$$L^\alpha R_l(F(z), \overline{G(z)}) = 0, \quad l = 1, \ldots, D, \ \alpha \in \mathbb{N}^m, \ z \in M. \tag{3.1}$$

Since F is CR, we may rewrite (3.1) in the form

$$H_l^\alpha(z, \bar{z}, \overline{D^{|\alpha|}G(z)}, F(z)) = 0, \quad l = 1, \ldots, D, \ \alpha \in \mathbb{N}^m, \ z \in M, \tag{3.2}$$

where the H_l^α are holomorphic functions near $(p, \bar{p}, \overline{D^{|\alpha|}G(p)}, P)$ and where

$$D^A G = \left(\frac{\partial^{|\beta|} G_\nu}{\partial z^\beta}\right)_{|\beta| \le A, \ \nu = 1, \ldots, M}$$

denotes the partial derivatives of G up to order A, sometimes called the *jet of order* A of G. The equations of $\mathcal{V}_p(S)$ are clearly equivalent to the following ones:

$$H_l^\alpha(p, \bar{p}, \overline{D^{|\alpha|}G(p)}, Z) = 0, \quad l = 1, \ldots, D, \ \alpha \in \mathbb{N}^m.$$

In view of (3.2), these equations are verified for $Z = P$ and therefore $P \in \mathcal{V}_p(S)$. Since the ring $\mathcal{O}_P$ of germs at P of holomorphic functions in $\mathbb{C}^N$ is Nœtherian, there exists a positive integer A such that $\mathcal{V}_p(S)$ is given near P by the equations

$$H_l^\alpha(p, \bar{p}, \overline{D^{|\alpha|}G(p)}, Z) = 0, \quad l = 1, \ldots, D, \ |\alpha| \le A. \tag{3.3}$$

Modifying slightly the functions H_l^α, we may rewrite (3.3) in the more convenient form

$$H_l^\alpha(p, \bar{p}, \overline{D^A G(p)}, Z) = 0, \quad l = 1, \ldots, D, \ |\alpha| \le A.$$

Let $\mathcal{V}$ be the complex-analytic variety defined near $\Pi := (p, \bar{p}, \overline{D^A G(p)}, P)$ by the equations

$$H_l^\alpha(z, \zeta, \Delta, Z) = 0, \quad l = 1, \ldots, D, \ |\alpha| \le A,$$

where (z, ζ, Δ, Z) denotes the canonical coordinates in $\mathbb{C}^{2n+\kappa+N}$ and the integer κ is the length of the vector $D^A G$; that is,

$$\kappa = N' \binom{n + A}{n}.$$

Notice that $\mathcal{V}_p(S)$ coincides with the fiber

$$\mathcal{V}_{(p, \bar{p}, \overline{D^A G(p)})} = \{Z \text{ near } P : (p, \bar{p}, \overline{D^A G(p)}, Z) \in \mathcal{V}\}.$$

Since this fiber is assumed to be 0-dimensional at Π, we may apply the fundamental theorem on local representation of complex-analytic sets (see [9, Sec. 5.6, Prop. 4]). This theorem states that $\mathcal{V}$ is contained in a complex-analytic variety $\mathcal{Q}$ defined near Π by the equations

$$Q_j(z, \zeta, \Delta)(Z_j) = 0, \quad j = 1, \ldots, N, \tag{3.4}$$

where $Q_j(z, \zeta, \Delta)(Z_j)$ is a Weierstrass polynomial in Z_j with coefficients holomorphic in (z, ζ, Δ). Combining (3.2), (3.4), and the relation $\mathcal{V} \subset \mathcal{Q}$, we obtain

$$Q_j(z, \bar{z}, \overline{D^A G(z)})(F_j(z)) = 0, \quad j = 1, \ldots, N, \ z \in M. \tag{3.5}$$

This result can be seen as a finite analytic determination of F by the jet of order A of G.

Condition (3.5) means that each F_j annihilates on M a polynomial with coefficients in $\mathcal{R}_p(M)$; hence, the proof of Lemma 3.3 is complete. $\square$

3.3. *Analyticity of the Graph*

LEMMA 3.4. *Let ϕ be a smooth CR function defined in a neighborhood of p in M. If M is minimal at p and if there exist $d \geq 1$ and $\alpha_k \in \hat{\mathcal{R}}_p(M)$ $(k = 0, \ldots, d-1)$ such that*

$$\phi^d + \alpha_{d-1}\phi^{d-1} + \cdots + \alpha_0 = 0 \text{ on } M \text{ near } p \tag{3.6}$$

outside the singular locus of the α_k, then ϕ is real-analytic near p.

Proof. The proof is divided into two steps.

Step 1: Algebraicity over the field of meromorphic functions. Let

$$\phi^\delta + \beta_{\delta-1}\phi^{\delta-1} + \cdots + \beta_0 = 0 \text{ on } M \text{ near } p \tag{3.7}$$

be a polynomial equation of the form (3.6) of minimal degree δ. For every $j = 1, \ldots, m$ we apply the CR operator L_j, which is a derivation of the field $\hat{\mathcal{R}}_p(M)$, to (3.7). Since ϕ is CR, we obtain

$$(L_j\beta_{\delta-1})\phi^{\delta-1} + \cdots + (L_j\beta_0) = 0 \text{ on } M \text{ near } p. \tag{3.8}$$

Necessarily, for all $k = 0, \ldots, \delta - 1$, $L_j\beta_k \equiv 0$ on M near p outside the singular locus of β_k. Otherwise, let $k_0 \geq 1$ denote the larger integer such that $L_j\beta_{k_0} \not\equiv 0$ and then divide (3.8) by $L_j\beta_{k_0}$. We obtain a contradiction with the fact that (3.7) is of minimal degree. Therefore, the β_k are in $\mathcal{S}_p(M)$ and by Proposition 2.5 there exists a meromorphic extension m_k of β_k near p for all k. Hence, ϕ satisfies the polynomial equation with *meromorphic* coefficients

$$\phi^\delta + m_{\delta-1}\phi^{\delta-1} + \cdots + m_0 = 0 \text{ on } M \text{ near } p \tag{3.9}$$

outside the singular locus of the m_k; that is, ϕ is algebraic over the field $\mathcal{M}_p$ of germs at p of meromorphic functions in $\mathbb{C}^n$.

Step 2: Analyticity of the graph. Multiplying (3.9) by the least common multiple of the denominators of the coefficients, we obtain

$$h_\delta\phi^\delta + \cdots + h_0 = 0 \text{ on } M \text{ near } p, \tag{3.10}$$

where the h_j are holomorphic functions near p. Let $\Psi\colon (M, p) \to (\mathbb{R}^l, 0)$, $l = 2m + d$, be a local real-analytic diffeomorphism. The function $\psi := \phi \circ \Psi^{-1}$ is smooth near 0 and the $a_j := h_j \circ \Psi^{-1}$ are real-analytic near 0. In these new coordinates, (3.10) is changed into

$$a_\delta\psi^\delta + \cdots + a_0 = 0 \text{ on } \mathbb{R}^l \text{ near } 0. \tag{3.11}$$

We may assume that $\psi(0) = 0$ and that the a_j have no common factors (as elements of the ring of convergent power series $\mathbb{R}\{x_1, \ldots, x_l\}$). Denote by Γ_ψ the graph of ψ over a neighborhood of 0 in $\mathbb{R}^l$; Γ_ψ is a smooth submanifold of $\mathbb{R}^{l+2}$ passing through 0. Denote by Y the real-analytic subset of $\mathbb{R}^l_x \times \mathbb{C}_w \simeq \mathbb{R}^{l+2}$ defined in a neighborhood of 0 by the equation

$$a_\delta(x)w^\delta + \cdots + a_0(x) = 0. \tag{3.12}$$

In view of (3.11), $\Gamma_\psi \subset Y$ in a neighborhood of 0.

CLAIM 3.5. *Γ_ψ and Y have the same dimension at* 0.

Therefore, by a result of Malgrange [24, Chap. VI, Prop. 3.11], Γ_ψ is a real-analytic submanifold and consequently ψ is real-analytic near 0.

The proof of Lemma 3.4 is complete. □

The method that we used in step 2 follows the idea of [2, Lemma 2.7]. However, the following short proof of Claim 3.5 uses only basic notions of commutative algebra (see e.g. [32]) and simplifies the elimination method applied in [3, Lemma 5.1].

Proof of Claim 3.5. The dimension of Γ_ψ at 0 is l. Notice that (3.12) is divided into two real-valued equations. Therefore, the dimension of Y at 0 is l or $l+1$. Let S denote the common zeros of the a_j. For $x \notin S$, (3.12) determines w up to finitely many possibilities—that is, Y is a d-sheeted ramified analytic cover over $\mathbb{R}^l \setminus S$ near 0. Therefore, the dimension of Y at such points is l. Now we deal with the singular set S. Clearly, $S \times \mathbb{C} \subset Y$ and so, in order to prove that $\dim Y = l$, it suffices to prove that $\dim S \le l-2$. It is easier (and sufficient) to prove that $\dim_{\mathbb{C}} \mathcal{S} \le l-2$, where $\mathcal{S}$ is the complex-analytic subset of $\mathbb{C}^l$ defined near 0 by the equations $a_j(z) = 0,\ j = 0, \dots, \delta$. In these equations, we consider the a_j to be elements of the ring of convergent power series $\mathbb{C}\{z_1, \dots, z_l\}$. Without loss of generality, we may assume for all $j = 0, \dots, \delta$ that $a_j \not\equiv 0$, $a_j(0) = 0$, and a_j is irreducible (otherwise, we would use the following reasoning with each irreducible factor of a_j). Let A_j be the irreducible complex-analytic subset $\{a_j(z) = 0\}$ of dimension $l-1$ in $\mathbb{C}^l$. Since the a_j have no common factors, there exist two indices $j_1 \ne j_2$ such that $a_{j_1} \not\equiv a_{j_2}$ up to a unit of $\mathbb{C}\{z_1, \dots, z_l\}$, that is, A_{j_1} and A_{j_2} do not coincide near 0. Therefore, $\dim_0 A_{j_1} \cap A_{j_2} = l-2$ (see [32, Chap. VIII, Sec. 9, Cor. 2]) and $\dim_0 \mathcal{S} \le l-2$ as desired. □

Proof of Theorem 3.2 (cont.). For each $j = 1, \dots, N$, Lemma 3.3 shows that F_j is algebraic over the field $\hat{\mathcal{R}}_p(M)$. Then, Lemma 3.4 shows that each F_j is real-analytic near p.

The proof of Theorem 3.2 is complete. □

4. Characteristic Variety and Essential Finiteness

The notion of essential finiteness is related to the characteristic variety of a smooth CR diffeomorphism as follows.

LEMMA 4.1. *If $f\colon M \to M'$ is a smooth CR diffeomorphism between generic real-analytic submanifolds $M, M' \subset \mathbb{C}^n$, with $p \in M$, $p' \in M'$, and $f(p) = p'$, then $A'_{p'}$ coincides with $\mathcal{V}_p(f)$ in a neighborhood of p'.*

Proof. Let $M' \subset \mathbb{C}^{n'}$ be a generic real-analytic CR submanifold of codimension d' and CR dimension m'. As in Section 1.1, we may write the equations of M' near the point $p' \in M'$ in the form

$$\overline{y'_k} = \phi'_k(\overline{x'}, x', y'), \quad k = 1, \ldots, d', \tag{4.1}$$

where $\mathbb{C}^{n'} \ni z' = (x', y') \in \mathbb{C}^{m'} \times \mathbb{C}^{d'}$ is a system of local holomorphic coordinates near $p' = (x'_p, y'_p)$ and where the $\phi'_k(\xi', x', y')$ are holomorphic functions near $(\overline{x'_p}, x'_p, y'_p)$ satisfying $\phi'_k(\overline{x'_p}, x', y') \equiv \phi'_k(\xi', x'_p, y') \equiv y'$. The operators

$$L'_j(z', \overline{z'}) = \frac{\partial}{\partial \overline{x'_j}} + \sum_{k=1}^{d'} \frac{\partial \phi'_k}{\partial \overline{x'_j}}(\overline{x'}, x', y') \frac{\partial}{\partial \overline{y'_k}}, \quad j = 1, \ldots, m',$$

form a (commuting) basis of the CR operators on M', and the complexified complex-conjugate operators

$$\mathcal{L}'_j(z') = \frac{\partial}{\partial x'_j} + \sum_{k=1}^{d'} \frac{\partial \overline{\phi'_k}}{\partial x'_j}(x', \overline{x'_p}, \overline{y'_p}) \frac{\partial}{\partial y'_k}, \quad j = 1, \ldots, m',$$

form a (commuting) basis of the holomorphic operators tangent to $Q'_{p'}$, the Segre variety of M' associated to the point p'. We have used the notation $\bar{h}(Z) := \overline{h(\bar{Z})}$ for a holomorphic function h. It is easy to prove the following "curved" version of the uniqueness theorem for holomorphic functions.

FACT 4.2. A function R holomorphic near p' vanishes identically on $Q'_{p'}$ if and only if $\mathcal{L}'^{\alpha} R|_{p'} = 0$ for all $\alpha \in \mathbb{N}^{m'}$.

Since $f \colon M \to M'$ is a smooth CR diffeomorphism, it follows that $n = n'$, $m = m'$, and $d = d'$. Notice that $A'_{p'}$ is the set of points z' near p' such that $\rho'_k(\cdot, \overline{z'})$ vanishes identically on $Q'_{p'}$ for all $k = 1, \ldots, d$. In view of Fact 4.2, $z' \in A'_{p'}$ if and only if

$$\mathcal{L}'^{\alpha} \rho'_k(\cdot, \overline{z'})|_{p'} = 0, \quad k = 1, \ldots, d, \ \alpha \in \mathbb{N}^m, \tag{4.2}$$

which is clearly equivalent (after complex conjugation) to

$$L'^{\alpha} \rho'_k(z', \bar{\cdot})|_{p'} = 0, \quad k = 1, \ldots, d, \ \alpha \in \mathbb{N}^m. \tag{4.3}$$

The pull-back $K_j := f^* L'_j$ $(j = 1, \ldots, m)$ of the L'_j form a basis of the CR operators on M. Since (4.3) is equivalent to

$$K^{\alpha} \rho'_k(z', \overline{f(\cdot)})|_p = 0, \quad k = 1, \ldots, d, \ \alpha \in \mathbb{N}^m,$$

we have that (4.3), and consequently (4.2), is equivalent to

$$L^{\alpha} \rho'_k(z', \overline{f(\cdot)})|_p = 0, \quad k = 1, \ldots, d, \ \alpha \in \mathbb{N}^m. \tag{4.4}$$

We thus conclude that $A'_{p'}$ coincides with $\mathcal{V}_p(f)$ near p'.

The proof of Lemma 4.1 is complete. □

REMARK 4.3. In the proof of Lemma 4.1, we replace the equality $Q'_{z'} = Q'_{p'}$ between complex analytic sets (cf. Section 1.2) by the infinite system of complex-analytic equations (4.4), which represents the equality of the *germs* at p' of the complex-analytic sets $Q'_{p'}$ and $Q'_{z'}$. Furthermore, by Nœther's theorem, we may replace (4.4) by a finite subsystem representing the equality of the *jets* at p' of the sets $Q'_{p'}$ and $Q'_{z'}$.

Lemma 4.1 yields the following new characterization of essential finiteness.

PROPOSITION 4.4. *The submanifold M is essentially finite at p if and only if the dimension of $\mathcal{V}_p(\mathrm{id}_M)$ at p is 0, where id_M is the identity mapping of M.*

Proof. This is clear in view of Lemma 4.1. □

We now give the proofs of the corollaries of Theorem 1.2.

Proof of Corollary 1.3. Lemma 4.1, together with Theorem 1.2, directly gives the conclusion. □

Proof of Corollary 1.4. We assume that $p = p' = 0$ and use normal coordinates (see (1.1)) $z' = (x', y') \in \mathbb{C}^{n-1} \times \mathbb{C}$, so that M' is given near 0 by $\overline{y'} = \phi'(\overline{x'}, x', y')$, where $\phi'(\xi', x', y')$ is holomorphic near $(0, 0, 0)$ and satisfies $\phi'(0, x', y') \equiv \phi'(\xi', 0, y') \equiv y'$. We write $f = (f', f_n)$ in the normal coordinates. By a reasoning on formal power series, Baouendi and Rothschild [4] proved that, if f is of finite multiplicity at p and if M' is essentially finite at p', then the complex-analytic subset $\mathcal{W}_0(f) \subset \mathbb{C}^{n-1}$ defined by the equations in x',

$$L^\alpha \phi'(\overline{f'(\cdot)}, x', 0)|_0 = 0 \quad \text{for all } \alpha \in \mathbb{N}^{n-1},$$

is of dimension 0. Recall that the characteristic variety $\mathcal{V}_0(f)$ is given by the equations in z',

$$L^\alpha \rho'(z', \overline{f(\cdot)})|_0 = 0 \quad \text{for all for all} \alpha \in \mathbb{N}^{n-1}, \tag{4.5}$$

where we can choose $\rho'(z', \overline{z'}) := \phi'(\overline{x'}, x', y') - \overline{y'}$ as a defining function of M'. For $\alpha = (0, \ldots, 0)$, (4.5) implies that $\phi'(0, x', y') - 0 = y' = 0$. Therefore, (4.5) is equivalent to $L^\alpha(\phi'(\overline{f'(\cdot)}, x', 0) - \overline{f_n(\cdot)})|_0 = 0$ for all $\alpha \in \mathbb{N}^{n-1}$. Since the coordinates are normal, $L^\alpha \overline{f_n}|_0 = 0$ for all $\alpha \in \mathbb{N}^{n-1}$, which implies that $\mathcal{V}_0(f)$ coincides with $\mathcal{W}_0(f)$ near 0 and is consequently of dimension 0. Furthermore, [4, Thm. 3] shows that M is necessarily essentially finite at p. Since it is a hypersurface, M is therefore minimal at p and Theorem 1.2 applies, proving that f is real-analytic near p. □

Proof of Corollary 1.5. That f is K-nondegenerate at p obviously implies, by the holomorphic implicit function theorem, that the dimension of the characteristic variety $\mathcal{V}_p(f)$ is 0 at p'; Theorem 1.2 then gives the conclusion. □

REMARK 4.5. In the situation of Corollary 1.5, the proof of Theorem 1.2 is highly simplified: in Section 3.2, the holomorphic implicit function theorem shows that

$f_j(z) = H_j(z, \bar{z}, \overline{D^A f(z)})$ for $j = 1, \ldots, n'$ and $z \in M$, where the H_j are holomorphic near $(p, \bar{p}, \overline{D^A f(p)})$ and $D^A f(z)$ denotes the jet of order A of f at z; in other words, f_j is in $\mathcal{R}_p(M)$. Since, moreover, f_j is CR on M, there is no need to use our main technical tool (Proposition 2.5) in this situation; the simplified version (Proposition 2.6), which deals with the nonsingular case directly, proves that each f_j extends holomorphically in a neighborhood of p.

References

[1] R. A. Aĭrapetyan, *Continuation of CR-functions from piecewise-smooth CR-manifolds,* Mat. Sb. (N.S.) 134 (1987), 108–118.

[2] M. S. Baouendi, X. Huang, and L. Rothschild, *Regularity of CR mappings between algebraic hypersurfaces,* Invent. Math. 125 (1996), 13–36.

[3] M. S. Baouendi, H. Jacobowitz, and F. Trèves, *On the analyticity of CR mappings,* Ann. of Math. (2) 122 (1985), 365–400.

[4] M. S. Baouendi and L. P. Rothschild, *Germs of CR maps between real analytic hypersurfaces,* Invent. Math. 93 (1988), 481–500.

[5] ———, *Directions of analytic discs attached to generic manifolds of finite type,* J. Funct. Anal. 125 (1994), 149–171.

[6] E. Bedford, *Holomorphic continuation at a totally real edge,* Math. Ann. 230 (1977), 213–225.

[7] T. Bloom and I. Graham, *On "type" conditions for generic real submanifolds of* $\mathbb{C}^n$, Invent. Math. 40 (1977), 217–243.

[8] A. Boggess, *The holomorphic extension of CR functions near a point of higher type,* Proc. Amer. Math. Soc. 103 (1988), 847–855.

[9] E. M. Chirka, *Complex analytic sets,* Kluwer, Dordrecht, 1989.

[10] B. Coupet, H. Gaussier, and A. Sukhov, *Regularity of CR maps between convex hypersurfaces of finite type,* Proc. Amer. Math. Soc. 127 (1999), 3191–3200.

[11] B. Coupet, S. Pinchuk, and A. Sukhov, *On boundary rigidity and regularity of holomorphic mappings,* Internat. J. Math. 7 (1996), 617–643.

[12] ———, *Analyticité des applications CR,* C. R. Acad. Sci. Paris Sér. I Math. 329 (1999), 489–494.

[13] ———, *On partial analyticity of CR mappings,* Math. Z. 235 (2000), 541–557.

[14] B. Coupet and A. Sukhov, *On CR mappings between pseudoconvex hypersurfaces of finite type in* $\mathbb{C}^2$, Duke Math. J. 88 (1997), 281–304.

[15] S. Damour, *Sur l'algébricité des applications holomorphes,* C. R. Acad. Sci. Paris Sér. I Math. 332 (2001), 491–496.

[16] M. Derridj, *Le principe de réflexion en des points de faible pseudo convexité, pour des applications holomorphes propres,* Invent. Math. 79 (1985), 197–215.

[17] K. Diederich and J. E. Fornæss, *Proper holomorphic mappings between real-analytic pseudoconvex domains in* $\mathbb{C}^n$, Math. Ann. 282 (1988), 681–700.

[18] K. Diederich and S. M. Webster, *A reflection principle for degenerate real hypersurfaces,* Duke Math. J. 47 (1980), 835–843.

[19] F. Forstnerič, *Extending proper holomorphic mappings of positive codimension,* Invent. Math. 95 (1989), 31–61.

[20] C. K. Han, *Analyticity of CR equivalences between some real hypersurfaces in* $\mathbb{C}^n$ *with degenerate levi forms,* Invent. Math. 73 (1983), 51–69.

[21] S. M. Ivashkovich, *The Hartogs-type extension theorem for meromorphic maps into compact Kähler manifolds,* Invent. Math. 109 (1992), 47–54.
[22] B. Lamel, *A reflection principle for real-analytic submanifolds of complex spaces,* J. Geom. Anal. (to appear).
[23] H. Lewy, *On the boundary behaviour of holomorphic mappings,* Accad. Naz. Lincei 35 (1977), 1–8.
[24] B. Malgrange, *Ideals of differentiable functions,* Tata Inst. Fund. Res. Stud. Math, 3, Oxford Univ. Press, London, 1967.
[25] J. Merker and E. Porten, *On removable singularities for integrable CR functions,* Indiana Univ. Math. J. 48 (1999), 805–856.
[26] S. I. Pinchuk, *Boundary uniqueness for holomorphic functions of several complex variables,* Mat. Zametki 15 (1974), 205–215.
[27] ———, *On the analytic continuation of holomorphic mappings,* Mat. Sb. (N.S.) 27 (1975), 345–392.
[28] S. I. Pinchuk and Sh. I. Tsyganov, *Smoothness of CR-mappings of strictly pseudoconvex hypersurfaces,* Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), 1120–1129, 1136.
[29] W. Rothstein, *Ein neuer Beweis des Hartogsschen Hauptsatzes und seine Ausdehnung auf meromorphe Funktionen*, Math. Z. 53 (1950), 84–95.
[30] B. Shiffman, *Separately meromorphic functions and separately holomorphic mappings,* Several complex variables and complex geometry, part 1 (Santa Cruz, CA, 1989), pp. 191–198, Amer. Math. Soc., Providence, RI, 1991.
[31] A. E. Tumanov, *Extension of CR-functions into a wedge from a manifold of finite type,* Math. USSR-Sb. 136 (1988), 128–139.
[32] O. Zariski and P. Samuel, *Commutative algebra*, vol. II, Van Nostrand, Princeton, NJ, 1960.

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